Lie Algebras Cohomology

Carmen A. Bruni

December 3, 2011

Our primary goal is to apply our knowledge of cochain complexes to study Lie algebras. As is often the case with cohomological theories, this will give us an effective language to express ideas in Lie algebras. To see more on the topic, one can look into Whitehead lemmas and their applications to proving Weyl's theorem. There is an alternative way to formulate Lie algebra cohomology using ideas related to de Rham cohomology and Lie groups, however for simplicity, the interest of time, and since this is a course on Lie algebras, we will approach this via category theory.

Definition 1. A cochain complex of $\mathfrak{g}$-modules is a family $\{C^n\}_{n\in\mathbb{Z}}$ of $\mathfrak{g}$-modules with maps $d^n : C^n \to C^{n+1}$ such that $d^n \circ d^{n-1} = 0$. In particular, this means that $\mathrm{im}(d^{n-1}) \subseteq \ker(d^n)$. We define

$$H^n(C^\cdot) = \ker(d^n)/\mathrm{im}(d^{n-1})$$

where $C^\cdot$ is the abbreviated notation used for cochain complexes. We call these $\mathfrak{g}$-modules cohomology modules.

What are some examples of cochain complexes of $\mathfrak{g}$-modules? One example comes from *exact sequences*, that is sequences where $\mathrm{im}(d^{n-1}) = \ker(d^n)$. However this brings on the question of where do exact sequences come from. Another more helpful example comes from left derived functors which we now shift our attention to.

Throughout, let L, M, N be arbitrary $\mathfrak{g}$-modules over a field k. Look at the maximal trivial $\mathfrak{g}$-submodule of M. That is,

$$M^{\mathfrak{g}} := \{m \in M| \quad xm = 0 \quad \forall x \in \mathfrak{g}\}$$

One can show that the map from the category of $\mathfrak{g}$-modules to the category of $\mathfrak{g}$-modules [1] defined by $F(M) = M^{\mathfrak{g}}$ is a functor.[2] Let's see what this functor does to exact sequences. Notice that

$$0 \xrightarrow{d_1} L \xrightarrow{d_2} M \xrightarrow{d_3} N \xrightarrow{d_4} 0$$

under the functor F gets mapped to

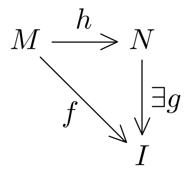
$$0 \xrightarrow{d_1'} L^{\mathfrak{g}} \xrightarrow{d_2'} M^{\mathfrak{g}} \xrightarrow{d_3'} N^{\mathfrak{g}}$$

where the d_i' are just the restriction maps. This last sequence turns out to be left exact (meaning the last map is not necessarily a surjection (so $\mathrm{im} d_3' \neq \ker d_4'$), but all the other maps are exact). We call F a *left exact functor*. Again however, this definition has started with a given exact sequence. So how do we form these sequences? The answer to this lies in a process known as injective resolution. This will give us right derived functors. They can be defined for categories in general but to keep it a bit grounded, I will define the concept over $\mathfrak{g}$-modules. First, we need some preliminary definitions

[1] Here, our category consists of $\mathfrak{g}$-modules and $\mathfrak{g}$-modules homomorphisms give us the maps between $\mathfrak{g}$-modules. If you have never seen category theory before, Wikipedia has some good information to get you started.

[2] It turns out that this is right adjoint to the trivial $\mathfrak{g}$-module functor and hence is a functor. This is sometimes called the submodule functor.

Definition 2. A $\mathfrak{g}$-module is said to be injective in the category of $\mathfrak{g}$-modules if $\hom(\cdot, I)$ is an exact functor. Equivalently, it is injective if for all morphisms $f : M \to I$ and injection $h : M \to N$, there exists a $\mathfrak{g}$-morphism $g : N \to I$ such that $g \circ h = f$. In other words, injections defined on submodules can be extended to the entire module. In a picture, this means that the following diagram commutes



Definition 3. A category is said to have enough injectives meaning that for each object A, there is a long exact sequence

$$0 \to M \to I^0 \to I^1 \to \ldots$$

with each I^j injective.

Note. The category of $\mathfrak{g}$-modules has enough injectives. This follows since every left $\mathfrak{g}$-module is naturally a left $U_{\mathfrak{g}}$-module, where $U_{\mathfrak{g}}$ is the universal enveloping algebra.

Note. This definition is usually written as the fact that every object can be embedded into an injective object.

With our new dictionary, let's see what happens when we apply a left exact functor like our $F(M) = M^{\mathfrak{g}}$ to an injective resolution of M. Notice that the exact sequence from an injective resolution

$$0 \to M \to I^0 \to I^1 \to \ldots$$

transforms into

$$0 \overset{d^{-1}}{\to} F(I^0) \overset{d^0}{\to} F(I^1) \overset{d^1}{\to} F(I^2) \to \ldots$$

where in the first map we compress the first two maps so this sequences uses just the injective modules. Notice that this second sequence no longer necessarily exact, which is actually a good thing because if it were exact, as we discussed before, we would have trivial cohomology. We define

$$H^i(\mathfrak{g}, M) := R^i(F)(N) := \ker(d^i)/\mathrm{im}(d^{i-1})$$

and we call this the Lie algebra cohomology of a $\mathfrak{g}$-module M. In general, this process is known as taking the right derived functors.

Note. It is at this point that I should note that functors take complexes to complexes so in fact the second sequence in the above definition is actually a complex.

Note. As a result of left exactness, one can actually show that $H^0(\mathfrak{g}, M) = M^{\mathfrak{g}}$, where here the equals denotes a canonical isomorphism. This is a direct check and is left as an exercise.

Note. Also notice that this process as stated is not clearly well defined. Recall that we started with an arbitrary injective resolution and we constructed these right derived functors. One can check that this is well defined however I felt that I would need to discuss maps between complexes and this is a direction I did not want to include in this brief talk. However I clearly feel morally obligated to at least mention this implicit fact.

As a special case, notice that $F_{\mathfrak{g},M}(N) := \hom_{\mathfrak{g}}(M, N)$ is also a left exact functor. In this case, denote

$$\mathrm{Ext}^i_{\mathfrak{g}}(M, N) := R^i(F_{\mathfrak{g},M})(N).$$

In fact, we have the following

Theorem 4. *Let k denote the trivial $\mathfrak{g}$-module defined on k (that is, $ga = 0$ for all $g \in \mathfrak{g}$ and $a \in k$). We have that $H^i(\mathfrak{g}, M) \cong \mathrm{Ext}^i_{U_\mathfrak{g}}(k, M)$.*

Proof. (Sketch) In class, we showed that every left g-module is a left $U_\mathfrak{g}$-module. This implies that

$$\hom_{U_\mathfrak{g}}(k, M) \cong \hom_g(k, M)$$

However, using the map

$$\begin{aligned} \hom_\mathfrak{g}(k, M) &\to M^\mathfrak{g} \\ \phi &\mapsto \phi(1) \end{aligned}$$

which is well defined since $g\phi(1) = \phi(g \cdot 1) = \phi(0) = 0$ as k is a trivial $\mathfrak{g}$-module, ϕ is a $\mathfrak{g}$-module homomorphism and $g \in \mathfrak{g}$ is arbitrary, can be used to give an isomorphism from $\hom_{U_\mathfrak{g}}(k, M)$ to $M^\mathfrak{g}$. Since the first term of the derived functor sequence is equal, then we have that all derived functors are the same which would complete the proof. ■

References

[Wei94] Charles A. Weibel. *An introduction to homological algebra*, volume 38 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1994.