Math 502. Problem Set 4. Due Tuesday March 15.

(1) *Gauss sums.* Let p be a prime, and let $\chi : (\mathbb{Z}/p\mathbb{Z})^\times \to S^1$ be a *nontrivial* character of the *multiplicative* group $(\mathbb{Z}/p\mathbb{Z})^\times$. We can extend χ to a function on $\mathbb{Z}/p\mathbb{Z}$ by letting $\chi(0) = 0$. Recall that the group $\mathbb{Z}/p\mathbb{Z}$ is self-dual (this was essentially proved in Homework 1). Consider the Fourier transform of the function χ on $\mathbb{Z}/p\mathbb{Z}$ (note that we started with a character of the multiplicative group to get this function, but the Fourier transform is happenning on the additive group $\mathbb{Z}/p\mathbb{Z}$). The Fourier transform $\widehat{\chi} : \mathbb{Z}/p\mathbb{Z} \to \mathbb{C}$ is:

$$\widehat{\chi}(x) = \sum_{y \in \mathbb{Z}/p\mathbb{Z}} \chi(y) e^{-2\pi i x y/p} = \sum_{y \in (\mathbb{Z}/p\mathbb{Z})^\times} \chi(y) e^{-2\pi i x y/p}.$$

Let

$$G(\chi) = \sum_{y \in (\mathbb{Z}/p\mathbb{Z})^\times} \chi(y) e^{2\pi i y/p}.$$

The sum $G(\chi)$ is called a Gauss sum.

(a) Prove that $\widehat{\chi}(x) = \chi(-1) G(\chi) \overline{\chi}(x)$.
(b) Prove that $\overline{G(\chi)} = \chi(-1) G(\overline{\chi})$.
(c) Prove that $|G(\chi)| = \sqrt{p}$ (note that this implies that there are a lot of cancellations in the sum: a naive estimate of its magnitude would be $|G(\chi)| \leq p - 1$, since it's a sum of $p - 1$ roots of unity).

(2) Let G be a compact group, and let $f \in C(G)$ (where $C(G)$ denotes the space of continuous functions on G) be a left and right G-finite function (that is, the subspace of $C(G)$ spanned by left and right translates of f is finite-dimensional). Prove that there are only finitely many irreducible representations (π, V) of G such that $\pi(f) \neq 0$.

(3) Let k be an arbitrary field. Let A, B be k-algebras. An (A, B)- *bimodule* is a k-vector space V with both left A-module structure and a right B-module structure which satisfy $(av)b = a(vb)$ for all $a \in A, b \in B, v \in V$. Note that any left A-module is automatically an (A, k)-bimodule, and any right A-module is a (k, A)-bimodule.

Recall that if V is an (A, B)-bimodule, and W is a left B-module, then one can form the tensor product $V \otimes_B W$ – it is the k-vector space

$$(V \otimes_k W) / \langle vb \otimes w - v \otimes bw \mid v \in V, b \in B \rangle,$$

and $V \otimes_B W$ has a left A-module structure.

If A, B, C are three k-algebras, and if V is an (A, B)-bimodule, and W is an (A, C)-bimodule, then the vector space $\mathrm{Hom}_A(V, W)$ (this is the space of all left A-module homomorphisms from V to W) becomes a (B, C)-bimodule (in a canonical way) by setting $(bf)(v) = f(vb)$ and $(fc)(v) = f(v)c$ for all $b \in B$, $f \in \mathrm{Hom}_A(V, W)$, $v \in V$, and $c \in C$.

Now let A, B, C, D be four k-algebras, and let V be a (B, A)-bimodule, W be a (C, B)-bimodule, and X – a (C, D)-bimodule. Prove that

$$\mathrm{Hom}_B(V, \mathrm{Hom}_C(W, X)) \cong \mathrm{Hom}_C(W \otimes_B V, X) \quad \text{as } (A, D)-\text{bimodules}.$$

Hint: The isomorphism is given by $f \mapsto (w \otimes_B v \mapsto f(v)w)$ for all $v \in V$, $w \in W$, and $f \in \mathrm{Hom}_B(V, \mathrm{Hom}_C(W, X))$.

(4) Exercise 6.4 in Serre (p. 50).

(5) For what values of $a, b \in \mathbb{Q}$ is $x = a + b\sqrt{d}$ an algebraic integer, if
  (a) d=2
  (b) d=3
  (c) d=5 ?