1. Maximal toral subalgebra \( \mathfrak{h} \):

Let's take the split one, consisting of diagonal matrices of trace 0. It is spanned by \( h_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \) and \( h_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix} \).

In particular, \( \mathfrak{h}^* \) is 2-dimensional.

2. Compute \( \text{ad} \ h_1 \) and \( \text{ad} \ h_2 \).

(These would be 8x8 matrices, since \( \mathfrak{sl}_3 \) is 8-dimensional.)

In this basis:

\[
\text{ad} \ h_1 = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

\[
\text{ad} \ h_2 = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

So, we have 6 root spaces:

<table>
<thead>
<tr>
<th>( \mathfrak{g}_\alpha )</th>
<th>( \alpha(h_1) )</th>
<th>( \alpha(h_2) )</th>
<th>new names for roots (see p. 2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \mathfrak{e}_{12} )</td>
<td>2</td>
<td>-1</td>
<td>( \alpha )</td>
</tr>
<tr>
<td>( \mathfrak{e}_{13} )</td>
<td>1</td>
<td>1</td>
<td>( \alpha + \beta )</td>
</tr>
<tr>
<td>( \mathfrak{e}_{21} )</td>
<td>-2</td>
<td>1</td>
<td>( -\alpha )</td>
</tr>
<tr>
<td>( \mathfrak{e}_{23} )</td>
<td>-1</td>
<td>2</td>
<td>( \beta )</td>
</tr>
<tr>
<td>( \mathfrak{e}_{31} )</td>
<td>-1</td>
<td>-1</td>
<td>( -(\alpha + \beta) )</td>
</tr>
<tr>
<td>( \mathfrak{e}_{32} )</td>
<td>1</td>
<td>-2</td>
<td>( -\beta )</td>
</tr>
</tbody>
</table>

3. We can observe some linear relations, right away:

\( \alpha_3 = -\alpha_1 \)
\( \alpha_5 = -\alpha_2 \)
\( \alpha_6 = -\alpha_4 \)

and \( \alpha_1 + \alpha_4 = \alpha_2 \)

etc.
We will represent the 6 roots $\alpha_1, \ldots, \alpha_6$ by vectors in a 2-dim Euclidean space ($\mathfrak{g}^*$ is 2-dimensional).

Let's change notation: if we denote $\alpha_1 = \alpha$ and $\alpha_4 = \beta$ then the other roots are $-\alpha, -\beta, \alpha + \beta, \alpha - \beta$.

It remains to compute $(\alpha, \alpha)$, $(\alpha, \beta)$, $(\beta, \beta)$.

Recall that the inner product in our Euclidean space is the dual form to the Killing form on $\mathfrak{g}$.

First, compute

\[ K(h_1, h_1) = \text{Tr} \left( (\text{ad} h_1)_2 \right) = 12 \]
\[ K(h_2, h_2) = \text{Tr} \left( (\text{ad} h_2)_2 \right) = 12 \]
\[ K(h_1, h_2) = \text{Tr} \left( (\text{ad} h_1)(\text{ad} h_2) \right) = -6. \]

This tells us that the dual basis to $(h_1, h_2)$ consists of two vectors of equal length at the angle of $60^\circ$ to each other.

Here's the painful linear algebra justifying this:

Suppose $(h_1^*, h_2^*)$ is the dual basis of $\mathfrak{g}^*$ (dual to the basis $(h_1, h_2)$ of $\mathfrak{g}$).

Recall the notation $\tau_\alpha$ for any $\alpha \in \mathfrak{g}^*$: $\tau_\alpha$ is the element of $\mathfrak{g}$ s.t.

\[ \alpha(h) = K(\tau_\alpha, h) \quad \forall \ h \in \mathfrak{g}. \]

By definition, we have

\[ (h_1^*, h_2^*) = K(\tau_{h_1^*}, \tau_{h_2^*}), \]

inner product on $\mathfrak{g}^*$ dual to $K(\cdot, \cdot)$.
By definition of a dual basis, we have:
\[
\begin{align*}
K(t_{h_1^*}, h_1) &= 1 \\
K(t_{h_2^*}, h_1) &= 0 \\
K(t_{h_1^*}, h_2) &= 0 \\
K(t_{h_2^*}, h_2) &= 1.
\end{align*}
\]

Let \( t_{h_1^*} = ah_1 + bh_2 \). We can find \((a, b)\):
\[
\begin{cases}
 a \cdot K(h_1, h_1) + b \cdot K(h_2, h_1) = 1 \\
 a \cdot K(h_1, h_1) + b \cdot K(h_2, h_2) = 0
\end{cases} \implies \begin{cases}
 12a - 6b = 1 \\
 -6a + 12b = 0
\end{cases}
\]

\( \implies a = \frac{2}{18}, b = \frac{1}{18} \).

So, \( t_{h_1^*} = \frac{2}{18} h_1 + \frac{1}{18} h_2 \).

Similarly, \( t_{h_2^*} = \frac{1}{18} h_1 + \frac{2}{18} h_2 \).

Now we can compute \( K(t_{h_1^*}, t_{h_1^*}) = \frac{4}{18^2} \cdot K(h_1, h_1) \)
\[+ 2 \cdot \frac{2}{18} \cdot \frac{1}{18} \cdot K(h_1, h_2) \]
\[+ \frac{1}{18^2} \cdot K(h_2, h_2) = \frac{2}{18}. \]

The result of this calculation is:
\[
\begin{align*}
(h_1^*, h_1^*) &= K(t_{h_1^*}, t_{h_1^*}) = \frac{2}{18} \\
(h_2^*, h_2^*) &= \frac{2}{18} \\
(h_1^*, h_2^*) &= \frac{1}{18}
\end{align*}
\]

(This justifies the claim made above that \( h_1^*, h_2^* \) are of equal length and make an angle of 60°).

Finally, we can compute \((\alpha, \alpha'), (\alpha, \beta)\) and \((\beta, \beta')\).

Since \( \alpha(h_1) = 2, \alpha(h_2) = -1 \), the root \( \alpha \) has coordinates \((2, -1)\) in the basis \( \{h_1^*, h_2^*\} \).
Then 

\[(\alpha, \alpha) = 4(h_1^*, h_1^*) + (h_2^*, h_2^*) - 22(h_1^*, h_2^*)\]

\[= 4 \cdot \frac{2}{18} + \frac{2}{18} - \frac{4}{18} = \frac{6}{18} = \frac{1}{3}\]

\[(\beta, \beta) = (h_1^*, h_1^*) + (h_2^*, h_2^*) + 2(h_1^*, h_2^*)\]

\[= \frac{2}{18} + \frac{2}{18} + 2 \cdot \frac{1}{18} = \frac{6}{18} = \frac{1}{3}\]

\[(\alpha, \beta) = (2h_1^* - h_2^*, -h_1^* + 2h_2^*) \alpha\]

\[= -2(h_1^*, h_1^*) - 2(h_1^*, h_2^*) + 2(h_1^*, h_2^*)\]

\[= -2 \cdot \frac{2}{18} - \frac{2}{18} + \frac{5}{18} = -\frac{3}{18} = -\frac{1}{6}\]

Let \(\Theta\) be the angle between \(\alpha\) and \(\beta\). Then 

\[\cos \Theta = \frac{\langle \alpha, \beta \rangle}{||\alpha|| ||\beta||} = \frac{-3/18}{\sqrt{3} \cdot \sqrt{3}} = \frac{-1}{\sqrt{3}} = -\frac{1}{2}\]

Hence, we get the picture:

\[\text{Diagram of } \alpha, \beta, \alpha + \beta, -\alpha, -\beta, -(\alpha + \beta)\]

\[\text{6. Cartan matrix:}\]

Recall the notation 

\[\langle \beta, \alpha \rangle = \frac{2 \langle \beta, \alpha \rangle}{\langle \alpha, \alpha \rangle} \]

This is always an integer.

Note that trivially, \(\langle \alpha, \alpha \rangle = 2\) always.

Recall that a base of a root system is a collection of roots s.t. each root is their linear combination with
either nonnegative or nonpositive coefficients.

In our example, \( \{ \alpha, \beta \} \) form a base. Elements of a base are called simple roots (There are other choices of bases!).

Cartan matrix by definition is a matrix \( (\langle \alpha_i, \alpha_j \rangle) \) where \( \alpha_i \) with \( \alpha_i \) is a base.

In our case, if we choose the base \( \{ \alpha, \beta \} \), then Cartan matrix is:

\[
\begin{pmatrix}
2 & -1 \\
-1 & 2
\end{pmatrix}
\]

\( \langle \alpha, \beta \rangle = 2 \frac{\langle \alpha, \beta \rangle}{\langle \beta, \beta \rangle} = -2 \cdot \frac{1/3}{1/3} = -1 \)

The corresponding Dynkin diagram is:

\[
\begin{array}{c}
\text{---}
\end{array}
\]

(1 dot for each simple root: \( \alpha, \beta \) in our case)

connected by one edge because

\( \langle \alpha, \beta \rangle, \langle \beta, \alpha \rangle = 1 \).