Lie Theory - Lecture 4 - Quotients

First we apply the theorem from the last lecture to the case where a lie group acts on another lie group.

$$f : G \to H \text{ a lie group map, then } G \text{ acts on } H \text{ by } \alpha(g)(h) = f(g) \cdot h$$

The kernal of f is then equal to the stabilizer of e_H and the image of f is equal to the orbit of e_H. Hence we can apply the theorem about orbits and stabilizers from the last lecture,

Theorem Let $f : G \to H$ be a lie group map then,
1. $\ker(f)$ is a normal lie subgroup and $T_e \ker(f) = \ker(d_e f)$
2. If $f(G)$ is a lie subgroup in H then $T_e f(G) = \operatorname{im}(d_e f)$

In order to define quotients of lie groups we introduce the following notion,

Definition We call a smooth map $p : X \to Y$ a *factorization map* if it has the properties such that

a) $U \subset X$ is open in X iff $p(U)$ is open in Y

b) Given a function f on X, $p^*(f) = f \circ p$ is smooth $\Rightarrow$ f is smooth

Here is an important fact about factorization maps,

Lemma Supose we have the following comutative diagram $\begin{array}{ccc} X & \xrightarrow{p} & Y \\ {\scriptstyle q}\downarrow & \swarrow{\scriptstyle \varphi} & \\ Z & & \end{array}$ if p is a factorization map, and q is smooth, then φ is smooth.

Corollary Given $\begin{array}{ccc} X & \xrightarrow{p} & Y \\ {\scriptstyle q}\downarrow & \swarrow{\scriptstyle \varphi} & \\ Z & & \end{array}$ if p, q are both factorization maps and φ is a bijection then φ is a diffeomorphism.

proof: By the lemma both φ and its inverse are smooth. □

An equivalent statement of this would be that given a map $p : X \to Y$ there is a unique differential sturcture on Y that makes p a factorization map. This is seen from the diagram,

$\begin{array}{ccc} X & \xrightarrow{p} & Y_1 \\ {\scriptstyle p}\downarrow & /\!\!/{\scriptstyle id} & \\ Y_2 & & \end{array}$ the identity map on the underling top. sp. must be a diffeo

Theorem (Quotients) Let H be a lie subgroup of G. Then the set G/H has the structure of a smooth manifold such that the projection,

$$p : G \to \frac{G}{H} \text{ is a factorization map and}$$

1. p is a locally trivial fibration with fibre H.
2. The canonical action of G on G/H is smooth.
3. If H is normal in G then G/H is a lie group.

proof:

Part 1. Define the topology on G/H as U is open iff $p^{-1}(U) \subset G$ is open. This topology must be hausdorff. Take $g_1 H$ and $g_2 H$ to be two distinct cosets, so $g_1^{-1} g_2 \notin H$. Now since H is closed we have by continuity of the multiplication and inverse functions that there exist nbds of g_1, g_2 such that

$$O(g_1)^{-1} \cdot O(g_2) \text{ doesn't meet } H$$

therefore $O(g_1)H \cap O(g_2)H = \emptyset$ and the space is hausdorff. Now we want to explain the differential structure. We will consider a nbd $O(e_G) \ni e_G$, take a submanifold S transversal to H at e_G and define the smooth map

$$\phi : S \times H \to G \text{ sending } (s, h) \mapsto s \cdot h$$

at the identity the derivative is $d_e \phi = d_e s + d_e h$ which is a linear isomorphism, therefore locally ϕ is a diffeomorphism. We can now give an open nbd of the identity in G/H the differential structure obtained from S. Then we can use shifts to transport the differential structure elsewhere over all of G/H. Finally p can be seen to be a factorization map from its local behavior.

Part 2. Consider the following commutative diagram

$$\begin{array}{ccc} G \times G & \xrightarrow{\mu} & G \\ {\scriptstyle p\times id}\downarrow & \searrow^{p\circ\mu} & \downarrow{\scriptstyle p} \\ G \times G/H & \xrightarrow[\lambda]{} & G/H \end{array}$$

the map $p \circ \mu$ is smooth and the map $p \times id$ is a factorization map so applying the lemma to the lower triangle gives that the action λ of G on G/H is smooth.

Part 3. Must show group operations are smooth, this is exactly the same as Part 2, e.g. we replace $p \times id$ with the factorization map $p \times p$ to get,

$$\begin{array}{ccc} G \times G & \xrightarrow{\mu} & G \\ {\scriptstyle p\times p}\downarrow & \searrow^{p\circ\mu} & \downarrow{\scriptstyle p} \\ G/H \times G/H & \longrightarrow & G/H \end{array} \qquad \square$$