Growing list of problems for Math534.

All Lie algebras are assumed to be finite-dimensional and over $\mathbb{C}$, unless otherwise specified. The references are: [H] – Humphreys; [FH] – Fulton & Harris.

The exercises not marked with a check mark are to be discussed during the next problem session in class.

1.✓ (a) Show that the Lie algebras $\mathfrak{sl}_n$, $\mathfrak{so}_n$ (with $n > 2$), $\mathfrak{sp}_n$ have the property $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$. (this is Exercise 9 on p. 5 of [H]).

   (b) Show that the derived algebra of $\mathfrak{gl}_n$ is $\mathfrak{sl}_n$ (this is [H], Exercise 2 on p.9)

2.✓ (a) Humphreys, Exercise 7 on p.5.

   (b) Humphreys, Exercise 3 on p.10.

3.✓ Humphreys, Exercise 7 on p.24.

4.✓ Let $G = \mathrm{SL}_2(\mathbb{R})$ be the group of real 2×2 matrices with determinant 1, and let $K = \mathrm{SO}_2(\mathbb{R})$ be the subgroup of real matrices preserving the quadratic form $Q(x, y) = x^2 + y^2$; we consider the both groups with their natural (real) topology. Let $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{R})$ be the Lie algebra of real 2×2 matrices of trace 0. Let $\mathbf{H} = \{x + iy \mid y > 0\}$ be the upper half-plane in $\mathbb{C}$. Let $C^\infty(\mathbf{H})$ be the space of (real)-smooth functions on $\mathbf{H}$. Recall that $\mathrm{SL}_2(\mathbb{R})$ acts on $\mathbf{H}$ by:

$$\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \cdot z = \frac{az + b}{cz + d}.$$

   (a) Prove that there is a natural homeomorphism from the quotient $\mathrm{SL}_2(\mathbb{R})/\mathrm{SO}_2(\mathbb{R})$ to $\mathbf{H}$. (Hint: consider the stabilizer of i under the action of $\mathrm{SL}_2(\mathbb{R})$ on $\mathbf{H}$). (This is in fact a diffeomorphism of real manifolds).

   (b) The homeomorphism from part (a) induces the isomorphism between the space of continuous functions on $\mathbf{H}$ and continuous K-invariant (i.e., such that $f(k^{-1}g) = f(g)$ for every $k \in K$, $g \in G$) functions on G.

   (c) Prove that when $X \in \mathfrak{g}$ is sufficiently close to 0, the matrix $\exp(X)$ is defined and is an element of G close to 1.

   (d) Prove that the action $X \cdot f = \frac{d}{dt}|_{t=0} f(\exp(-tX) \cdot z)$ makes $C^\infty(\mathbf{H})$ into a $\mathfrak{g}$-module. Denote this representation by ρ.

   (e) Let $\{X, Y, H\}$ be the standard basis of $\mathfrak{g}$. Prove that the Casimir operator $\rho(X)\rho(Y) + \rho(Y)\rho(X) + \frac{1}{2}\rho(H)^2$ coincides with the Laplace operator on $\mathbf{H}$, i.e. that it acts on $C^\infty(\mathbf{H})$ by:

$$f \mapsto cy^2 \left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} \right),$$

5.✓ (a) Humphreys, Exercise 6 on p.31

(b) Humphreys, Exercise 7 on p.31

The next three problems are basically the same:

7.✓ Humphreys, Exercise 4 on p.34

8.✓ Humphreys, Exercise 6 on p.34

9.✓ (a) For a Lie algebra $\mathfrak{g}$ and an $\mathfrak{g}$-module V, let $\operatorname{Sym}^n V$ be the nth symmetric power of V. Prove that it is a $\mathfrak{g}$-module.

(b) Prove that every irreducible representation of $\mathfrak{sl}_2(\mathbb{C})$ is a symmetric power of the standard 2-dimensional representation.

(c) Let V be the standard 2-dimensional representation of $\mathfrak{sl}_2(\mathbb{C})$. Prove that for $a \geq b$,

$$\operatorname{Sym}^a V \otimes \operatorname{Sym}^b V = \operatorname{Sym}^{a+b} V \oplus \operatorname{Sym}^{a+b-2} V \oplus \cdots \oplus \operatorname{Sym}^{a-b} V.$$

10.✓ Humphreys, Exercise 7 on p.34.

From now on, in the references to Humphreys, the first number denotes the number of the section, the second – to the number of the problem (to remove ambiguities if there are any typos in the page numbers :)

11.✓ Exercises about the dual root system:

(a) Humphreys, Problem 9.2 on p. 46;

(b) Humphreys, Problem 10.1 on p. 54;

(c) Humphreys, Problem 10.11 on p.55.

12.✓ Humphreys, Problem 9.9 on p.46.

13.✓ Humphreys, Problem 10.14 on p.55.

14.✓ Humphreys, Problem 11.6 on p.63.

In the next few problems, (Φ, E) is an irreducible root system, Δ is a base for Φ, W is the Weyl group, σ_α is the reflection about the hyperplane P_α orthogonal to α.

15.✓ Prove that if the group $\Gamma = \operatorname{Aut}(\Delta)$ of the graph automorphisms of the Dynkin diagram is trivial, then the element $-\operatorname{Id}$ belongs to W (where Id is the identity on the space E).

16.✓ Prove that for the types A_n $(n \geq 2)$, D_{2n+1}, and E_6, the element $-\operatorname{Id}$ is not in W.

17.✓ (a) Prove that the longest element in the Weyl group is unique.

(b) Prove that the longest element in W has order 2.

(c) Prove that if Φ is not of type A_n $(n \geq 2)$, D_{2n+1}, or E_6, then the longest element is $-\operatorname{Id}$.

18.✓ Let Φ be a root system that has vectors of two different lengths. Then denote by $\Phi_{\max}$ the root system that consists of the long roots, and by $\Phi_{\min}$ – the root system that consists of the short roots.

(a) Prove that the rank of $\Phi_{\min}$ and of $\Phi_{\max}$ equals the rank of Φ.

(b) Prove that the highest root in Φ lies in $\Phi_{\max}$.

19. (a)✓ Prove that if Φ is of type F_4, then both $\Phi_{\max}$ and $\Phi_{\min}$ are of type D_4.

(b) Prove that the Weyl group of type F_4 is isomorphic to $W_{D_4} \rtimes S_3$, where W_{D_4} is the Weyl group of D_4.

20.✓ Prove that if Φ is of type B_l, then $\Phi_{\max}$ is of type D_l, and $\Phi_{\min}$ is $A_1 \times \cdots \times A_1$ (l copies).

21. Construction of a free Lie algebra on a set X:

(a) Let X be any set, and let M_X be the set of non-associative words on the alphabet X, with the operation of concatenation (the "free magma on X"). Let A_X be the vector space of finite linear combinations of the set M_X. We can think of A_X as an algebra, by extending the opertaion on M_X by bi-linearity. Let I be the ideal in A_X generated by aa for $a \in A_X$, and $(ab)c + (bc)a + (ca)b$. Let $L_X = A_X/I$. Prove that L_X is a Lie algebra, and satisfies the universal property from the definition of the free Lie algebra.

(b) Prove that the universal enveloping algebra $U(L_X)$ is canonically isomorphic to $T(V_X)$ – the tensor algebra on the space V_X on the basis X.

(c) Prove (without using the uniqueness of a free Lie algebra) that the above construction gives an obejct canonically isomorphic to the construction of Section 17.5 in [H]. See also Exercise 6 on p. 95 in [H].

22. Compute the order of P/Q (the quotient of the weight lattice by the root lattice) for each type of irreducible root system.

23. [H], Exercise 13.4 (p.71)

24. [H], Exercise 13.5 (p.71) – note that this is the same as the exercises we discussed above – but now we can use the weight lattice to get the answer in an easier way.

25. [H], Exercise 13.6 (p.72).

26.* This is a continuation of Problem 2 from the written homework. It is about the representations of $\mathfrak{sp}_4(\mathbb{C})$. Draw the weight diagram (with multiplicities) for the irreducible representation with highest weight $3\alpha + 2\beta$. (Hint: look at the relationship with $\mathrm{Sym}^2 V \otimes W$, where V and W are from Problem 2 in the written homework).

THE END.