Lie Groups. Problem Set 4. Due Monday October 22.

1. Let $\mathfrak{g}$ be a Lie algebra that is a semidirect sum of $\mathfrak{g}_1$ and $\mathfrak{g}_2$ (that is, as a vector space it is a direct sum, and $\mathfrak{g}_1$ is an ideal). Let us define the map $\beta : \mathfrak{g}_2 \to \mathfrak{gl}(\mathfrak{g}_1)$ by the formula $[X_2, X_1]_{\mathfrak{g}} = \beta(X_2)X_1$. Show that for every $X_2 \in \mathfrak{g}_2$, $\beta(X_2)$ is a derivation of $\mathfrak{g}_1$.
2. (a) Show that $(GL_n(K))' = SL_n(K)$.
   (b) Show that $O_n(K)' = SO_n(K)$.
   (c) Show that $U_n' = SU_n$.
3. Show that $SL_n(K)$ is semisimple.
4. Show that the radical of a complex Lie group coincides with the radical of the same group if you considered it as a real Lie group.
5. Show that any nontrivial connected solvable Lie group has a connected normal subgroup of codimension 1.
6. Show that weight subspaces corresponding to different weights are linearly independent.
7. Let H be a normal subgroup of G, and let χ be a character of H, and R – a representation of G. Prove that for any $g \in G$, $R(g)V_\chi(H) = V_{\chi^g}(H)$, where χ^g is another character of H defined by $\chi^g(h) = \chi(g^{-1}hg)$.