Math 502. A couple "bonus problems".

For your enjoyment. No need to write this up unless you really want to.

(1) This is the problem that started representation theory as we know it (see the links at the top of the page, where this problem is actually solved, in case you want to read, rather than think, about it). It was proposed by Dedekind and solved by Frobenius.

Let us take a finite group G and write down its multiplication table. Now replace each element g in this multiplication table with a variable labelled x_g. Compute the determinant of the resulting matrix – it is a polynomial P in the variables $(x_g)_{g\in G}$. Consider the decomposition of P into irreducible factors. Prove that every irreducible polynomial P_i that appears in this decomposition appears with multiplicity equal to its degree.

(2) Fourier transform in general is known to have the property that the more concentrated the function is around a point, the more "spread out" its Fourier transform has to be (the precise formulation of this statement is called the uncertainty principle). The classical example: on $\mathbb{R}$, $\widehat{e^{-x^2/a^2}} = ae^{-a^2x^2}$. Prove the "uncertainty principle for finite groups":

$$\#\operatorname{supp}(f)\#\operatorname{supp}(\widehat{f}) \geq \#G,$$

where supp denotes the support of a function (the set of all points where the function is nonzero), and # denotes the cardinality of a set. Show that this inequality is sharp.