Math 423/502. Optional problem set on linear algebra.
Please do not hand in.

V will always denote a complex vector space, and $I : V \to V$ – the identity map. If we say $A : V \to V$, we always mean a linear operator on V. Let V, W be two vector spaces. We denote by $\mathrm{Hom}(V, W)$ the vector space of all linear operators from V to W.

1. Jordan canonical form, and miscellaneous problems

(1) Let $A : V \to V$ be a linear map. Assume that $A^n = I$ for some n. Show that V has a basis of eigenvectors for A (that is, the matrix of A can be diagonalized).

(2) * Let A_s be the diagonal part of the canonical Jordan form of A, and let $A_n = A - A_s$. Prove that there exist polynomials P and Q, such that $A_s = P(A)$, $A_n = Q(A)$.

(3) Commuting linear operators.
   (a) Suppose $A, B : V \to V$ are diagonalizable linear operators (i.e. each of them has a basis of eigenvectors). Show that there exists a common basis of eigenvectors for A and B.
   (b) Suppose a linear operator A has distinct eigenvalues, and suppose $AB = BA$. Prove that there exists a polynomial P, such that $B = P(A)$. Is this assertion true if we do not assume that the eigenvalues of A are distinct?
   (c) In general, let $V_\lambda = \cup_m \ker(A - \lambda I)^m$ (call it the generalized eigenspace of A), and suppose $B : V \to V$ commutes with A. Show that the generalized eigenspaces of A are B-invariant.

(4) Projectors.
   (a) Let $p : V \to V$ be a linear operator satisfying $p^2 = p$ (such operators are called projectors). Show that there is a direct sum decomposition $V = \ker(p) \oplus \mathrm{Im}(p)$. (Thus, you can think of p as a projection onto its image along its kernel).
   (b) Let W be a linear subspace of V. Show that there is a one-to-one correspondence between projectors p with $\mathrm{Im}(p) = W$, and direct complements of W.
   (c) Suppose $A : V \to V$ commutes with p. Show that $\ker(p)$ and $\mathrm{Im}(p)$ are A-invariant subspaces.

2. Dual vector spaces and bilinear forms

Let V^* denote the linear dual of V, i.e., the space of linear functionals on V.

(5) (a) Let $\{e_1, \dots, e_n\}$ be a basis of V. Prove that there exists a basis $\{e_1^*, \dots, e_n^*\}$ of V^* with the property $e_i^*(e_j) = \delta_{ij}$. Such a basis is called the *dual basis* to $\{e_1, \dots, e_n\}$.

(b) Let $\{e_1, \ldots, e_n\}$ and $\{e_1^*, \ldots, e_n^*\}$ be dual bases of V and V^*, respectively. Suppose that $A : V \to V$ is a linear operator with the matrix $M = (a_{ij})$ with respect to the basis $\{e_1, \ldots, e_n\}$. Let $A^* : V^* \to V^*$ be the *dual* linear operator, defined by the property:

$$A^*(w)(v) = w(Av), \quad \forall w \in W, v \in V.$$

Show that the matrix of A^* with respect to the basis $\{e_1^*, \ldots, e_n^*\}$ is $(a_{ji}) = M^T$.

(6) Prove that for any matrix A, the rank of A equals the rank of A^T.

(7) A sequence of linear maps $V \xrightarrow{A} W \xrightarrow{B} U$ is called *exact* (in the middle term) if $\ker(B) = \mathrm{Im}(A)$. A longer sequence is called exact if it is exact in every term.

Prove that the sequence $0 \to V \xrightarrow{A} W \xrightarrow{B} U \to 0$ is exact if and only if the dual sequence $0 \to U^* \xrightarrow{B^*} W^* \xrightarrow{A^*} V^* \to 0$ is exact.

(8) Let $B : V \times V \to \mathbb{C}$ be a linear functional (such linear functionals are called bilinear forms on V). Find the condition on B that guarantees that the map $w \mapsto (v \mapsto B(v, w))$ is an isomorphism from V to V^*. (Note that there is no *canonical* isomorphism from V to V^*, but any nice enough bilinear form can be used to make such an isomorphism).

(9) Prove that $\mathrm{Hom}(V, W) \cong \mathrm{Hom}(W^*, V^*)$.

(10) Show that there is a *canonical* isomorphism $V^{**} \to V$.

3. Tensor products

(11) Let $f : V \times W \to V \otimes W$ be the canonical map: $f(v, w) = v \otimes w$. Prove that it is *universal* in the following sense:

for any vector space U, and any bilinear map $B : V \times W \to U$, there exists a unique linear operator $C : V \otimes W \to U$ such that $B = C \circ f$.

This is called the universal property of the tensor product. It is not hard to prove that any two objects satisfying such a universal property have to be isomorphic, and thus one can use the universal property as the *definition* of the tensor product.

(12) Prove that $V^* \otimes W$ is canonically isomorphic to $\mathrm{Hom}(V, W)$. (Hint: use the universal property of the tensor product).

(13) (a) Let $A : V_1 \to V_2$ be a linear map of vector spaces. Let W be an arbitrary vector space. Then we can construct the linear map

$$A \otimes I : V_1 \otimes W \to V_2 \otimes W,$$

where $I : W \to W$ is the idenity map. Prove that if $A : V_1 \to V_2$, $B : V_2 \to V_3$ are linear operators, then $(B \circ A) \otimes I = (B \otimes I) \circ (A \otimes I)$. (Note: this property tells us that "tensoring with W" is a *functor* from the category of vector spaces over $\mathbb{C}$ to itself.)

(b) Suppose $0 \to V_1 \xrightarrow{A} V_2 \xrightarrow{B} V_3 \to 0$ is an exact sequence of linear maps of vector spaces, and let W be an arbitrary vector space. Prove that the sequence

$$0 \to V_1 \otimes W \xrightarrow{A \otimes I} V_2 \otimes W \xrightarrow{B \otimes I} V_3 \otimes W \to 0$$

is exact as well. (In the language of functors and categories, this says that "*the tensor multiplication functor is exact*". Note that this is true for vector spaces over a field, but *not* for modules over a ring).

4. Symmetric and exterior powers

For the definitions of higher symmetric and exterior powers, please see, for example, Sections 5 and 6 in Kostrikin-Manin (the posted reference).

(14) Recall our definition of the space $\operatorname{Alt}^2 V$ (see Serre, Section 1.6). Prove that $\operatorname{Alt}^2 V \cong \wedge^2 V$.

(15) Prove that

$$\operatorname{Sym}^m(V \oplus W) = \bigoplus_{a=0}^{m} \operatorname{Sym}^a V \otimes \operatorname{Sym}^{m-a} W;$$
$$\wedge^m(V \oplus W) = \bigoplus_{a=0}^{m} \wedge^a V \otimes \wedge^{m-a} W.$$

(16) Let V be an n-dimensional vector space, and $A : V \to V$ – a linear map. Then $\wedge^n V$ is a 1-dimensional vector space, and thus $\wedge^n A : \wedge^n V \to \wedge^n V$ is multiplication by scalar. Prove that this scalar equals $\det(A)$.