The space of central functions

Def: $Z(G) \subset \mathbb{C}[G]$ = space of conjugation-invariant elements
↑ central functions. (class functions) ↖ constant on conjugacy classes.

Remark: Characters belong to $Z(G)$ $\quad \chi_\rho \in Z(G)$ for ρ rep.

[because: $\chi_\rho(xgx^{-1}) = Tr(\rho(xgx^{-1})) = Tr\,\rho(g) = \chi_\rho(g)$

Tr is conj. invariant. $\square$

Theorem: Let $\{\rho_i\}$ be representations irreducible of G.

$\Rightarrow$ χ_{ρ_i} form an orthonormal basis of $Z(G)$
↑ as $L^2(G)$

Proof: We already know they are orthogonal, need to show: they span $Z(G)$.

Lemma: Let $f \in Z(G)$. Then $\rho(f) : V \longrightarrow V$

$$\rho(f) = \frac{1}{|G|} \sum_{g \in G} f(g) \cdot \rho(g)$$

$\rho(f)$ is $\lambda\, Id_V$ where $\lambda = \frac{1}{\dim V} (\bar{f}, \chi_\rho)_{L^2}$

Proof of theorem (assuming the Lemma):

Assume $\{\chi_{\rho_i}\}_i$ don't span $Z(G)$

$\Rightarrow \exists f \in Z(G)$ that is orthogonal to all χ_{ρ_i} (a non zero)

Take π = right regular representation.

complex conjugate of f. $\pi(\bar{f}) = \quad V \longrightarrow V = \bigoplus_i (\dim V_i)\, V_i$

$\Rightarrow \pi(\bar{f})$ is a linear operator that acts on each V_i

by the scalar from the Lemma: $\frac{1}{\dim V_i}(\bar{\bar{f}}, \chi_{\rho_i}) = \frac{1}{\dim V_i}(f, \chi_{\rho_i}) = 0$ orthog.

$\Rightarrow \pi(\bar{f})$ is the 0 operator $\pi(\bar{f}) : \mathbb{C}[G] \longrightarrow \mathbb{C}[G]$

$\Rightarrow \quad \bar{f}(g) = 0 \quad \forall g \in G.$ $\square$

<u>Proof of the Lemma:</u> (ρ, V) irreducible representation

$$\rho(f) : V \longrightarrow V$$

Let's show that if $f \in Z(G)$, then $\rho(f)$ commutes with $\rho(g)$ for $g \in G$.

$$\rho(g)^{-1} \rho(f) \rho(g) = \rho(g^{-1}) \frac{1}{|G|} \left(\sum_{x \in G} f(x) \rho(x) \right) \rho(g)$$

$f \in Z(G)$

$$= \frac{1}{|G|} \sum_{x \in G} f(x) \rho(g^{-1} x g) = \frac{1}{|G|} \sum_{y \in G} f(g y g^{-1}) \rho(y) = \rho(f)$$

$y = g^{-1} x g$

<u>Schur's Lemma</u>

$\Rightarrow \quad \rho(f) = \lambda \cdot Id. \qquad$ (now: What is λ?)

$$Tr\, \rho(f) = (\dim V) \cdot \lambda$$

$$\overset{\shortparallel}{(\bar{f}, \chi_\rho)_{L^2}} \longleftarrow Tr\, \rho(f) = \frac{1}{|G|} \sum_{g \in G} f(g) \overbrace{\overline{Tr\, \rho(g)}}^{\chi_\rho(g)} = (f, \overline{\chi_\rho})_{L^2} = (\bar{f}, \chi_\rho)_{L^2}$$

$$\Rightarrow \quad \lambda = \frac{1}{\dim V} \cdot (\bar{f}, \chi_\rho)_{L^2}$$ $\square$

<u>Corrollary:</u> # of irrep of G = # conj. classes of G.

<u>Proof:</u> $Z(G)$ has two bases:

- characters of irr. rep — ~~we just showed that.~~
- characteristic functions of conjugacy classes. $\square$

"Fourier transform interchanges these bases."

(at least for abelian groups.

Algebraic "perspective": (using finite groups)

Think of $\mathbb{C}[G]$ as a ring (with convolution as operation)

$Z(G)$ is the centre of $\mathbb{C}[G]$ (exercise)

$\mathbb{C}[G]$ is even an algebra over $\mathbb{C}$.

Def: An algebra A is called *simple*, if it is *not* nilpotent.
(i.e. $\{\sum_i a_i b_i \mid a_i b_i \in A\} = A^2 \neq 0$)

and has *no* proper 2-sided ideals.

An algebra A over a field k is called *central* if $Z(A) = k$.

An algebra A is called *semisimple* if it is a direct sum of simple algebras

Fact: A semisimple $\Leftrightarrow$ $\mathrm{Rad}(A) = \{ a \mid a \text{ kills all } A\text{-modules} \} = 0$.

Key examples: ① k any field:

$M_n(k)$ is a central simple algebra
$\overset{\shortparallel}{k^{n \times n}}$

(Exercise*: 2-sided ideals of $M_n(R)$ are of the form $M_n(J)$ for ideals J in commutative rings R...

② $\mathbb{H}$ Quaternions over $\mathbb{R}$, division Ring
$\Rightarrow$ simple algebra over $\mathbb{R}$.

Remark: Theorem (by Frobenius) The "only" division rings over $\mathbb{R}$ are $\mathbb{R}, \mathbb{C}, \mathbb{H}$ up to isomorphism.

(no proof)

Thm: (Wedderburn) Suppose k is a field. (no proof)

$\Rightarrow$ Every central and simple algebra over k is of the form $M_n(D)$, where D is a division ring over k.
that is a k-algebra

Corollary: k alg. closed $\wedge$ $k \subset D$ division ring. then $D = k$.

Proof: $\alpha \in D$ $k(\alpha) \leftarrow$ field extension of k $\Rightarrow$ equals k.
alg. closed

$\square$

So over $\mathbb{C}$, the only simple, central algebras are $M_n(\mathbb{C})$

Theorem: (Maschke) (from last time)

$\mathbb{C}[G]$ is a semisimple $\mathbb{C}$-algebra: $\mathbb{C}[G] = \bigoplus M_{n_i}(\mathbb{C})$
$= \bigoplus \operatorname{End}(V_i)$

Remark: One can prove this algebraically and then follows Maschke from it.

$$\mathbb{C}[G] \cong \bigoplus M_{n_i}(\mathbb{C}) = \bigoplus_{\substack{V_i \text{ irreps} \\ \text{of } G}} \operatorname{End} V_i$$

$\cup$

this is actually commutative, yay. :)

$Z(G) \longrightarrow \mathbb{C}^r$ for $r = \#$ of conj. classes of G.
$= \#$ irreps.

$f \longmapsto (\lambda_1, \dots, \lambda_r)$ by $(\lambda_1 Id_{V_1}, \dots, \lambda_r Id_{V_r})$

λ_i = scalar by which $\rho_i(f)$ acts on V_i (see Lemma)

$$\lambda_i = \frac{1}{\dim V_i} (\overline{f}, \chi_\rho)_{L^2}$$

Theorem: ρ irrep of G, $\Rightarrow \dim \rho \mid |G|$

Proof idea: (Tate) Note: $\rho(g)$ is matrix of finite order.
$\Rightarrow$ diagonalizable $\Rightarrow$ eigenvalues are roots of 1.

Then $\chi_\rho(g)$ is an algebraic integer.

Then if $f \in Z(G)$ is algebraic integer-valued, then $\sum_{g \in G} f(g) \chi_{\rho_i}(g)$ is also

Take $f = \chi_{g_i} \in Z(G)$ in integral closure of $\mathbb{Z}$ in $Z(G)$.

$\Rightarrow$ its image in $\mathbb{C}^n$ is integral over $\mathbb{Z}$.

image of $\chi_{g_i} = \frac{|G|}{\dim V} \in \mathbb{Q}$ has to be integral over $\mathbb{Z}$

$\mathbb{Z}$ integrally closed.

$\rightsquigarrow$ If didn't divide by $|G|$ ..... $\Rightarrow \frac{|G|}{\dim V} \in \mathbb{Z}$

"cool, crazy, efficient proof." $\square$

<u>Upshot:</u> For finite groups: $n_1, \dots, n_r = \dim V_1, \dots, \dim V_r$

- $\sum_i n_i^2 = |G|$
- $n_i \mid |G|$
- $r = \#$ conj. classes of G.

Use this if you actually want to find decomposition of a rep of a group.

<u>Remark:</u> <u>Fact:</u> $n_i \mid \frac{|G|}{|Z|}$ ← centre of G, (proof uses $\underbrace{G \times \dots \times G}_{m}$)

<u>Later:</u> If A normal abelian subgroup of $G \Rightarrow n_i \mid \frac{|G|}{|A|}$.

<u>Remark</u> Representations of $G \Leftrightarrow \mathbb{C}[G]$ - modules.

If V vectorspace: $\rho: G \to GL(V) \Leftrightarrow$ action of $\mathbb{C}[G]$ on V

f acts by $\rho(f)$

<u>but:</u> "non-commutative ring $\mathbb{C}[G]$"

<u>Compare with:</u> $(V, T) \longleftrightarrow k[x]$ - modules.

V: k-vectorspace, $T: V \to V$

<u>Examples:</u> 1) $G = \mathbb{Z}$: $\mathbb{C}[G] = \boxed{\mathbb{C}[\mathbb{Z}] = \mathbb{C}[z, z^{-1}]}$ Laurent-polynomials.

~~Representations of~~ $\mathbb{Z}$

2) $G = \mathbb{Z}/n\mathbb{Z} = C_n$, $\mathbb{C}[G] = \boxed{\mathbb{C}[x]/(x^n - 1) = \mathbb{C}[\mathbb{Z}/n\mathbb{Z}]}$

Relation with Fourier transform (Harmonic analysis)

see: www.tauda

Fourier analysis: f periodic function on $\mathbb{R}$ with period $\boxed{\tau := 2\pi}$

$$f(x) = \sum_{n=-\infty}^{\infty} c_n \sin(\tau n x) + d_n \cos(\tau n x)$$

ignore assumptions on f... (convergence...)

$$\mathbb{R} \longrightarrow S^1 \subset \mathbb{C}$$

unit circle with Haar-measure (invariant under S^1 as group) $\mu(g \cdot A) = \mu(A)$)

$$x \longmapsto e^{\tau i x}$$

$$f : S^1 \longrightarrow \mathbb{C}$$

$d\mu$ = arc length $d\vartheta$

Consider $L^2(\mu) = \{ f : S^1 \to \ \mid \frac{1}{\tau}\int_0^\tau |f(e^{i\vartheta})|^2 d\vartheta < \infty \}$

<u>Claim</u>: Fourier analysis can be understood as the decomposition of the right regular representations of S^1 into irreducible representations.

Most important thing from rep-theory.

1) Irr rep's of S^1: $n \in \mathbb{Z}$

$$z \longmapsto z^n$$

$$e^{i\vartheta} \longmapsto e^{in\vartheta} \quad \Rightarrow \text{ lin. op } \quad \mathbb{C} \to \mathbb{C}, \quad x \mapsto z^n x$$

This gives us infinitely many 1-dim-representations of S^1 indexed by $n \in \mathbb{Z}$.

<u>Exercise</u>: For $n \neq m$ they are <u>not</u> isomorphisms.

$$\rho : S^1 \longrightarrow GL_1(\mathbb{C}) \equiv \mathbb{C}^\times \quad \text{has to be group homo.}$$

$$z_1 z_2 \longmapsto \rho(z_1)\cdot\rho(z_2)$$

We require ρ to be continuous (as S^1 is a topological group)

- With this requirement these are <u>all</u> representations.

<u>Proof</u>: • 1-dim: $\rho(S^1)$ has to be compact. $\Rightarrow \rho(S^1) \subset S^1 \subset \mathbb{C}$

Look at roots of 1: use continuity $\Rightarrow$ homomorphism.

Any homom. $S^1 \to S^1$ has to be $z \mapsto z^n$ for some $n \in \mathbb{Z}$.

• More dim: reps of S^1: If V is irr then it is 1-dim

Fact: V - vector space

have a commuting family of lin. operators such that each is diagonalizable then they are simultaneously diagonalizable.

i.e. $\exists$ Basis $\{e_i\}$ $\rho(t)$ is diagonal

Then each $\langle e_i \rangle$ is S^1-invariant $\Rightarrow$ decomposes in 1-dim.

General fact: ~~A~~ Abelian Groups have 1-dim irreps.

$\square$

Back to S^1: irreps $\longleftrightarrow$ $n \in \mathbb{Z}$

$$L^2(S^1) \overset{\text{Peter-Weyl}}{=} \widehat{\bigoplus_{n\in\mathbb{Z}}} \mathbb{C}_{z^n}$$

1-dim reps, where S^1 acts by $z \longrightarrow z^n$.

Ψ

$f = \sum c_i \chi_n$, ONBasis of $L^2(S^1)$ given by the characters $\chi_n : z \to z^n$

Then the coefficients of f are $(f, \chi_n)_{L^2} = \frac{1}{\tau}\int_0^1 f(e^{\tau i x})\, \overline{\chi_n}(e^{\tau i x})\, dx$

$$= \frac{1}{\tau}\int_0^1 f(e^{\tau i x})\, e^{-\tau i n x}\, dx$$

(Fact: $\overline{\chi_n} = \chi_{-n}$)