Exercises for Math535.

Some of these exercises venture into the world of Lie groups, even though otherwise we are not discussing Lie groups. This is done for the purposes of developing/testing intuition for such concepts as connectedness and simple connectedness. A Lie group is a smooth (real or complex) manifold with compatible group structure (i.e. the multiplication and inverse maps are smooth). If $\mathbf{G}$ is an algebraic group (say, defined over $\mathbb{R}$), then $\mathbf{G}(\mathbb{R})$ and $\mathbf{G}(\mathbb{C})$ are real Lie groups; $\mathbf{G}(\mathbb{C})$ is a complex Lie group for any algebraic group $\mathbf{G}$ defined over $\mathbb{C}$. A very useful resource (and the source of many exercises below) on Lie groups is the book by Onischik and Vinberg, "Lie groups and algebraic groups" (it is referred to as "OV" below). Please talk to me if you'd like to borrow it.

1$\checkmark$. Check that the co-multiplication map defined in Examples 2.1.4 (Springer) for $\mathbb{G}_a$ and $\mathbb{G}_m$ indeed corresponds to addition and multiplication (respectively) for the points $\mathbb{G}_a(k) = k$ and $\mathbb{G}_m(k) = k^\times$.

2$\checkmark$. Prove that $\{\exp(t\left[\begin{smallmatrix} 1 & 0 \\ 0 & i \end{smallmatrix}\right]) \mid t \in \mathbb{C}\}$ of $\mathrm{GL}_2(\mathbb{C})$ is not an algebraic subgroup. (Hint: there's no polynomial $f(x,y)$ such that $f(e^t, e^{it}) = 0$ for all t). The reason this example is interesting is that it is a Lie subgroup (that is, a subgroup that is an image of a Lie group under an injective immersion), but not an algebraic subgroup.

3$\checkmark$. Prove that there are no nontrivial homomorphisms of algebraic groups from $\mathbb{G}_m$ to $\mathbb{G}_a$, over an algebraically closed field of characteristic zero.

4$\checkmark$. Show that $\mathrm{GL}_n(\mathbb{C})$ is connected, but $\mathrm{GL}_n(\mathbb{R})$ has two connected components (in the real topology).

5$\checkmark$. Show that $\mathrm{Sp}_{2n}(\mathbb{C})$, $Sp_{2n}(\mathbb{R})$, and $\mathrm{SO}_n(\mathbb{C})$ are connected in the real topology.

Hint: one of the ways to prove such statements is using the following Lemma:

> Let G be a Lie group that acts transitively on a smooth manifold X. Suppose the stabilizer of some point $x \in X$ is connected. Then G is connected.

Then consider the action of these groups on some convenient manifolds (e.g. spheres in the case of the orthogonal group), so that the stabilizer of a point would be a smaller orthogonal (respectively, symplectic) group; and use induction.

6$\checkmark$. Find the dimensions of SO_n and Sp_{2n}. (Hint: consider the action of GL_n on the space of all symmetric (respectively, skew-symmetric) bilinear forms; it is easy to prove there is an open orbit of that action).

7✓. Prove that the algebraic groups SO_n and Sp_{2n} are connected (in the sense of algebraic groups), by finding a good set of generators (see Springer, Exercises 2.2.9 (1) and (2)).

8✓. We have shown that for algebraic groups, if G is connected, then the commutator subgroup $[G, G]$ is a closed algebraic subgroup. For Lie groups over $\mathbb{C}$ a similar statement doesn't hold. Find a better example, or show that the following example works: Let H be the group of 3×3 upper triangular martices, with 1's on the diagonal. Let $S^1 = \{z \in \mathbb{Z} \mid |z| = 1\}$ be the unit circle on the complex plane, with the natural group structure. Consider the subgroup N of $H \times S^1$ generated by the element:

$$n = \left(\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, c \right),$$

where c is an element of S^1 of infinite order. Let $G = H \times S^1/N$. Then $[G, G]$ is not a Lie subgroup. (this example comes from OV, Exercise 4 on p.56).

9✓. Show that the projective space $\mathbb{P}^n$ is an irreducible variety.

10✓. Let $G = \mathrm{PGL}_2 = \mathrm{GL}/Z$, where Z is the centre of GL_2. It is an algebraic group; there are two ways to think about it as an algebraic variety: as a principal open subset of $\mathbb{P}^3$ consisting of points $[a : b : c : d]$ with $ad - bc \neq 0$, or as an affine variety with the coordinate ring $k[G]$ being the subring of $k[a, b, c, d]/(ad - bc - 1)$ that consists of even polynomials (we will discuss later why this is so).

1. Construct any faithful finite-dimensional representation of G (Hint: use the Lie algebra – to be discussed soon).
2. Let $f(a, b, c, d) = a^2/(ad - bc)$ – note that it is a well-defined regular function on PGL_2. Use this function to construct a faithful finite-dimensional representation, as in the proof of Proposition 2.3.6 of Springer.
3. Where does the argument of Theorem 2.3.7 fail for an elliptic curve?

11✓. Prove that for an algebraic group G over the field $\bar{\mathbb{F}}_p$ (algebraic closure of $\mathbb{F}_p$), an element of $G(\bar{\mathbb{F}}_p)$ is semisimple if and only if its order is prime to p, and unipotent if and only if its order is a power of p (first, show that every element of $G(\bar{\mathbb{F}}_p)$ has finite order).

12*. Let V be a vector space over an imperfect field F. Let us say that a linear operator $T : V \to V$ is semisimple if every T-invariant subspace of V has a T-invariant complement. Make an example of a linear operator that does not have a Jordan decomposition, with this definition of "semisimple".

13✓. Prove that any algebraic torus contains elements that are not contained in any proper algebraic subgroup.

14. Exercises 3.2.10 from Springer (everything that we did not prove in class). Mostly, number (6).

15. Consider the algebraic group $G = \mathrm{GL}_2$.

    (a) Prove that $T_e G$ is the 4-dimensional affine space (hint for the next part: it is convenient to identify this space with the space M_2 of 2×2-matrices).

    (b) Let $g = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \mathrm{GL}_2(k)$. Let $\phi_g : \mathrm{GL}_2 \to GL_2$ be the inner automorphism defined by g: $\phi_g(x) = gxg^{-1}$. Find (in terms of g) the differential $d_e \phi_g$.

16. Prove that $d_e(Ad) = ad$ (that is, the differential at the identity of the adjoint representation of an algebraic group on its Lie algebra is the adjoint representation of the Lie algebra).

In the next few problems, (Φ, E) is an irreducible root system, Δ is a base for Φ, W is the Weyl group, σ_α is the reflection about the hyperplane P_α orthogonal to α. The exercises 17-22 mostly come from Onischik-Vinberg, p.175.

17. Prove that if the group $\Gamma = \mathrm{Aut}(\Delta)$ of the graph automorphisms of the Dynkin diagram is trivial, then the element $-\mathrm{Id}$ belongs to W (where Id is the identity on the space E).

18. Prove that for the types A_n ($n \geq 2$), D_{2n+1}, and E_6, the element $-\mathrm{Id}$ is not in W.

19. (a) Prove that the longest element in the Weyl group is unique.

    (b) Prove that the longest element in W has order 2.

    (c) Prove that if Φ is not of type A_n ($n \geq 2$), D_{2n+1}, or E_6, then the longest element is $-\mathrm{Id}$.

20. Let Φ be a root system that has vectors of two different lengths. Then denote by $\Phi_{\max}$ the root system that consists of the long roots, and by $\Phi_{\min}$ – the root system that consists of the short roots.

    (a) Prove that the rank of $\Phi_{\min}$ and of $\Phi_{\max}$ equals the rank of Φ.

    (b) Prove that the highest root in Φ lies in $\Phi_{\max}$.

21. (*a*) Prove that if Φ is of type F_4, then both $\Phi_{\max}$ and $\Phi_{\min}$ are of type D_4.

    (b) Prove that the Weyl group of type F_4 is isomorphic to $W_{D_4} \rtimes S_3$, where W_{D_4} is the Weyl group of D_4.

22. Prove that if Φ is of type B_l, then $\Phi_{\max}$ is of type D_l, and $\Phi_{\min}$ is $A_1 \times \cdots \times A_1$ (l copies).