Lie Groups. Problem Set 3. Due Wednesday October 9.

1. Let k, l be positive integers, $n = k + l$. Let us define the group $O(k, l)$ to be the group of all (real) matrices that preserve the quadratic form $x_1^2 + \cdots + x_k^2 - x_{k+1}^2 - \cdots - x_n^2$. This is a Lie group of dimension $n(n-1)/2$. Let us denote by $SO(k, l)$ the subgroup that consists of matrices with determinant 1.
   (a) Show that $SO(k, l)$ is an open subgroup of index 2 in $O(k, l)$.
   (b) Show that every element $A \in O(k, l)$ has nonzero $k \times k$-minor that is made from the first k rows and the first k columns.
   (c) Show that this minor can be either positive or negative, and therefore $SO(k, l)$ is not connected (there are two connected components).

2. Show that $GL_n(\mathbb{C})$ is connected, but $GL_n(\mathbb{R})$ has two connected components.

3. Show that U_n and Sp_n are connected.

4. Show that SU_n and Sp_n are simply connected.

5. Show that $\pi_1(SO_n) = \mathbb{Z}/2\mathbb{Z}$ when $n \geq 3$.