Exercises for Math535.

1✓. Write down a map of rings that gives the addition map on the $\mathbb{C}$-points of $\mathbb{G}_a$. (Hint: this has to be a ring homomorphism $k[x] \to k[x] \otimes k[x]$.)

2✓. Let $g = \left[\begin{smallmatrix} 1 & i \\ 0 & 1 \end{smallmatrix}\right] \in \mathfrak{sl}_2(\mathbb{C})$. Write down the map of the coordinate rings that gives the left translation by g on $\mathfrak{sl}_2(\mathbb{C})$.

3✓. Prove that $\{\exp(t \left[\begin{smallmatrix} 1 & i \\ 0 & 1 \end{smallmatrix}\right]) \mid t \in \mathbb{C}\}$ is a Lie subgroup of $\mathrm{GL}_2(\mathbb{C})$ but not an algebraic subgroup. (Hint: you can use the hint from the textbook. It is Problem 7 in Chapter 3, Section 1).

4✓. Prove that there are no nontrivial homomorphisms of algebraic groups from $\mathbb{G}_m$ to $\mathbb{G}_a$, over an algebraically closed field of characteristic zero.

5✓. Show that $\mathrm{GL}_n(\mathbb{C})$ is connected, but $\mathrm{GL}_n(\mathbb{R})$ has two connected components (in the real topology).

6✓. Show that U_n and $\mathrm{Sp}_{2n}(\mathbb{C})$, $Sp_{2n}(\mathbb{R})$ are connected in the real topology.

7✓. Show that SU_n and $\mathrm{Sp}_{2n}(\mathbb{C})$, $\mathrm{Sp}_{2n}(\mathbb{R})$ are simply connected (as real manifolds).

8✓. Show that $\pi_1(\mathrm{SO}_n(\mathbb{R})) = \mathbb{Z}/2\mathbb{Z}$ when $n \geq 3$ (the fundamental group is in the usual, topological, sense). Also note that $\pi_1(\mathrm{SO}_2(\mathbb{R})) = \mathbb{Z}$.

9✓. Prove that if H is a subgroup of an alegbraic group G, and $\bar{H}$ is its Zariski closure, then $\bar{H}$ is also an algebraic subgroup.

10*. Recall that for algebraic groups, if G is connected, then the commutator subgroup $[G, G]$ is a closed algebraic subgroup. For Lie groups over $\mathbb{C}$ a similar statement doesn't hold. Find a better example, or show that the following example works: Let H be the group of 3×3 upper triangular martices, with 1's on the diagonal. Let $S^1 = \{z \in \mathbb{Z} \mid |z| = 1\}$ be the unit circle on the complex plane, with the natural group structure. Consider the subgroup N of $H \times S^1$ generated by the element:

$$n = \left(\left[\begin{smallmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right], c \right),$$

where c is an element of S^1 of infinite order. Let $G = H \times S^1/N$. Then $[G, G]$ is not a Lie subgroup.

11✓. Show that the projective space $\mathbb{P}^n$ is an irreducible variety.

12$\checkmark$. Prove that a subvariety X of $\mathbb{P}^n$ over an algebraically closed field K of characteristic zero is irreducible iff $K[X]^{pr}$ has no zero divisors.

13$\checkmark$. Let V be a vector space (over an algebraically closed field of characteristic zero). Prove that the natural action of $\mathrm{GL}(V)$ on $\mathbb{P}(V)$ is algebraic.

14$\checkmark$. Find $\pi_1(\mathrm{SO}_2(\mathbb{C}))$, and then $\pi_1(\mathrm{SO}_n(\mathbb{C}))$ (in their real manifold topology).

15$\checkmark$. Prove that any commutative linear algebraic group is a direct product of a quasitorus and a vector group. Prove that any *connected* commutative linear algebraic group is a direct product of a torus and a vector group.

16$\checkmark$. Prove that any algebraic torus contains elements that are not contained in any proper algebraic subgroup.

17$\checkmark$. Show that the representation $\tilde{\lambda}$ of G in $\mathcal{D}_G$ that corresponds to the representation of G in $\mathfrak{g}$ by left translations is given by the formula

$$\tilde{\lambda}(g)D = \lambda(g) \circ D \circ \lambda(g)^{-1},$$

where $D \in \mathcal{D}_G$, $g \in G$, and $\lambda(g)$ if the map $\lambda(g) : K[G] \to K[G]$ defined by $\lambda(g)(f)(x) = f(gx)$.

18$\checkmark$. (exrcise 6 in Springer, Section 4.4.15 (p. 75 in Birkhäuser's "Modern Classics" edition)).
Let $s \in \mathrm{M}_n$ be an arbitrary matrix, and let $G = \{g \in \mathrm{GL}_n \mid gs({}^t g) = s\}$. Then G is a closed subgroup of GL_n. Prove that its Lie algebra is contained in $\{X \in \mathfrak{gl}_n \mid Xs + s({}^t X) = 0\}$.

19$\checkmark$. **Reductive Lie algebras and nondegenerate bilinear forms**

1. (OV Problem 5, Section 4.1.1, p. 137) Prove that if $\mathfrak{n} \subset \mathfrak{gl}(V)$ is a Lie algebra, and the bilinear form $\mathrm{Tr}(XY)$ vanishes on $\mathfrak{n}$, then $\mathfrak{n}$ is solvable.
2. (OV Problem 6, p.137) Prove that if $\mathfrak{n}$ is a unipotent ideal in $\mathfrak{g} \subset \mathfrak{gl}(V)$ (that is, an ideal consisting of nilpotent elements), then $(\mathfrak{n}, \mathfrak{g}) = 0$ (where $(X, Y) = \mathrm{Tr}(XY)$).
3. (OV Problems 10-11, p.138) Prove that if $\mathfrak{g}$ is a reductive Lie algebra, then $\mathfrak{g} = \mathfrak{z}(\mathfrak{g}) \oplus [\mathfrak{g}, \mathfrak{g}]$.
4. (Jerome's example). There can exist a nondegenerate bilinear form on a solvable non-commutative Lie algebra. (Thus, it is essential that we are using the specific form $\mathrm{Tr}(XY)$ everywhere above). Consider the Lie algebra $\mathfrak{g}$ that, as a vector space, is

$$\mathfrak{g} = \langle x \rangle \oplus \langle y \rangle \oplus \langle z \rangle \oplus \langle h \rangle,$$

with the Lie bracket defined by:

$$[x, y] = z \quad [h, x] = x \quad [h, y] = -y, \quad [z, \cdot] = 0.$$

One can define a nondegenerate bilinear form on it, with $(x, y) = 1$, and $(h, z) = 1$.

20. Compute the root lattice, coroot lattice, and π_1 for the root system of type A_2.
21. Compute π_1 for the root system of type B_2.
22. Assume that we know that the special orthogonal group SO_n is of type B_n when n is odd, and of type D_n when n is even. Assume also that: $\pi_1(\Phi)$ is $\mathbb{Z}/2\mathbb{Z}$ when Φ is a root system of type B_n, and for Φ of type D_n, we have $\pi_1(\Phi) = \mathbb{Z}/4\mathbb{Z}$ when n is odd, $\pi_1(\Phi) = \mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$ when n is even. Prove that there exists a simply connected algebraic group that is a double cover of SO_n. This group is denoted by Spin_n.
22. *Accidental isomorphisms.* Everything is over $\mathbb{C}$. Prove that:
    (a) Spin_3 is isomorphic to $\mathrm{SL}_2 \simeq \mathrm{Sp}_2$.
    (b) Spin_5 is isomorphic to Sp_4.
    (c) Spin_6 is isomorphic to SL_4.