Growing list of problems for Math534.

All Lie algebras are assumed to be finite-dimensional and over $\mathbb{C}$, unless otherwise specified. The references are: [H] – Humphreys; [FH] – Fulton & Harris.

The exercises not marked with a check mark are to be discussed during the next problem session in class.

1.$\checkmark$ (a) Show that the Lie algebras $\mathfrak{sl}_n$, $\mathfrak{so}_n$ (with $n > 2$), $\mathfrak{sp}_n$ have the property $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$. (this is Exercise 9 on p. 5 of [H]).

   (b) Show that the derived algebra of $\mathfrak{gl}_n$ is $\mathfrak{sl}_n$ (this is [H], Exercise 2 on p.9)

2.$\checkmark$ (a) Humphreys, Exercise 7 on p.5.

   (b) Humphreys, Exercise 3 on p.10.

3$\checkmark$. Humphreys, Exercise 3 on p.20

4$\checkmark$. Humphreys, Exercise 4 on p.20

5$\checkmark$. Humphreys, Exercise 1 on p.24.

6. Let $G = \mathrm{SL}_2(\mathbb{R})$ be the group of real 2×2 matrices with determinant 1, and let $K = \mathrm{SO}_2(\mathbb{R})$ be the subgroup of real matrices preserving the quadratic form $Q(x, y) = x^2 + y^2$; we consider the both groups with their natural (real) topology. Let $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{R})$ be the Lie algebra of real 2×2 matrices of trace 0. Let $\mathbf{H} = \{x + iy \mid y > 0\}$ be the upper half-plane in $\mathbb{C}$. Let $C^\infty(\mathbf{H})$ be the space of (real)-smooth functions on $\mathbf{H}$. Recall that $\mathrm{SL}_2(\mathbb{R})$ acts on $\mathbf{H}$ by:

$$\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \cdot z = \frac{az + b}{cz + d}.$$

   (a)$\checkmark$ Prove that there is a natural homeomorphism from the quotient $\mathrm{SL}_2(\mathbb{R})/\mathrm{SO}_2(\mathbb{R})$ to $\mathbf{H}$. (Hint: consider the stabilizer of i under the action of $\mathrm{SL}_2(\mathbb{R})$ on $\mathbf{H}$). (This is in fact a diffeomorphism of real manifolds).

   (b)$\checkmark$ The homeomorphism from part (a) induces the isomorphism between the space of continuous functions on $\mathbf{H}$ and continuous K-invariant (i.e., such that $f(k^{-1}g) = f(g)$ for every $k \in K$, $g \in G$) functions on G.

   (c)$\checkmark$ Prove that when $X \in \mathfrak{g}$ is sufficiently close to 0, the matrix $\exp(X)$ is defined and is an element of G close to 1.

   (d)$\checkmark$ Prove that the action $X \cdot f = \frac{d}{dt}|_{t=0} f(\exp(-tX) \cdot z)$ makes $C^\infty(\mathbf{H})$ into a $\mathfrak{g}$-module. Denote this representation by ρ.

(e) Let $\{X, Y, H\}$ be the standard basis of $\mathfrak{g}$. Prove that the Casimir operator $\rho(X)\rho(Y) + \rho(Y)\rho(X) + \frac{1}{2}\rho(H)^2$ coincides with the Laplace operator on $\mathbf{H}$, i.e. that it acts on $C^\infty(\mathbf{H})$ by:

$$f \mapsto cy^2 \left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} \right),$$

7. (a) Humphreys, Exercise 6 on p.31

   (b) Humphreys, Exercise 7 on p.31

8. Draw the weight diagram for the tensor product of the standard and adjoint representations of ${}_2(\mathbb{C})$ and find its decomposition as a direct sum of irreducible representations.

   The next three problems are basically the same:

9. Humphreys, Exercise 4 on p.34

10. Humphreys, Exercise 6 on p.34

11. (a) For a Lie algebra $\mathfrak{g}$ and an $\mathfrak{g}$-module V, let $\operatorname{Sym}^n V$ be the nth symmetric power of V. Prove that it is a $\mathfrak{g}$-module.

    (b) Prove that every irreducible representation of $\mathfrak{sl}_2(\mathbb{C})$ is a symmetric power of the standard 2-dimensional representation.

    (c) Let V be the standard 2-dimensional representation of $\mathfrak{sl}_2(\mathbb{C})$. Prove that for $a \geq b$,

$$\operatorname{Sym}^a V \otimes \operatorname{Sym}^b V = \operatorname{Sym}^{a+b} V \oplus \operatorname{Sym}^{a+b-2} V \oplus \cdots \oplus \operatorname{Sym}^{a-b} V.$$

***** End of problems to be discussed on November 8 *****

From here on, in the references to Humphreys, the first number denotes the number of the section, the second – to the number of the problem (to remove ambiguities if there are any typos in the page numbers :) These are the future exercises:

Root systems and Weyl groups.

12. Humphreys, Problem 9.9 on p.46.

13. Humphreys, Problem 10.14 on p.55.

14. Humphreys, Problem 11.6 on p.63.

15. Exercises about the dual root system:

    (a) Humphreys, Problem 9.2 on p. 46;

    (b) Humphreys, Problem 10.1 on p. 54;

    (c) Humphreys, Problem 10.11 on p.55.

In the next few problems, (Φ, E) is an irreducible root system, Δ is a base for Φ, W is the Weyl group, σ_α is the reflection about the hyperplane P_α orthogonal to α. Note that Problems 16-18 will be discussed again once we cover abstract theory of weights (see problem 23 below).

16. Prove that if the group $\Gamma = \mathrm{Aut}(\Delta)$ of the graph automorphisms of the Dynkin diagram is trivial, then the element $-\mathrm{Id}$ belongs to W (where Id is the identity on the space E).

17. Prove that for the types A_n $(n \geq 2)$, D_{2n+1}, and E_6, the element $-\mathrm{Id}$ is not in W.

18. (a) Prove that the longest element in the Weyl group is unique.
    (b) Prove that the longest element in W has order 2.
    (c) Prove that if Φ is not of type A_n $(n \geq 2)$, D_{2n+1}, or E_6, then the longest element is $-\mathrm{Id}$.

19. Let Φ be a root system that has vectors of two different lengths. Then denote by $\Phi_{\max}$ the root system that consists of the long roots, and by $\Phi_{\min}$ – the root system that consists of the short roots.
    (a) Prove that the rank of $\Phi_{\min}$ and of $\Phi_{\max}$ equals the rank of Φ.
    (b) Prove that the highest root in Φ lies in $\Phi_{\max}$.

20. (*a*) Prove that if Φ is of type F_4, then both $\Phi_{\max}$ and $\Phi_{\min}$ are of type D_4.
    (b) Prove that the Weyl group of type F_4 is isomorphic to $W_{D_4} \rtimes S_3$, where W_{D_4} is the Weyl group of D_4.

21. Prove that if Φ is of type B_l, then $\Phi_{\max}$ is of type D_l, and $\Phi_{\min}$ is $A_1 \times \cdots \times A_1$ (l copies).

The weight lattice and the fundamental group.

22. Compute the order of P/Q (the quotient of the weight lattice by the root lattice) for each type of irreducible root system.

23. [H], Exercise 13.4 (p.71)

24. [H], Exercise 13.5 (p.71) – note that this is the same as the exercises we discussed above – but now we can use the weight lattice to get the answer in an easier way.

25. [H], Exercise 13.6 (p.72).

Accidental isomorphisms.

26. (a) Describe the Lie algebra $\mathfrak{so}_2$.
    (b) Prove that over any algebraically closed field of characteristic zero, $\mathfrak{sl}_2 \simeq \mathfrak{so}_3 \simeq \mathfrak{sp}_2$.

(c) Prove that over any algebraically closed field of characteristic zero, $\mathfrak{sl}_3 \simeq \mathfrak{so}_4$.

(d) Prove that over any algebraically closed field of characteristic zero, $\mathfrak{sp}_4 \simeq \mathfrak{so}_5$, and $\mathfrak{sl}_4 \simeq \mathfrak{so}_6$.

(e) Prove that the group $\mathrm{SL}_2(\mathbb{C})$ is the simply connected cover of the group $\mathrm{SO}_3(\mathbb{C})$.