Problem Set 1. Due Thursday January 12.

AM = Atiyah-Macdonald.

(1) (a) Prove that if $p \neq 2$ and -1 is not a square in $\mathbb{F}_p$, then the function $t \mapsto (\frac{t^2-1}{t^2+1}, \frac{2t}{t^2+1})$ defines a bijection between $\mathbb{F}_p$ and the set of solutions to the equation $x^2 + y^2 = 1$ over $\mathbb{F}_p$ with the exception of the solution $(1, 0)$.
    (b) Prove that if -1 is a square in $\mathbb{F}_p$, then the equation $x^2 + y^2 = 1$ has $p - 1$ solutions in $\mathbb{F}_p^2$.

(2) (a) Prove that if in a ring R every element is either nilpotent or invertible, then R is a local ring.
    (b) Prove that $\mathbb{C}[x]/(x^2)$ is a local ring.

(3) Let $C(X, x)$ be the ring of *germs of real-valued continuous functions at a point* x on a topological space X. By definition, a germ of continuous functions at x is the equivalence class of functions, where functions $f : X \to \mathbb{R}$ and $g : X \to \mathbb{R}$ are called equivalent if there exists an open neighbourhood U containing x such that f and g coincide on U. The set of germs of continuous functions at $x \in X$ can be given the ring structure by point-wise addition and multiplication of functions (check that this gives well-defined operations on germs!) Prove that $C(X, x)$ is a local ring.

(4) Exercise 2 on p.10 of AM.

(5) Exercise 4 on p.11 of AM.

(6) Exercise 5 on p. 11 of AM.

(7) Exercise 7 on p.11 of AM.

(8) Exercise 9 on p. 11 of AM.

(9) Exercise 10 on p.11 of AM.

(10) Exercise 12 on p.11 of AM.