Math 502. Problem Set 2. Due Tuesday February 1.

(1) Exercise 2.6 in Serre (it is in Section 2.3, p.17).

(2) Let $Z(G) = \{g \in G \mid gh = hg\ \forall h \in G\}$ be the centre of the group G. Let (ρ, W) be a finite-dimensional irreducible representation of G. Prove that for any element $g \in Z(G)$, $\rho(g)$ is multiplication by a scalar.

(3) Given a group G, let R_G denote its right regular representation. Prove that

$$R_{G\times H} \cong R_G \otimes R_H.$$

Hint: see 1.6 in the posted chapter of Kostrikin-Manin.

(4) Let V be a 3-dimensional vector space. Show that $\operatorname{Sym}^2 V$ is isomorphic to the space of homogeneous polynomials of degree 2 in 3 variables.

(5) Let (ρ, W) be the 3-dimensional irreducible representation of A_4. Find the decomposition of $\operatorname{Sym}^2 W$ into irreducibles.

(6) Let (ρ, V) be a representation of G, and let K be a subgroup of G.
  (a) Prove that the vectors fixed by $\rho(k)$ for every $k \in K$ form a linear subspace of G. Denote this subspace by V^K.
  (b) Let $P : V \to V$ be the linear operator defined by the formula

$$Pv := \frac{1}{|K|} \sum_{k\in K} \rho(k)v.$$

  Prove that P is a projector onto V^K.
  (c) Prove that P is the *unique* projector onto V^K that commutes with $\rho(k)$ for all $k \in K$.
  (d) Show that $\ker(P)$ is the linear span of $\{\pi(k)v - v \mid v \in V\}$.