Growing list of problems for Math534.

All Lie algebras are assumed to be finite-dimensional and over $\mathbb{C}$, unless otherwise specified. The references are: [H] – Humphreys; [FH] – Fulton & Harris.

The exercises not marked with a check mark are to be discussed during the next problem session in class.

1.$\checkmark$
   1. Show that the Lie algebras $\mathfrak{sl}_n$, $\mathfrak{so}_n$ (with $n > 2$), $\mathfrak{sp}_n$ have the property $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$. (this is Exercise 9 on p. 5 of [H]).
   2. Show that the derived algebra of $\mathfrak{gl}_n$ is $\mathfrak{sl}_n$ (this is [H], Exercise 2 on p.9)

2.$\checkmark$
   1. Humphreys, Exercise 7 on p.5.
   2. Humphreys, Exercise 3 on p.10.

4. Humphreys, Exercise 3 on p.20

5. Humphreys, Exercise 4 on p.20

6. Humphreys, Exercise 1 on p.24.

7. Let $G = \mathrm{SL}_2(\mathbb{R})$ be the group of real 2×2 matrices with determinant 1, and let $K = \mathrm{SO}_2(\mathbb{R})$ be the subgroup of real matrices preserving the quadratic form $Q(x, y) = x^2 + y^2$; we consider the both groups with their natural (real) topology. Let $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{R})$ be the Lie algebra of real 2×2 matrices of trace 0. Let $\mathbf{H} = \{x + iy \mid y > 0\}$ be the upper half-plane in $\mathbb{C}$. Let $C^\infty(\mathbf{H})$ be the space of (real)-smooth functions on $\mathbf{H}$. Recall that $\mathrm{SL}_2(\mathbb{R})$ acts on $\mathbf{H}$ by:
$$\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \cdot z = \frac{az + b}{cz + d}.$$
   (a) Prove that there is a natural homeomorphism from the quotient $\mathrm{SL}_2(\mathbb{R})/\mathrm{SO}_2(\mathbb{R})$ to $\mathbf{H}$. (Hint: consider the stabilizer of i under the action of $\mathrm{SL}_2(\mathbb{R})$ on $\mathbf{H}$). (This is in fact a diffeomorphism of real manifolds).

   (b) The homeomorphism from part (a) induces the isomorphism between the space of continuous functions on $\mathbf{H}$ and continuous K-invariant (i.e., such that $f(k^{-1}g) = f(g)$ for every $k \in K$, $g \in G$) functions on G.

   (c) Prove that when $X \in \mathfrak{g}$ is sufficiently close to 0, the matrix $\exp(X)$ is defined and is an element of G close to 1.

   (d) Prove that the action $X \cdot f = \frac{d}{dt}|_{t=0} f(\exp(-tX) \cdot z)$ makes $C^\infty(\mathbf{H})$ into a $\mathfrak{g}$-module. Denote this representation by ρ.

(e) Let $\{X, Y, H\}$ be the standard basis of $\mathfrak{g}$. Prove that the Casimir operator $\rho(X)\rho(Y) + \rho(Y)\rho(X) + \frac{1}{2}\rho(H)^2$ coincides with the Laplace operator on $\mathbf{H}$, i.e. that it acts on $C^\infty(\mathbf{H})$ by:

$$f \mapsto cy^2 \left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} \right),$$