Lie Theory - Lecture 3 - Orbits and Stabilizers

In the last lecture we saw what it meant for a lie group G to act on a manifold X. We said that G acts on X if there exists a group homomorphism $\alpha : G \to \mathrm{Diff}(X)$, such that the action map,

$$G \times X \to X \quad \text{sending} \quad (g, x) \mapsto \alpha(g)(x) \ \text{ is smooth.}$$

Choosing a point x on the manifold X gives,

$$\begin{array}{cccc} \alpha_x & : & G & \to & X \\ & : & g & \mapsto & \alpha(g)(x) \end{array}$$

Then the image of α_x is equal to the orbit of x and the inverse image $\alpha_x^{-1}(x)$ is equal to the stabilizer of x. Moreover α has the following property.

Lemma For a given $x \in X$, a_x is smooth of constant rank. By constant rank we mean that the rank of $d\alpha_x$ is the same at all points of G.

proof: α_x is the restriction of a smooth map and so smooth. So we have the following commutative diagram of smooth maps,

$$\begin{array}{ccc} G & \xrightarrow{\alpha_x} & X \\ {\scriptstyle l_g}\big\downarrow & & \big\downarrow{\scriptstyle \alpha(g)} \\ G & \xrightarrow{\alpha_x} & X \end{array}$$

But since both of the vertical maps are diffeomorphisms the rank of α_x must be constant, by the chain rule. □

Any map with this property has some nice features.

Facts Let $f : X \to Y$ have constant rank, then it is locally linearizable. That is in some local coordinates of the form $f : \mathbb{R}^n \to \mathbb{R}^m$,

$$f(x_1, \ldots, x_n) = (x_1, \ldots, x_k, 0, \ldots, 0)$$

where k is the rank of α_x. In particular we have that,

1. $\forall y \in Y, f^{-1}(y)$ is a submfld of X and if $x \in f^{-1}(y) \to T_x f^{-1}(y) = \ker(d_x f)$

2. Let $x \in X$ then $\exists$ small open nbd $O(x) \ni x$ such that $f(O(x))$ is a submfld of Y. Moreover $T_{f(x)} f(O(x)) = \mathrm{im}(d_x f)$

Note: As we saw with the embedding of the real line into the torus, $f(X)$ doesn't have to be a submanifold of Y.

3. $\text{codim} f^{-1}(p) = \text{rank of } f \quad \text{and} \quad \dim f(O(x)) = \text{rank of } f$

Using these three facts we get the following theorem for α_x.

Theorem Let α be an action on X as before. Then α_x has constant rank k and,

1. The stabilizer $G_x = \alpha_x^{-1}(x)$ is a lie subgroup of codim k.
2. There exists an open nbd $O(e) \ni e$ of the identity such that $\alpha_x(O(e))$ is a submfld of X, moreover,

$$T_x \alpha_x(O(e)) = \text{im}(d_e \alpha_x)$$

3. If the orbit of x is a submfld then its dimention is k.

Proposition If G is a compact lie group acting on a manifold X then all orbits are submanifolds of X.

proof: First of all recall that the orbit of $x \in X$ is the image of the map α_x. Therefore it's sufficent to prove that the image of α_x is locally a submanifold. Also by conjugation it is sufficient just to check a local neighborhood of one point, let's chose $\alpha_x(e) = x$.

By the theorem above $\alpha_x(O(e))$ is a submanifold, moreover $\alpha_x(O(e)) = \alpha_x(O(e) \cdot G_x)$. Now let's consider the rest of the image. The set $C = G \setminus O(e) \cdot G_x$ is a closed set in G so it is compact, hence $\alpha_x(C)$ is compact and so forms a closed subset of X.

The final step is to check that the sets $\alpha_x(C)$ and $\alpha_x(O(e) \cdot G_x)$ are disjoint, otherwise C contains elements of $O(e) \cdot G_x$. Hence $\alpha_x(O(e))$ is contained in the compliment of the closed set $\alpha_x(C)$ and so there is an open set $U \subset X$ such that $U \cap \alpha_x(G) = \alpha_x(O(e))$ $\square$

Examples We have seen some compact lie groups before,
- $S^1 \times \cdots \times S^1$ the product of n circles is a compact lie group.
- $O_n(\mathbb{R})$ is compact see homework.
- $O(f, V)$ the matrices preserving some symmetric bilinear form f, is a comact ;lie group for the same reason

$O_n(\mathbb{R})$ can be realised as a stabilizer of the $GL_n(\mathbb{R})$ action of $(X, B) \mapsto X^t B X$ on the symetric matricies, and since the orbit of a nondegenerate form is open in the symetric matricies can deduce by the rank nullity theorem that $\dim(O_n(\mathbb{R})) = \frac{n(n-1)}{2}$.

In a similar way the symplectic group $Sp(n, V)$ can be produced as the stabilizer of a $GL_n(V)$ action on the skew symmetric forms.