A calculation for $\mathfrak{sl}_3$

(1.) Maximal toral subalgebra $\mathfrak{h}$:
let's take the split one, consisting of diagonal matrices of trace 0.
It is spanned by $h_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and $h_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$

In particular, $\mathfrak{h}^*$ is 2-dimensional.

(2.) Compute $\operatorname{ad} h_1$ and $\operatorname{ad} h_2$.
(These would be 8×8-matrices, since $\mathfrak{sl}_3$ is 8-dimensional)
basis: $h_1, h_2, e_{12}, e_{13}, e_{21}, e_{23}, e_{31}, e_{32}$.

In this basis,

$$\operatorname{ad} h_1 = \begin{array}{c} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & & & & & \\ 0 & 0 & & 1 & & & 0 & \\ 0 & 0 & & & -2 & & & \\ 0 & 0 & & & & -1 & & \\ 0 & 0 & & 0 & & & -1 & \\ 0 & 0 & & & & & & 1 \end{pmatrix} \\ \begin{array}{cccccc} e_{12} & e_{13} & e_{21} & e_{23} & e_{31} & e_{32} \end{array} \end{array} \qquad \operatorname{ad} h_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & & & & & \\ 0 & 0 & & +1 & & & 0 & \\ 0 & 0 & & & +1 & & & \\ 0 & 0 & & & & 2 & & \\ 0 & 0 & & 0 & & & -1 & \\ 0 & 0 & & & & & & -2 \end{pmatrix}$$

So, we have 6 root spaces:

	$\mathfrak{g}_\alpha$	$\alpha(h_1)$	$\alpha(h_2)$	new names for roots (see p. 2)
α_1	$\langle e_{12} \rangle$	2	-1	α
α_2	$\langle e_{13} \rangle$	1	$+1$	$\alpha + \beta$
α_3	$\langle e_{21} \rangle$	-2	$+1$	$-\alpha$
α_4	$\langle e_{23} \rangle$	-1	2	β
α_5	$\langle e_{31} \rangle$	-1	-1	$-(\alpha+\beta)$
α_6	$\langle e_{32} \rangle$	1	-2	$-\beta$

(3) We can observe some linear relations right away:

$\alpha_3 = -\alpha_1$
$\alpha_5 = -\alpha_2$ and $\alpha_1 + \alpha_4 = \alpha_2$
$\alpha_6 = -\alpha_4$ etc.

(4.) We will represent the 6 roots $\alpha_1, \dots, \alpha_6$ by vectors in a 2-dim Euclidean space ($\mathfrak{h}^*$ is 2-dimensional)

Let's change notation: if we denote $\alpha_1 =: \alpha$ and $\alpha_4 =: \beta$

then the other roots are $-\alpha, -\beta, \alpha+\beta$, and $(-\alpha-\beta)$.

It remains to compute (α,α), (α,β); (β,β).

Recall that the inner product in our Euclidean space is the dual form to the Killing form on $\mathfrak{h}$.

First, compute
$$K(h_1,h_1) = \mathrm{Tr}\left((\mathrm{ad}\, h_1)^2\right) = 12$$
$$K(h_2,h_2) = \mathrm{Tr}\left((\mathrm{ad}\, h_2)^2\right) = 12$$
$$K(h_1,h_2) = \mathrm{Tr}\left((\mathrm{ad}\, h_1)(\mathrm{ad}\, h_2)\right) = -6.$$

This tells us that the dual basis to (h_1,h_2) consists of two vectors of equal length at the angle of $60°$ to each other.

Here's the painful linear algebra justifying this:

suppose $\{h_1^*, h_2^*\}$ is the dual basis of $\mathfrak{h}^*$ (dual to the basis $\{h_1, h_2\}$ of $\mathfrak{h}$).

Recall the notation t_α for any $\alpha \in \mathfrak{h}^*$:

t_α is the element of $\mathfrak{h}$ s.t.
$$\alpha(h) = K(t_\alpha, h) \quad \forall\, h \in \mathfrak{h}.$$

By definition, we have $(h_1^*, h_2^*) = K(t_{h_1^*}, t_{h_2^*})$, etc.

inner product on $\mathfrak{h}^*$ dual to $K(\cdot,\cdot)$.

By definition of a dual basis, we have:

$$\begin{cases} K(t_{h_1^*}, h_1) = 1 \\ K(t_{h_2^*}, h_2) = 0 \end{cases} \quad \text{and} \quad \begin{cases} K(t_{h_2^*}, h_1) = 0 \\ K(t_{h_2^*}, h_2) = 1. \end{cases}$$

Let $t_{h^*} = a h_1 + b h_2$. we can find (a, b):

$$\begin{cases} a\, K(h_1, h_1) + b\, K(h_2, h_1) = 1 \\ a\, K(h_2, h_1) + b\, K(h_2, h_2) = 0 \end{cases} \Rightarrow \begin{cases} 12a - 6b = 1 \\ -6a + 12b = 0 \end{cases}$$

$$\Rightarrow a = 2/18, \quad b = 1/18.$$

So, $t_{h_1^*} = \frac{2}{18} h_1 + \frac{1}{18} h_2$

Similarly, $t_{h_2^*} = \frac{1}{18} h_1 + \frac{2}{18} h_2$

Now we can compute $K(t_{h_1^*}, t_{h_1^*}) = \frac{4}{18^2} \cdot K(h_1, h_1)$

$$+ 2 \cdot \frac{2}{18} \cdot \frac{1}{18} K(h_1, h_2)$$

$$+ \frac{1}{18^2} K(h_2, h_2) = \frac{2}{18}.$$

The result of this calculation is:

$$(h_1^*, h_1^*) = K(t_{h_1^*}, t_{h_1^*}) = \frac{2}{18}$$

$$(h_2^*, h_2^*) = \frac{2}{18}$$

$$(h_1^*, h_2^*) = \frac{1}{18}$$

(This justifies the claim made above that h_1^*, h_2^* are of equal length and make an angle of $60°$).

(5) Finally, we can compute (α, α), (α, β) and (β, β).

Since $\alpha(h_1) = 2$, $\alpha(h_2) = -1$, the root α has coordinates $(2, -1)$ in the basis $\{h_1^*, h_2^*\}$.

Then $(\alpha,\alpha) = 4(h_1^*, h_1^*) + (h_2^*, h_2^*) - 2\cdot 2(h_1^*, h_2^*)$

$$= 4\cdot\frac{2}{18} + \frac{2}{18} - \frac{4}{18} = \frac{6}{18} = \frac{1}{3}$$

$$(\beta,\beta) = (h_1^*, h_1^*) + (h_2^*, h_2^*) + 2(h_1^*, h_2^*)$$

$$= \frac{2}{18} + \frac{2}{18} + 2\cdot\frac{1}{18} = \frac{6}{18} = \frac{1}{3}$$

$$(\alpha,\beta) = (2h_1^* - h_2^*, -h_1^* + 2h_2^*)$$

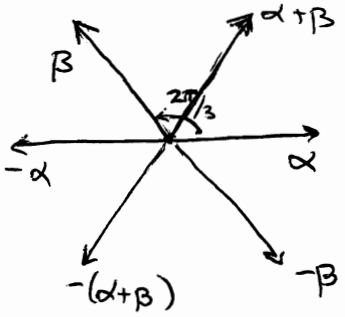
$$= -2(h_1^*, h_1^*) - 2(h_2^*, h_2^*) + 5(h_1^*, h_2^*)$$

$$= -2\cdot\frac{2}{18} - \frac{2^2}{18} + \frac{5}{18} = \frac{-3}{18} = -1/6$$

Let θ be the angle between α and β.

Then $\cos\theta = \dfrac{(\alpha,\beta)}{\|\alpha\|\,\|\beta\|} = \dfrac{-3/18}{\sqrt{1/3}\cdot\sqrt{1/3}} = \dfrac{-9}{18} = \dfrac{-1}{2}$

Hence, we get the picture:



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⑥ Cartan matrix:

Recall the notation $\langle\beta,\alpha\rangle := \dfrac{2(\beta,\alpha)}{(\alpha,\alpha)}$.

This is always an integer.

Note that trivially, $\langle\alpha,\alpha\rangle = 2$ always.

Recall that a <u>base</u> of a root system is a collection of roots s.t. each root is their linear combination with

either nonnegative or nonpositive coefficients.

In our example, $\{\alpha, \beta\}$ form a base.
Elements of a base are called simple roots
(There are other choices of bases!).

Cartan matrix by definition is a matrix $(\langle \alpha_i, \alpha_j \rangle)$

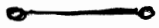
where $\{\alpha_i\}$ is a base.

In our case, if we choose the base $\{\alpha, \beta\}$, then

Cartan matrix is:

$$\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}. \qquad \left(\text{because } \langle \alpha, \beta \rangle = \frac{2(\alpha, \beta)}{(\beta, \beta)} = \frac{-2 \cdot 1/6}{1/3} = -1 \right)$$

The corresponding Dynkin diagram is:



(1 dot for each simple root: α, β in our case)

connected by one edge because

$$\langle \alpha, \beta \rangle \cdot \langle \beta, \alpha \rangle = 1.$$