Modular forms, Homework 2.

1. *(You do not have to write up this solution if you already know this calculation). Hint: see Example 2.23 in Milne for hints.*

The goal of this problem is it to compute the index $[\Gamma(1):\Gamma(N)]$. For a ring R, we denote by $\mathrm{GL}_2(R)$ the group

$$\mathrm{GL}_2(R) := \{X \in M_2(R) \mid \det(X) \in R^\times\},$$

where $M_2(R)$ is the set of 2×2 matrices with entries in R, and $R^\times$ is the group of units of R.

(a) Let $\mathbb{F}_p$ be the field of p elements. Prove that $\#\,\mathrm{GL}_2(\mathbb{F}_p) = (p^2-1)(p^2-p)$.
(b) Let $r \in \mathbb{N}$. Prove that $\#\,\mathrm{GL}_2(\mathbb{Z}/p^r\mathbb{Z}) = p^{4(r-1)}(p^2-1)(p^2-p)$.
(c) Suppose $N = \prod_i p_i^{r_i}$ is the prime factorization of N. Prove that $\mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z}) \simeq \prod_i \mathrm{GL}_2(\mathbb{Z}/p_i^{r_i}\mathbb{Z})$.
(d) Find $\#\,\mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$.
(e) Prove that $\#\,\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z}) = \varphi(N)^{-1}\#\,\mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$, where φ is Euler's φ-function.
(f) Find $[\Gamma(1):\Gamma(N)]$.
(g) $[\bar{\Gamma}(1):\bar{\Gamma}(N)]$, where $\bar{\cdot}$ denotes the quotient by $\{\pm I\}$ if $-I$ is in the group. Consider the case $N=2$ separately.

2. Exercise 2.24 on p.39 in Milne's notes.

3. Algebraic description of ramification:

(a) Consider a smooth curve on the affine plane, defined by the equation $f(x,y)=0$, where f is a degree 2 polynomial. Consider the projection onto the x-axis. Prove that a point (x_0,y_0) on the curve is a ramification point for this projection map iff $\frac{\partial f}{\partial y}|_{(x_0,y_0)} = 0$.
(*Hint: you can use implicit differentiation and consider it a calculus problem.*)
(b) Recall that $\mathbb{CP}^3$ is the complex projective space, with homogeneous coordinates $[z_0:z_1:z_2:z_3]$, where $[z_0:z_1:z_2:z_3]$ stands for the equivalence class of triples $(z_0,z_1,z_2,z_3)\in\mathbb{C}^4$ with the usual equivalence $(z_0,z_1,z_2,z_3)\sim(\lambda z_0,\lambda z_1,\lambda z_2,\lambda z_3)$ with $\lambda\in\mathbb{C}^\times$ (i.e. the space of lines through the origin in $\mathbb{C}^4$). Let X be a curve defined by a system of two polynomial equations in $\mathbb{CP}^3$: $p_1(z_0,z_1,z_2,z_3) = p_2(z_0,z_1,z_2,z_3) = 0$, where p_i are homogeneous polynomials with complex coefficients. Consider a projective line L in $\mathbb{CP}^3$ given by $L_1 = L_2 = 0$ where L_1, L_2 are linear homogeneous polynomials. Prove that there is a (natural) projection from X onto L such that a point on X is a ramification point for this projection iff the following Jacobian determinant vanishes at that point:

$$J := \begin{vmatrix} \frac{\partial p_1}{\partial z_0} & \cdot\cdot & \frac{\partial p_1}{\partial z_3} \\ \frac{\partial p_2}{\partial z_0} & \cdot\cdot & \frac{\partial p_2}{\partial z_3} \\ \frac{\partial L_1}{\partial z_0} & \cdot\cdot & \frac{\partial L_1}{\partial z_3} \\ \frac{\partial L_2}{\partial z_0} & \cdot\cdot & \frac{\partial L_2}{\partial z_3} \end{vmatrix} = 0.$$

4.* Using Riemann-Hurwitz formula, prove that the intersection of two generic quadric surfaces in $\mathbb{CP}^3$ is an elliptic curve.

More precisely, A *quadric surface* is a surface defined by a degree 2 homogeneous polynomial in these coordinates:

$$\sum_{0\le i,j\le 3} a_{ij} z_i z_j = 0, \tag{1}$$

where $a_{ij} \in \mathbb{C}$. By *generic* we mean a property that holds for *almost all* coefficients (a_{ij}) (here the notion of 'almost all' means, the exceptions form a hypersurface, defined by some polynomial equations, in the space of all coefficients).

For this problem, an *elliptic curve* is a complex projective curve of genus 1. It is OK to work with complex manifolds (and Riemann surfaces) instead of the algebraic surfaces/curves. Thus the problem is asking the following: consider the curve in $\mathbb{CP}^3$ obtained as the intersection of two surfaces defined by equations of the form (1). It is OK to assume without proof that for two generic surfaces, you do get a Riemann surface (i.e. a smooth curve) as the intersection. Then you only need to prove that it has genus 1.

Hint: use the previous problem. To count ramification points, you can use Bezout's theorem.