1. The set-up

All of this is basically copied from various places in the article by Kottwitz in the Clay volume; but the goal is to write it without recourse to the tree.

Let F be a Non-Archimedean local field, either of characteristic zero or large (unspecified at the moment) positive characteristic, $\mathcal{O}_F$ – its ring of integers, and ϖ – the uniformizer. Let $G = \mathrm{GL}_2(F)$, $K_0 = \mathrm{GL}_2(\mathcal{O}_F)$.

Notation for some standard subgroups: Z is the centre of G, B – the Borel subgroup consisting of the upper-triangular matrices, $B = AN = NA$, where N is the unipotent upper-triangular matrices, and A is the diagonal torus. (If we want to generalize this to such groups as the unitary group, we'll need to make the distinction between a maximal torus T and a maximal split torus A – hence the unusual notation for the torus, following Kottwitz).

The goal of this note is to compute (directly, using the double cosets) the values of the orbital integral $\mathcal{O}_\gamma(f)$ for the functions f that are bi-invariant under K_0 (in particular, for the characteristic function of K_0), for the specific element

$$\gamma_u = \left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right].$$

The whole calculation is, in a sense, based on two decompositions of G: Cartan decomposition and Iwasawa decompostion.

(1) Cartan decompostion in its crude form states that $G = K_0 A K_0$, but in fact we have $G = K_0 A^+ K_0$, where $A^+ = \left\{ \left[\begin{smallmatrix} \varpi^m & 0 \\ 0 & \varpi^n \end{smallmatrix}\right] \mid m \geq n, m, n \in \mathbb{Z} \right\}$. (In general, two double cosets $K_0 a K_0$ and $K_0 a' K_0$ are the same if and only if the images of a and a' in $A/A \cap K_0$ are conjugate under the Weyl group. We will talk about the set $A/A\cap K_0$ below.) We use Cartan decomposition to label the elements of the basis of the Hecke algebra of K_0 bi-invariant functions: we denote by $f_{m,n}$ the characteristic function of the double coset $K_0 \left[\begin{smallmatrix} \varpi^m & 0 \\ 0 & \varpi^n \end{smallmatrix}\right] K_0$.

(2) Iwasawa decomposition states that $G = BK_0 = NAK_0$. It is particularly useful for the computation of the orbital integral of the unipotent element γ_u, because the centralizer of γ_u is $G_{\gamma_u} = ZN = NZ$.

Definition 1.1. Let K be an arbitrary open compact subgroup of G, let $X = G/K$; G acts on X by $g \cdot xK = gxK$. Let x_0 be a base point in X. Define a map $\mathrm{inv} : X \times X \to K\backslash G/K$ by $(g_1x_0, g_2x_0) \mapsto Kg_2^{-1}g_1K$. It is easy to see that this map is well-defined, and gives a bijection from the set of G-orbits on $X \times X$ to the set of double cosets $K\backslash G/K$.

Using this notation, we have the general formula for an orbital integral of a characteristic function of a double coset (which works for any γ and any K, and will be used for computing the semisimple orbital integrals as well):

$$\mathcal{O}_\gamma(\mathbf{1}_{KaK}) = \sum_{x:\mathrm{inv}(\gamma x, x) = KaK} \frac{\mathrm{vol}(K)}{\mathrm{vol}(\mathrm{Stab}_{G_\gamma}(x))},$$

where x runs over the set of representatives of G_γ-orbits on X satisfying the condition $\mathrm{inv}(\gamma x, x) = KaK$.

2. Computing the orbital integral of γ_u on the basis elements of the spherical Hecke algebra

Take $K = K_0$, $\gamma = \gamma_u = \left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right]$. First, a trivial observation: for any $\gamma \in K_0$, $\det(\gamma)$ is a unit, so if $m \neq -n$, then the orbit of γ does not intersect the double coset $K_0 \left[\begin{smallmatrix} \varpi^m & 0 \\ 0 & \varpi^n \end{smallmatrix}\right] K_0$, and therefore $\mathcal{O}_\gamma(f_{m,n}) = 0$. To make notation a bit shorter, let $f_m = f_{m,-m}$. Thus, we only need to compute $\mathcal{O}_{\gamma_u}(f_m)$, $m \in \mathbb{Z}$. Note that f_0 is by definition the characteristic function of K_0.

We have: $G_{\gamma_u} = ZN$; the measure on it is $dt/|t| \wedge dx$ (where we think of the elements of Z as $\left[\begin{smallmatrix} t & 0 \\ 0 & t \end{smallmatrix}\right]$, and the elements of N as $\left[\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right]$). We normalize the measure on G such that $\mathrm{vol}(K_0) = 1$.

Then we get:

$$\mathcal{O}_\gamma(f_m) = \sum_{x:\mathrm{inv}(\gamma x, x) = K_0 \left[\begin{smallmatrix} \varpi^m & 0 \\ 0 & \varpi^{-m} \end{smallmatrix}\right] K_0} \frac{1}{\mathrm{vol}(\mathrm{Stab}_{ZN}(x))}.$$

Thus, we need to first understand the orbits of ZN on the set G/K_0. This is where the building (i.e. the tree) is very helpful, but for now let us do it naively. First, observe that $ZN\backslash G/K_0 = N\backslash G/(ZK_0)$. Let $V = G/ZK_0$. Then we can rewrite the formula as:

$$\mathcal{O}_\gamma(f_m) = \sum_{v:\mathrm{inv}(\gamma v, v) = K_0 \left[\begin{smallmatrix} \varpi^m & 0 \\ 0 & \varpi^{-m} \end{smallmatrix}\right] K_0} \frac{1}{\mathrm{vol}(\mathrm{Stab}_N(x))},$$

where v now runs over the set of representatives of N-orbits on V.

So, we are now interested in the N-orbits on $V = G/ZK_0$. (the elements of V correspond to vertices in the tree, but we can do the calculation without using that).

By Iwasawa decomposition, $G = NAK$, and therefore $N\backslash G/ZK_0 = A/Z(A \cap K_0)$. The set of cosets $A/Z(A \cap K_0)$ has a very explicit set of representatives: $\left[\begin{smallmatrix} \varpi^j & 0 \\ 0 & 1 \end{smallmatrix}\right]$, $j \in \mathbb{Z}$. Let v_j be corresponding elements of V.

Remark 2.1. in general $A/(A \cap K_0) \simeq X_*(A)$, and this is what connects this discussion with the standard apartment.

In short, the vertices v_j explicitly represented by the elements $\left[\begin{smallmatrix} \varpi^j & 0 \\ 0 & 1 \end{smallmatrix}\right]$, $j \in \mathbb{Z}$ form a set of representatives for the N-orbits on V.

Next, we need to compute the volumes of stabilizers of v_j, and given $m \in \mathbb{Z}$, figure out which v_j satisfy the condition

$$\mathrm{inv}(\gamma v, v) = K_0 \left[\begin{smallmatrix} \varpi^m & 0 \\ 0 & \varpi^{-m} \end{smallmatrix}\right] K_0.$$

First, an easy calculation shows that the stabilizer of v_j in N is the set $\left[\begin{smallmatrix} 1 & \varpi^j \mathcal{O}_F \\ 0 & 1 \end{smallmatrix}\right]$, and its volume is q^{-j}. Second, another easy calculation is that of $\mathrm{inv}(\gamma_u v_j, v_j)$. We compute:

$$v_j^{-1} \gamma_u v_j = \left[\begin{smallmatrix} 1 & \varpi^{-j} \\ 0 & 1 \end{smallmatrix}\right].$$

Then $\mathrm{inv}(\gamma_u v_j, v_j) = K_0 v_j^{-1} \gamma_u v_j K_0$, and therefore, for every $j \leq 0$, we get $\mathrm{inv}(\gamma_u v_j, v_j) = K_0$, and if $j = m > 0$, then $\mathrm{inv}(\gamma_u v_j, v_j) = K_0 \left[\begin{smallmatrix} \varpi^m & 0 \\ 0 & \varpi^{-m} \end{smallmatrix}\right] K_0$.

Finally, we get:

$$\mathcal{O}_{\gamma_u}(f_0) = \sum_{v_j : j \leq 0} \frac{1}{q^{-j}} = \frac{1}{1 - q^{-1}}; \quad \mathcal{O}_{\gamma_u}(f_m) = q^m.$$

3. In Laura's and Radhika's talks:

- The set V is interpreted as the set of vertices of an infinite tree (the "reduced building" for GL_2).
- G acts on this tree.
- the vertices v_j that appeared here are exactly the vertices in the apartment of A.
- The condition that $\mathrm{inv}(\gamma v, v)$ is a particular coset is interpreted as the condition on the distance from v to the fixed set of γ.
- The tree helps count the vertices that appear in the indexing set of the sum (for the semisimple elements γ, the naive counting presented here is much messier, and the tree provides the necessary help).