CHAPTER 1

Lie Groups

1. Connected and simply connected Lie groups

Lemma 1.1. *Let G be a Lie group and G^0 the connected component of the identity e of G. Then G^0 is a normal subgroup of G and G/G^0 is a discrete group.*

Proof. For all $g \in G^0$ we have that gG^0 is connected, open and closed since G^0 has these properties and the product is an homeomorphism. Since $g \in gG^0$ we have that $gG^0 = G^0$. Similarly $(G^0)^{-1}$ is connected, open and closed containing e so that $(G^0)^{-1} = G^0$. It follows that G^0 is a subgroup of G. Moreover for all $g \in G$ we have that gGg^{-1} is connected, open and closed. Since $e \in gGg^{-1}$ we have that $gGg^{-1} = G$, i.e. G^0 is normal.

Again because of the fact that multiplication by an element $g \in G$ is an homeomorphism, we have that the cosets of G^0 are the connected components of G, so that G/G^0 is discrete. □

Lemma 1.2. *Any open subgroup H of a Lie group G is closed and therefore contains G^0.*

Proof. The complement $G \setminus H$ is the union of the cosets of H different from H itself, which means that it is an open subset of G. □

Lemma 1.3. *Any connected Lie group G is generated by any open neighborhood of the identity e.*

Proof. The subgroup generated by an open set is open, by the lemma above it is closed, thus equal to G (who is connected). □

Theorem 1.4. *Let G be a Lie group acting transitively via α on a smooth connected manifold X. Then we have that:*

(1) *G^0 acts transitively on X as well.*
(2) *For all $x \in X$ we have that $G/G^0 \cong G_x/G_x \cap G^0$.*
(3) *If G_x is connected for some $x \in X$, then G is connected.*

Proof. (1) The map $\alpha_x : G \to X; g \mapsto \alpha(g)x$ is surjective and of constant rank equal to $dim(X)$. Since the rank is a local notion, we have that α_x restricted to G^0 has full rank so that it stays surjective on a neighborhood of x. This means that the orbit of x under the action of G^0 contains a neighborhood of x so that each orbit is open and closed, thus equal to X (which is connected).

(2) The group G^0 acts transitively on X, hence for every $g \in G$ we can choose an element $g' \in G^0$ such that $g'x = g^{-1}x$. This means that $gg' \in gG^0 \cap G_x$ and so $G_xG^0 = G$. The conclusion follows.

(3) It should follow from point 2.

□

Example 1.5. Clearly $SL_1(\mathbb{K})$ is connected. Moreover the stabilizer of the vector $(1, 0, ..., 0) \in \mathbb{K}^n$ under the natural action of $SL_n(\mathbb{K})$ is homeomorphic to $SL_{n-1}(\mathbb{K}) \times \mathbb{K}^{n-1}$, since an element of this set is of the form:

$$\begin{pmatrix} 1 & v \\ 0 & A \end{pmatrix}$$

where $A \in SL_{n-1}(\mathbb{K})$ and $v \in \mathbb{K}^{n-1}$. The stabilizer of the point $(1, 0, ..., 0)$ is then connected by induction hypothesis, thus the theorem above says that $SL_n(\mathbb{K})$ is connected.

Likewise we have that $SO_n(\mathbb{R})$ is connected since the stabilizer of its action on the sphere is $SO_{n-1}(\mathbb{R})$.

2. Simply connected Lie groups and universal cover

Definition 1.6. A Lie group homomorphism $f : G \to H$ is a covering homomorphism if it satisfies one of the following equivalent conditions:

(1) The homomorphism f maps diffeomorphically a neighborhood of e_G into a neighborhood of e_H.
(2) The subgroup $Ker f$ is discrete.
(3) The homomorphism f is a covering map.
(4) The differential $d_{e_G} f$ is an isomorphism between the tangent algebras.

Example 1.7. Consider the adjoint representation $Ad : SL_2(\mathbb{C}) \to End(sl_2(\mathbb{C}))$. Since $Ad(A)X = AXA^{-1}$, we have that $Ad(A)$ preserve the quadratic form $a^2 - bc$, hence $Ad(SL_2(\mathbb{C})) \subset O_3(\mathbb{C})$. Observing that $Ker(Ad) = Z(SL_2(\mathbb{C})) = \{I, -I\}$ we find that $SL_2(\mathbb{C}) \to O_3(\mathbb{C})$ is a covering homomorphism.

Lemma 1.8. *Let G be a connected Lie group. If N is a normal discrete subgroup, then $N \subset Z(G)$.*

Proof. For any $n \in N$ consider the map $f_n : G \to N; g \mapsto gng^{-1}$. The set $Im f$ is connected, but N is discrete so that $Im f$ is a point. Since $n \in Im f$ we conclude that $Im f = \{n\}$, i.e. $N \subset Z(G)$. □

Theorem 1.9. *Any connected Lie group G is isomorphic to a group $\widetilde{G}/N$, where $\widetilde{G}$ is a simply connected Lie group and N is a discrete central subgroup of $\widetilde{G}$. Furthermore if $(\widetilde{G_1}, N_1)$ is another such pair, then there exists a Lie group isomorphism $f : \widetilde{G} \to \widetilde{G_1}$ sending N to N_1.*

Proof. Recall that if we are given two simply connected covers $p : \widetilde{X} \to X$, $q : \widetilde{Y} \to Y$, and if we are given a map between the base spaces $f : X \to Y$ and two points $\tilde{x_0} \in \widetilde{X}$, $\tilde{y}_0 \in \widetilde{Y}$ such that $f(p(\tilde{x}_0)) = q(\tilde{y}_0)$, then there exists a unique map

$\tilde{f} : \widetilde{X} \to \widetilde{Y}$ with $\tilde{f}(\tilde{x}_0) = \tilde{y}_0$ and making the following diagram commute:

$$\begin{array}{ccc} \widetilde{X} & \overset{\tilde{f}}{\dashrightarrow} & \widetilde{Y} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ X & \underset{f}{\longrightarrow} & Y \end{array}$$

In fact for any other point $\tilde{x}_1 \in \widetilde{X}$ we can choose a path from $\tilde{x}_0$ to $\tilde{x}_1$. The composite $f \circ p$ gives a path in Y from $f(p(\tilde{x}_0))$ to $f(p(\tilde{x}_1))$ which lifts to a path in $\widetilde{Y}$. This last path α gives a well defined image $\tilde{f}(x_1) = \alpha(1)$.

Let now $p : \widetilde{G} \to G$ be a topological smooth simply connected cover. Let $\tilde{e}$ be a point in $p^{-1}(e_G)$. Apply the fact above to get the two following diagrams:

$$\begin{array}{ccc} \widetilde{G} \times \widetilde{G} & \overset{\tilde{\mu}}{\longrightarrow} & \widetilde{G} \\ {\scriptstyle p\times p}\downarrow & & \downarrow{\scriptstyle p} \\ G \times G & \underset{\mu}{\longrightarrow} & G \end{array}$$

$$\begin{array}{ccc} \widetilde{G} & \overset{\tilde{i}}{\dashrightarrow} & \widetilde{G} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p} \\ G & \underset{i}{\longrightarrow} & G \end{array}$$

where μ is the product in G and i is the inverse map of G. Using the uniqueness of $\tilde{\mu}$ and $\tilde{i}$ together with the fact that μ is a product and i is the inverse, we can prove that $\tilde{\mu}$ defines a group law on $\widetilde{G}$ with inverse given by $\tilde{i}$: The maps $\tilde{\mu}_1 : \widetilde{G} \times \widetilde{G} \times \widetilde{G} \to \widetilde{G}$; $(\tilde{x}, \tilde{y}, \tilde{z}) \mapsto \tilde{\mu}(\tilde{\mu}(\tilde{x}, \tilde{y}), \tilde{z})$ and $\tilde{\mu}_2 : \widetilde{G} \times \widetilde{G} \times \widetilde{G} \to \widetilde{G}$; $(\tilde{x}, \tilde{y}, \tilde{z}) \mapsto \tilde{\mu}(\tilde{x}, \tilde{\mu}(\tilde{y}, \tilde{z}))$ both cover the map $\mu : G \times G \times G \to G$; $(x, y, z) \mapsto xyz$ so that they are the same. Similarly for $\tilde{i}$. It follows that $G \cong \widetilde{G}/N$ as announced.

We have still to prove the uniqueness up to isomorphism of $\widetilde{G}$ and N. Let $(\widetilde{G_1}, N_1)$ be another such pair. Applying again the fact stated at the beginning of this proof, we find the following diagram:

$$\begin{array}{ccc} \widetilde{G} & \overset{\tilde{f}}{\dashrightarrow} & \widetilde{G_1} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p_1} \\ \widetilde{G}/N & \underset{f}{\longrightarrow} & \widetilde{G_1}/N_1 \end{array}$$

where f is the obvious isomorphism. Switching the roles of $\widetilde{G}$ and $\widetilde{G_1}$ and using the uniqueness of these maps, we have that $\tilde{f}$ is an homeomorphism. Applying one more time the uniqueness of the map between the simply connected cover, we can see that $\tilde{f}$ is also a group homomorphism (sending N to N_1): The two maps $\widetilde{G} \times \widetilde{G} \to \widetilde{G_1}$ given by $(\tilde{x}, \tilde{y}) \mapsto \tilde{f}(\tilde{x})\tilde{f}(\tilde{y})$ and $(\tilde{x}, \tilde{y}) \mapsto \tilde{f}(\tilde{x}\tilde{y})$ both cover the map $\widetilde{G}/N \times \widetilde{G}/N \to \widetilde{G_1}/N_1$; $(x, y) \mapsto f(x, y) = f(x)f(y)$, hence they are the same. □

Corollary 1.10. *Under the same assumptions as the theorem above we have that* $\pi_1(G) \cong N$.

Proof. Obviously $\pi_1(G)$ =Deck transformations= N. □

We recall now some basic facts about the foundamental group. Let $p : X \to Y$ be locally trivial fibration with fiber Z. Assume that X and Y are connected. Let $i : Z \to X$ be an inclusion and $z_0 \in Z$ a base point. Set $x_0 = i(z_0)$ and $y_0 = p(i(z_0))$. We have then an exact sequence of homotopy groups:

$$\pi_1(Z) \xrightarrow{i_*} \pi_1(X) \xrightarrow{p_*} \pi_1(Y) \xrightarrow{d} \pi_0(Z) \longrightarrow 0$$

where d is defined as follows: given a loop α in Y based at y_0, choose a lift $\tilde{\alpha}$ of α in X (such a lift exists since the fibration is locally trivial). Then the connected component of $\tilde{\alpha}(1)$ does not depend on the homotopy type of α, for a homotopy from α to β will lift to a homotopy from $\tilde{\alpha}$ to $\tilde{\beta}$, which clearly has to stay in the same connected component.