Growing list of problems for Math534.

All Lie algebras are assumed to be finite-dimensional and over $\mathbb{C}$, unless otherwise specified. The references are: [H] – Humphreys; [FH] – Fulton & Harris.

The exercises not marked with a check mark are to be discussed during the next problem session in class.

1. Show that the Lie algebras $\mathfrak{sl}_n$, $\mathfrak{so}_n$ (with $n > 2$), $\mathfrak{sp}_n$ have the property $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$. (this is Exercise 9 on p. 5 of [H]).

2. Show that the derived algebra of $\mathfrak{gl}_n$ is $\mathfrak{sl}_n$ (this is [H], Exercise 2 on p.9).

2. Humphreys, Exercise 7 on p.5.

2. Humphreys, Exercise 3 on p.10.

4. Humphreys, Exercise 3 on p.20

5. Humphreys, Exercise 4 on p.20


7. Let $G = \text{SL}_2(\mathbb{R})$ be the group of real $2 \times 2$ matrices with determinant 1, and let $K = \text{SO}_2(\mathbb{R})$ be the subgroup of real matrices preserving the quadratic form $Q(x, y) = x^2 + y^2$; we consider the both groups with their natural (real) topology. Let $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{R})$ be the Lie algebra of real $2 \times 2$ matrices of trace 0. Let $H = \{ x + iy \mid y > 0 \}$ be the upper half-plane in $\mathbb{C}$. Let $C^\infty(H)$ be the space of (real)-smooth functions on $H$. Recall that $\text{SL}_2(\mathbb{R})$ acts on $H$ by:

\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az + b}{cz + d}.
\]

(a) Prove that there is a natural homeomorphism from the quotient $\text{SL}_2(\mathbb{R})/\text{SO}_2(\mathbb{R})$ to $H$. (Hint: consider the stabilizer of $i$ under the action of $\text{SL}_2(\mathbb{R})$ on $H$). (This is in fact a diffeomorphism of real manifolds).

(b) The homeomorphism from part (a) induces the isomorphism between the space of continuous functions on $H$ and continuous $K$-invariant (i.e., such that $f(k^{-1}g) = f(g)$ for every $k \in K$, $g \in G$) functions on $G$.

(c) Prove that when $X \in \mathfrak{g}$ is sufficiently close to 0, the matrix $\exp(X)$ is defined and is an element of $G$ close to 1.

(d) Prove that the action $X \cdot f = \frac{d}{dt} |_{t=0} f(\exp(-tX) \cdot z)$ makes $C^\infty(H)$ into a $\mathfrak{g}$-module. Denote this representation by $\rho$.
Let \( \{X, Y, H\} \) be the standard basis of \( g \). Prove that the Casimir operator \( \rho(X)\rho(Y) + \rho(Y)\rho(X) + \frac{1}{2}\rho(H)^2 \) coincides with the Laplace operator on \( H \), i.e. that it acts on \( C^\infty(H) \) by:

\[
f \mapsto cy^2 \left( \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} \right),
\]