Math 502. Problem Set 3. Due Tuesday February 22.

(1) Prove that the induced representation we defined in class (see p.19 of Prof. Casselman's notes) is an induced representation in the sense of Serre, Section 3.3. (Compare this with Exercise 3.5 in Serre, p.31).

(2) Let H be a normal subgroup of G, let (σ, W) be an irreducible representation of H. Recall that for $g \in G$, we denote by σ^g the representation $\sigma^g(h) = \sigma(ghg^{-1})$ of H. Prove that the induced representation $\mathrm{Ind}_H^G \sigma^g$ is isomorphic to $\mathrm{Ind}_H^G \sigma$ for all $g \in G$.

(3) Let p and ℓ be primes, with p dividing $\ell - 1$. Let $G = \mathbb{Z}/\ell\mathbb{Z} \rtimes \mathbb{Z}/p\mathbb{Z}$. Classify all irreducible representations of G.

(4) Give a proof of Frobenius reciprocity by means of computing characters. Namely, prove the formula (i) on p.31 of Serre.

(5) Exercise 3.6 in Serre.

(6) Prove that the values of the character of a representation (π, V) of a finite group G can be obtained as orbital integrals of matrix coefficients of π. More precisely, fix a G-invariant inner product on V, and let $\{e_1, \dots, e_d\}$ be an orthonormal basis of V with respect to this inner product. Let $f_{i,j}(g)$ be the (i, j)-entry of the matrix representing the linear operator $\pi(g)$ in this basis (recall that such a function on G is called a matrix coefficient). Prove that when $i = j$, we get

$$\frac{1}{|G|} \sum_{x \in G} f_{i,i}(xgx^{-1}) = \frac{1}{\dim(\pi)} \chi_\pi(g),$$

where χ_π is the character of π.