Math 423/502. Problem Set 5 (final problem set on Representation theory).

(1) (ex. 2.2 in Serre) Let X be a finite set on which G acts, and let ρ be the corresponding permutation representation, and let χ be its character. Show that for an element $g \in G$, $\chi(g)$ is the number of elements of X fixed by g.
   (a) Let V be a 3-dimensional vector space. Show that $\operatorname{Sym}^2 V$ is isomorphic to the space of homogeneous polynomials of degree 2 in 3 variables.
   (b) Let (ρ, W) be the 3-dimensional irreducible representation of A_4. Find the decomposition of $\operatorname{Sym}^2 W$ into irreducibles.

(2) *Gauss sums.* Let p be a prime, and let $\chi : (\mathbb{Z}/p\mathbb{Z})^\times \to S^1$ be a *nontrivial* character of the *multiplicative* group $(\mathbb{Z}/p\mathbb{Z})^\times$. We can extend χ to a function on $\mathbb{Z}/p\mathbb{Z}$ by letting $\chi(0) = 0$.

Prove that the group $\mathbb{Z}/p\mathbb{Z}$ is self-dual. Then consider the Fourier transform of the function χ on $\mathbb{Z}/p\mathbb{Z}$ (note that we started with a function that is a character of the multiplicative group, but the Fourier transform is on the additive group $\mathbb{Z}/p\mathbb{Z}$). The Fourier transform $\widehat{\chi} : \mathbb{Z}/p\mathbb{Z} \to \mathbb{C}$ is:

$$\widehat{\chi}(x) = \sum_{y \in \mathbb{Z}/p\mathbb{Z}} \chi(y) e^{-2\pi i x y/p} = \sum_{y \in (\mathbb{Z}/p\mathbb{Z})^\times} \chi(y) e^{-2\pi i x y/p}.$$

Let

$$G(\chi) = \sum_{y \in (\mathbb{Z}/p\mathbb{Z})^\times} \chi(y) e^{2\pi i y/p}.$$

The sum $G(\chi)$ is called a Gauss sum.
   (a) Prove that $\widehat{\chi}(x) = \chi(-1) G(\chi) \overline{\chi}(x)$.
   (b) Prove that $\overline{G(\chi)} = \chi(-1) G(\overline{\chi})$.
   (c) Prove that $|G(\chi)| = \sqrt{p}$ (note that this implies that there are a lot of cancellations in the sum: a naive estimate of its magnitude would be $|G(\chi)| \le p-1$, since it's a sum of $p-1$ roots of unity).

(3) Given a group G, let R_G denote its right regular representation. Prove that

$$R_{G \times H} \cong R_G \otimes R_H.$$

(4) Let k be an arbitrary field. Let A, B be k-algebras. An (A, B)- *bimodule* is a k-vector space V with both left A-module structure and a right B-module structure which satisfy $(av)b = a(vb)$ for all $a \in A, b \in B, v \in V$. Note that any left A-module is automatically an (A, k)-bimodule, and any right A-module is a (k, A)-bimodule.

Recall that if V is an (A, B)-bimodule, and W is a left B-module, then one can form the tensor product $V \otimes_B W$ – it is the k-vector space

$$(V \otimes_k W) / \langle vb \otimes w - v \otimes bw \mid v \in V, b \in B \rangle,$$

and $V \otimes_B W$ has a left A-module structure.

If A, B, C are three k-algebras, and if V is an (A, B)-bimodule, and W is an (A, C)-bimodule, then the vector space $\operatorname{Hom}_A(V, W)$ (this is the space of all left A-module homomorphisms from V to W) becomes a (B, C)-bimodule (in a canonical way) by setting $(bf)(v) = f(vb)$ and $(fc)(v) = f(v)c$ for all $b \in B$, $f \in \operatorname{Hom}_A(V, W)$, $v \in V$, and $c \in C$.

Now let A, B, C, D be four k-algebras, and let V be a (B, A)-bimodule, W be a (C, B)-bimodule, and X – a (C, D)-bimodule. Prove that

$$\mathrm{Hom}_B(V, \mathrm{Hom}_C(W, X)) \cong \mathrm{Hom}_C(W \otimes_B V, X) \quad \text{as } (A, D) - \text{bimodules.}$$

Hint: The isomorphism is given by $f \mapsto (w \otimes_B v \mapsto f(v)w)$ for all $v \in V$, $w \in W$, and $f \in \mathrm{Hom}_B(V, \mathrm{Hom}_C(W, X))$.

(5) *Projector onto the fixed vectors.* Let (ρ, V) be a representation of G, and let K be a subgroup of G.
  (a) Prove that the vectors fixed by $\rho(k)$ for every $k \in K$ form a linear subspace of G. Denote this subspace by V^K.
  (b) Let $P : V \to V$ be the linear operator defined by the formula
  $$Pv := \frac{1}{|K|} \sum_{k \in K} \rho(k) v.$$
  Prove that P is a projector onto V^K.
  (c) Prove that P is the *unique* projector onto V^K that commutes with $\rho(k)$ for all $k \in K$.
  (d) Show that $\ker(P)$ is the linear span of $\{\pi(k)v - v \mid v \in V\}$.