Exercises for Math600D.

Everywhere, F stands for a non-Archimedean local field with the ring of integers $\mathcal{O}$, the maximal ideal $(\varpi) \subset \mathcal{O}$, and the residue field $\mathbb{F}_q \equiv \mathcal{O}/(\varpi)$. (We can think of $\mathbb{Q}_p \supset \mathbb{Z}_p \supset p\mathbb{Z}_p$ as an example, then $\mathbb{F}_q = \mathbb{F}_p$).

1.✓ Prove that every continuous character of the additive group of F is trivial on $\varpi^n\mathcal{O}$ for some integer $n > 0$, and every continuous character of the multiplicative group F^* is trivial on $1 + \varpi^n F$ for some n.

2.✓ Prove Frobenius reciprocity for finite groups: If H is a subgroup of G, and (π, V), (σ, W) are finite-dimensional representations of G and H, respectively, then
$$\mathrm{Hom}_G(\pi, \mathrm{Ind}_H^G\sigma) \simeq \mathrm{Hom}_H(\mathrm{Res}_H^G\pi, \sigma).$$

3.✓ Let G be a finite group, and $H \subset G$ a subgroup. Let (σ, W) be a representation of H. Prove that $\mathrm{Ind}_H^G(\sigma, W)$ as a vector space is isomorphic to a direct sum of $[G : H]$ copies of W. Labels these copies by the cosets in G/H, and write down the action of G on this vector space.

The next three exercises are about Mackey theory:

4.✓ Prove Mackey's theorem for finite groups (Proposition 4.1.2 in Bump)

5.✓ Let G be a finite group, and H – a *normal* subgroup. Note that in this case G acts on $\hat{H}$ – the set of irreducible representations of H – by $\rho \mapsto \rho^g$, where $\rho^g(h) = \rho(ghg^{-1})$, and obviously, if $g \in H$, then $\rho^g \simeq \rho$.

Prove that if ρ is an irreducible representation of H, and $\mathrm{Stab}_G(\rho) = H$, then $\mathrm{Ind}_H^G\rho$ is irreducible.

6.✓ Let P be the subgroup of $GL_2(\mathbb{F}_q)$ that consists of matrices of the form $\left(\begin{smallmatrix} \alpha & \beta \\ 0 & 1 \end{smallmatrix}\right)$, with $\alpha \in {\mathbb{F}_q}^*, \beta \in \mathbb{F}_q$. Show that P has exactly one irreducible representation of degree $q-1$, and $q-1$ in-equivalent 1-dimensional representations, and no other irreducible representations.

From now on, G stands for a totally disconnected topological group, unless otherwise specified. We will always denote the centre of G by $Z(G)$.

7.✓ Prove that if (π, V) is an irreducible smooth representation of G, then there exists a character $\chi : Z(G) \to \mathbb{C}^*$ such that $\pi(z)v = \chi(z)v$ for all $z \in Z(G), v \in V$.

8.✓ Prove Cartan decomposition $G = K_0AK_0$ for $G = \mathrm{GL}_2(F)$. (here $K_0 = \mathrm{GL}_2(\mathcal{O})$, and A is a set of diagonal matrices of the form $\left(\begin{smallmatrix} \varpi^n & 0 \\ 0 & \varpi^m \end{smallmatrix}\right)$ with $n \geq m$.

9.✓ Let $a \in A$ as above. Show that K_0aK_0/K_0 is finite (and try to find the cardinality).

10.✓ Let (π, V) be an arbitrary representation of G. Prove that the set of smooth vectors in V is a linear subspace of V.

11.✓ Let $d\mu_l(x)$ be a left-invariant Haar measure on G, and let $\Delta_G(x)$ be the modulus character (defined by $d\mu_l(xg) = \Delta_G(g)d\mu_l(x)$.

(a) Prove that $\Delta_G(g)$ is indeed a character.

(b) Prove that $d\mu_r(x)\Delta_G(x)^{-1}d\mu_l(x)$ is a right-invariant Haar measure.

(c) Prove that $d\mu_l(x^{-1}) = \Delta_G(x^{-1})d\mu(x)$.

12.✓ Compute the modulus character for the standard Borel subgroup of $\mathrm{GL}_2(F)$ (*i.e.*, for the group B of upper-triangular invertible 2×2 matrices).

13.✓ Let $e_k \in C_c^\infty(G)$ be the function associated with a compact open subgroup K: $e_K = \frac{1}{\mathrm{vol}(K)}\mathrm{char}_K$. prove that e_K is an idempotent, *i.e.*, that $e_K * e_K = e_K$.

Homological algebra exercises have been cancelled, and will be discussed in a separate handout.

14.✓ Fill in the details in the proof of Frobenius reciprocity that appears on p. 29 of F. Murnaghan's notes (see also Remark on p.30).

15.✓ Let G be a finite group. Show that every conjugation-invariant function on G lies in the centre of the group algebra $\mathbb{C}[G]$.

16.✓ Let (ρ, W) be an irreducible representation of a finite group G. Then we can think of W as a $\mathbb{C}[G]$-module. The goal of this exercise is to prove that there exists unique central primitive idempotent $e_\rho \in \mathbb{C}[G]$ such that $e_\rho W \neq 0$.

(a) (surprisingly difficult; uses orthogonality relations for matrix coefficients, see the next problem.)
Let χ_ρ be the character of ρ. Then

$$\chi_\rho * \chi_\rho = \frac{1}{\dim(\rho)}\chi_\rho.$$

(b) Let $e_\rho = \left(\frac{\dim(\rho)}{|G|}\right)^{1/2} \chi_{\tilde{\rho}}$. Then e_ρ is the required idempotent. (It is not hard to see that it does not kill W, using orthogonality relations for characters; however, we need part (a) to establish that it is an idempotent.)

(c) Prove uniqueness of e_ρ. (I think, for this one needs to know that the complex dimension of $Z(\mathbb{C}[G])$ equals the number of conjugacy classes in G, and therefore the number of irreducible representations of G.)

17.✓ **Orthogonality relations for matrix coefficients.** Suppose (π, V) is an irreducible *unitary* representation of a finite group G, meaning that V is endowed with a Hermitian form $\langle , \rangle$ and $\pi(g)$ is a unitary operator with respect to it for every $g \in G$. Let $v, w \in V$. Then $\pi_{v,w}(g) = \langle \pi(g)v, w\rangle$ is a function on G (so it can also be thought of as an element of $\mathbb{C}[G]$).

Prove that

$$\int_G \pi_{v_1,w_1}(g)\overline{\pi_{v_2,w_2}(g)}dg = \frac{1}{\dim(\pi)}\langle v_1, v_2\rangle\overline{\langle w_1, w_2\rangle}.$$

(Of course, here $\int_G f(g)dg$ means $1/|G|$ times the sum of the values of f over all the group elements, i.e., the integral is with respect to the normalized counting measure).

18.✓ Compute the Fourier transform of the characteristic function of the ring of integers of F (with respect to any character ψ that has the property that its restriction to $\mathcal{O}_F$ is trivial, and its restriction to $(\varpi)^{-1}\mathcal{O}_f$ is nontrivial.

19.✓ Prove Fourier inversion formula for the Fourier transform on F.

20.✓ Prove that Fourier transform is an isomorphism of rings without unity, $(C_c^\infty(F), \cdot) \to (C_c^\infty(F), *)$.

21. Let (π, V) be a smooth representation, and let $\mathcal{K}(\pi)$ be its Kirillov model, and $\mathcal{W}(\pi)$ be its Whittaker model. Recall that $\mathcal{K}(\pi)$ is a representation in a space of functions on $F^\times$, and $\mathcal{W}(\pi)$ is a representation in a space of functions on G. For $\xi : F^\times \to \mathbb{C}$, define $W_\xi : G \to \mathbb{C}$ by

$$W_\xi(g) := (\pi(g)\xi)(1).$$

Show that for $\xi \in \mathcal{K}$, W_ξ is in $\mathcal{W}$, and the map $\xi \mapsto W_\xi$ is an isomorphism of G-representations.

22. Let $\phi : F^\times \to \mathbb{C}$ be a multiplicative character, and let $\chi = \chi_1 \otimes \chi_2$ be a character of T. Define $\phi \cdot \chi$ by $\phi \cdot \chi = \phi\chi_1 \otimes \phi\chi_2$. Show that

$$\mathrm{Ind}_B^G(\phi \cdot \chi) \simeq \phi \cdot \mathrm{Ind}_B^G \chi.$$

23. **Kirillov models for principal series.** Recall that by definition, the (normalized) induced representation $i_B^G\sigma$ consists of smooth functions on G satisfying $f(bg) = \delta(b)^{1/2}\sigma(b)f(g)$, $b \in B, g \in G$. Denote this space of functions by I. Recall that σ is a representation of B that is trivial on N, and for $t = \left[\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right]$, we have $\sigma(t) = \chi_1(a)\chi_2(b)$.

Let $w = \left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right]$ be yet another choice of the Weyl element (note that it again differs from the previous choices we made, for no good reason).

(a) Let

$$\Phi_f(x) = f\left(w \left[\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right]\right)$$

Then Φ_f is a function on F. Show that it is locally constant. Show that the map $f \mapsto \Phi_f$ is injective. (Hint: Use Bruhat decomposition).

(b) Show that $\Phi_f(x)|x|\chi_1(x)\chi_2^{-1}(x)$ is constant for large $|x|$. The space of functions Φ_f is sometimes called the *noncompact model* of π. Let us denote this space by $\mathcal{F}_{\chi_1\chi_2^{-1}}$.

(c) Recall that we have fixed an additive character $\psi : F \to \mathbb{C}$ (this choice of the character was needed to construct the Kirillov model). We can use the same ψ to identify the additive group of our field F with its Pontryagin dual, and therefore we have Fourier transform

$$\hat{f}(y) = \int_F f(x)\psi(xy)dx.$$

Let

$$\xi_f(x) = \chi_2(x)|x|^{1/2} \int_F \Phi_f(y)\psi(xy)dy = \chi_2(x)|x|^{1/2}\widehat{\Phi_f}(x).$$

Then the space $\{\xi_f \mid f \in I\}$ is the Kirillov model of π.

To prove this, one needs to see that the mirabolic subgroup acts on this space exactly by the same formulas that we got for the action of the mirabolic subgroup on the Kirillov model, and use uniqueness of Kirillov model to arrive at the conclusion.

(d) Show, by pure analysis, that for $\Phi \in \mathcal{F}_{\chi_1\chi_2^{-1}}$, the function $\widehat{\Phi}$ has the following asymptotic behaviour near 0:

$$\widehat{\Phi}(x) = \begin{cases} a\chi_1(x)\chi_2^{-1}(x) + b, & \chi_1\chi_2^{-1} \neq 1, |x|^{-1} \\ a\mathrm{ord}(x) + b & \chi_1\chi_2^{-1}(x) = 1 \\ b & \chi_1\chi_2^{-1}(x) = |x|^{-1}, \end{cases}$$

where a, b are complex constants.

Compare this statement with the asymptotic statement for Kirillov model that appeared in class on March 17 (Bump, Theorem 4.7.2)!

(e) The map from noncompact model to Kirillov model is injective iff $\chi_1\chi_2^{-1} \neq |x|^{-1}$. If $\chi_1\chi_2^{-1}(x) = |x|^{-1}$, then the kernel of this map consists of constant functions.

Notice that from here one can get an alternative proof of the irreducibility criterion for our induced representations i_B^G.