

### Modular forms, Homework 3.

1. Prove Theorem 3.8 in Milne.
2. Prove that Weierstrass  $\wp$ -function satisfies the differential equation:

$$\wp'(z)^2 = 4\wp(z)^3 - g_2\wp(z) - g_3.$$

3. In this problem we establish that an elliptic curve is an *algebraic* group with respect to addition, i.e. that the operation of addition is algebraic. Note that  $\mathbb{C}/\Lambda$  is, naturally, a group with respect to addition.

- (a) Prove Milne's Proposition 3.9:

$$\wp(z + z') = \frac{1}{4} \left( \frac{\wp'(z) - \wp'(z')}{\wp(z) - \wp(z')} \right)^2 - \wp(z) - \wp(z').$$

- (b) Prove that if  $E$  is the elliptic curve with  $E(\mathbb{C}) = \mathbb{C}/\Lambda$  then the maps  $E \times E \rightarrow E$ ,  $(x, y) \mapsto x + y$  and  $E \rightarrow E$ ,  $x \mapsto -x$  are algebraic.

**4.\* Inversion of an elliptic integral.** As a warm-up, consider the stereographic projection for a circle: you can check (or believe – do not need to write it up) that with a suitable choice of coordinates it is given by  $t \mapsto \left( \frac{t^2-1}{t^2+1}, \frac{2t}{t^2+1} \right)$  (think of it as a map from the line to the circle). What this formula does is: it provides a *rational* bijection between the set of solutions to the equation  $x^2 + y^2 = 1$  and the points of the projective line (note that  $\left( \frac{t^2-1}{t^2+1}, \frac{2t}{t^2+1} \right)$  is the affine representation of the point  $(t^2 - 1 : 2t : t^2 + 1)$ ). Note that this bijection works over *any* field of characteristic not 2 (if you do it over  $\mathbb{Q}$ , you get Pythagorean triples; over a finite field it is useful for counting solutions to  $x^2 + y^2 = 1$ , and over  $\mathbb{C}$  it gives us yet another way to think of the Riemann sphere: namely, as a *curve* in  $\mathbb{P}^2(\mathbb{C})$  defined by the equation  $x^2 + y^2 = 1$ ).

- (a) Consider the Riemann sphere again, this time, think of it as above – as the curve in  $\mathbb{P}^2(\mathbb{C})$  with the equation  $z^2 + w^2 = 1$ . Let  $z_0$  be some fixed point, and let  $F(u) = \int_{z_0}^u \frac{1}{w} dz$ . (Does the integral depend on the path?). Find the inverse function of  $F(u)$ .

*Hint: this question is essentially trivial if you recall trig substitution. It actually explains why the trig substitution works: you parametrize the curve as  $z = \sin(u)$ ,  $w = \cos(u)$ . Note also that  $\sin$  is a function with one period.*

- (b) Now consider an elliptic curve  $w^2 = z^3 - az - b$ . Again consider the integral  $F(u) = \int_{z_0}^u \frac{1}{w} dz$  – the integral is along a path on our curve. Observe that this integral is a multi-valued function (why?). However, it has a well-defined inverse: prove that its inverse is, in fact, a suitable Weierstrass  $\wp$ -function!