

### Growing list of problems for Math534.

All Lie algebras are assumed to be finite-dimensional and over  $\mathbb{C}$ , unless otherwise specified. The references are: [H] – Humphreys; [FH] – Fulton & Harris.

The exercises not marked with a check mark are to be discussed during the next problem session in class.

1. ✓ (a) Show that the Lie algebras  $\mathfrak{sl}_n$ ,  $\mathfrak{so}_n$  (with  $n > 2$ ),  $\mathfrak{sp}_n$  have the property  $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$ . (this is Exercise 9 on p. 5 of [H]).  
(b) Show that the derived algebra of  $\mathfrak{gl}_n$  is  $\mathfrak{sl}_n$  (this is [H], Exercise 2 on p.9)
2. ✓ (a) Humphreys, Exercise 7 on p.5.  
(b) Humphreys, Exercise 3 on p.10.
3. Humphreys, Exercise 3 on p.20
4. ✓. Humphreys, Exercise 4 on p.20
5. ✓. Humphreys, Exercise 1 on p.24.
6. (*This problem will be discussed later, when we cover the universal enveloping algebra.*) Let  $G = \mathrm{SL}_2(\mathbb{R})$  be the group of real  $2 \times 2$  matrices with determinant 1, and let  $K = \mathrm{SO}_2(\mathbb{R})$  be the subgroup of real matrices preserving the quadratic form  $Q(x, y) = x^2 + y^2$ ; we consider the both groups with their natural (real) topology. Let  $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{R})$  be the Lie algebra of real  $2 \times 2$  matrices of trace 0. Let  $\mathbf{H} = \{x + iy \mid y > 0\}$  be the upper half-plane in  $\mathbb{C}$ . Let  $C^\infty(\mathbf{H})$  be the space of (real)-smooth functions on  $\mathbf{H}$ . Recall that  $\mathrm{SL}_2(\mathbb{R})$  acts on  $\mathbf{H}$  by:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot z = \frac{az + b}{cz + d}.$$

- (a) Prove that there is a natural homeomorphism from the quotient  $\mathrm{SL}_2(\mathbb{R})/\mathrm{SO}_2(\mathbb{R})$  to  $\mathbf{H}$ . (Hint: consider the stabilizer of  $i$  under the action of  $\mathrm{SL}_2(\mathbb{R})$  on  $\mathbf{H}$ ). (This is in fact a diffeomorphism of real manifolds).
- (b) The homeomorphism from part (a) induces the isomorphism between the space of continuous functions on  $\mathbf{H}$  and continuous  $K$ -invariant (i.e., such that  $f(k^{-1}g) = f(g)$  for every  $k \in K, g \in G$ ) functions on  $G$ .
- (c) Prove that when  $X \in \mathfrak{g}$  is sufficiently close to 0, the matrix  $\exp(X)$  is defined and is an element of  $G$  close to 1.
- (d) Prove that the action  $X \cdot f = \left. \frac{d}{dt} \right|_{t=0} f(\exp(-tX) \cdot z)$  makes  $C^\infty(\mathbf{H})$  into a  $\mathfrak{g}$ -module. Denote this representation by  $\rho$ .

- (e) Let  $\{X, Y, H\}$  be the standard basis of  $\mathfrak{g}$ . Prove that the Casimir operator  $\rho(X)\rho(Y) + \rho(Y)\rho(X) + \frac{1}{2}\rho(H)^2$  coincides with the Laplace operator on  $\mathbf{H}$ , i.e. that it acts on  $C^\infty(\mathbf{H})$  by:

$$f \mapsto cy^2 \left( \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} \right),$$

7. (a) Humphreys, Exercise 6 on p.31  
 (b) Humphreys, Exercise 7 on p.31
8. Draw the weight diagram for the tensor product of the standard and adjoint representations of  $\mathfrak{sl}_2(\mathbb{C})$  and find its decomposition as a direct sum of irreducible representations.

**The next three problems are basically the same:**

9. Humphreys, Exercise 4 on p.34
10. Humphreys, Exercise 6 on p.34
11. (a) For a Lie algebra  $\mathfrak{g}$  and an  $\mathfrak{g}$ -module  $V$ , let  $\text{Sym}^n V$  be the  $n$ th symmetric power of  $V$ . Prove that it is a  $\mathfrak{g}$ -module.  
 (b) Prove that every irreducible representation of  $\mathfrak{sl}_2(\mathbb{C})$  is a symmetric power of the standard 2-dimensional representation.  
 (c) Let  $V$  be the standard 2-dimensional representation of  $\mathfrak{sl}_2(\mathbb{C})$ . Prove that for  $a \geq b$ ,

$$\text{Sym}^a V \otimes \text{Sym}^b V = \text{Sym}^{a+b} V \oplus \text{Sym}^{a+b-2} V \oplus \dots \oplus \text{Sym}^{a-b} V.$$

**\*\*\* End of problems to be discussed on October 31 \*\*\*,  
 but take a look at the problems below – by now almost all their statements should make sense.**

From here on, in the references to Humphreys, the first number is the number of the section, the second – the number of the problem (to remove ambiguities if there are any typos in the page numbers.) These are the future exercises:

**Root systems and Weyl groups.**

12. Humphreys, Problem 9.9 on p.46.
13. Humphreys, Problem 10.14 on p.55.
14. Humphreys, Problem 11.6 on p.63.
15. Exercises about the dual root system:  
 (a) Humphreys, Problem 9.2 on p. 46;  
 (b) Humphreys, Problem 10.1 on p. 54;

(c) Humphreys, Problem 10.11 on p.55.

In the next few problems,  $(\Phi, E)$  is an irreducible root system,  $\Delta$  is a base for  $\Phi$ ,  $W$  is the Weyl group,  $\sigma_\alpha$  is the reflection about the hyperplane  $P_\alpha$  orthogonal to  $\alpha$ . Note that Problems 16-18 will be discussed again once we cover abstract theory of weights (see problem 23 below).

16. Prove that if the group  $\Gamma = \text{Aut}(\Delta)$  of the graph automorphisms of the Dynkin diagram is trivial, then the element  $-\text{Id}$  belongs to  $W$  (where  $\text{Id}$  is the identity on the space  $E$ ).
17. Prove that for the types  $A_n$  ( $n \geq 2$ ),  $D_{2n+1}$ , and  $E_6$ , the element  $-\text{Id}$  is not in  $W$ .
18. (a) Prove that the longest element in the Weyl group is unique.  
(b) Prove that the longest element in  $W$  has order 2.  
(c) Prove that if  $\Phi$  is not of type  $A_n$  ( $n \geq 2$ ),  $D_{2n+1}$ , or  $E_6$ , then the longest element is  $-\text{Id}$ .
19. Let  $\Phi$  be a root system that has vectors of two different lengths. Then denote by  $\Phi_{\max}$  the root system that consists of the long roots, and by  $\Phi_{\min}$  – the root system that consists of the short roots.  
(a) Prove that the rank of  $\Phi_{\min}$  and of  $\Phi_{\max}$  equals the rank of  $\Phi$ .  
(b) Prove that the highest root in  $\Phi$  lies in  $\Phi_{\max}$ .
20. (a) Prove that if  $\Phi$  is of type  $F_4$ , then both  $\Phi_{\max}$  and  $\Phi_{\min}$  are of type  $D_4$ .  
(b) Prove that the Weyl group of type  $F_4$  is isomorphic to  $W_{D_4} \rtimes S_3$ , where  $W_{D_4}$  is the Weyl group of  $D_4$ .
21. Prove that if  $\Phi$  is of type  $B_l$ , then  $\Phi_{\max}$  is of type  $D_l$ , and  $\Phi_{\min}$  is  $A_1 \times \cdots \times A_1$  ( $l$  copies).

**The weight lattice and the fundamental group.**

22. Compute the order of  $P/Q$  (the quotient of the weight lattice by the root lattice) for each type of irreducible root system.
23. [H], Exercise 13.4 (p.71)
24. [H], Exercise 13.5 (p.71) – note that this is the same as the exercises we discussed above – but now we can use the weight lattice to get the answer in an easier way.
25. [H], Exercise 13.6 (p.72).

**Accidental isomorphisms.**

26. (a) Describe the Lie algebra  $\mathfrak{so}_2$ .

- (b) Prove that over any algebraically closed field of characteristic zero,  $\mathfrak{sl}_2 \simeq \mathfrak{so}_3 \simeq \mathfrak{sp}_2$ .
- (c) Prove that over any algebraically closed field of characteristic zero,  $\mathfrak{sl}_3 \simeq \mathfrak{so}_4$ .
- (d) Prove that over any algebraically closed field of characteristic zero,  $\mathfrak{sp}_4 \simeq \mathfrak{so}_5$ , and  $\mathfrak{sl}_4 \simeq \mathfrak{so}_6$ .
- (e) Prove that the group  $\mathrm{SL}_2(\mathbb{C})$  is the simply connected cover of the group  $\mathrm{SO}_3(\mathbb{C})$ .