Math 257/316 Assignment 5 Solutions

1. For the "triangle wave" function

$$f(x) = \begin{cases} x & 0 \le x \le 1 \\ 2-x & 1 \le x \le 2 \end{cases}$$

defined on $[0,2]$:

(a) compute its Fourier sine series

Doing some integration by parts,

$$\begin{aligned} b_k &= \frac{2}{2}\int_0^2 f(x)\sin(k\pi x/2)dx = \int_0^1 x\sin(k\pi x/2)dx + \int_1^2 (2-x)\sin(k\pi x/2)dx \\ &= -x\frac{2}{k\pi}\cos(k\pi x/2)|_0^1 + \frac{2}{k\pi}\int_0^1 \cos(k\pi x/2)dx \\ &\quad -(2-x)\frac{2}{k\pi}\cos(k\pi x/2)|_1^2 - \frac{2}{k\pi}\int_1^2 \cos(k\pi x/2)dx \\ &= -\frac{2}{k\pi}\cos(k\pi/2) + \frac{4}{k^2\pi^2}\sin(k\pi x/2)|_0^1 + \frac{2}{k\pi}\cos(k\pi/2) - \frac{4}{k^2\pi^2}\sin(k\pi x/2)|_1^2 \\ &= \frac{8}{k^2\pi^2}\sin(k\pi/2) = \frac{8}{k^2\pi^2}\begin{cases} 0 & k \textit{ even} \\ (-1)^{(k-1)/2} & k \textit{ odd} \end{cases} \end{aligned}$$

so

$$\textit{F.S.S. of } f(x) \quad = \frac{8}{\pi^2}\sum_{k=1,k \textit{ odd}}^{\infty} \frac{((-1)^{(k-1)/2}}{k^2}\sin\left(\frac{k\pi x}{2}\right) = \frac{8}{\pi^2}\sin\left(\frac{\pi x}{2}\right) - \frac{8}{9\pi^2}\sin\left(\frac{3\pi x}{2}\right) + \cdots .$$

(b) computes its Fourier cosine series

Begin with

$$a_0 = \frac{2}{2}\int_0^2 f(x)dx = \int_0^1 xdx + \int_1^2 (2-x)dx = \frac{1}{2} + \frac{1}{2} = 1$$

(twice the average value of f on $[0,2]$). For $k \geq 1$, integrate by parts:

$$\begin{aligned}
a_k &= \frac{2}{2}\int_0^2 f(x)\cos(k\pi x/2)dx = \int_0^1 x\cos(k\pi x/2)dx + \int_1^2 (2-x)\sin(k\pi x/2)dx \\
&= \frac{2}{k\pi}x\sin(k\pi x/2)|_0^1 - \frac{2}{k\pi}\int_0^1 \sin(k\pi x/2)dx + \frac{2}{k\pi}(2-x)\sin(k\pi x/2)|_1^2 + \frac{2}{k\pi}\int_1^2 \sin(k\pi x/2)dx \\
&= \frac{2}{k\pi}\sin(k\pi/2) + \frac{4}{k^2\pi^2}\cos(k\pi x/2)|_0^1 - \frac{2}{k\pi}\sin(k\pi/2) - \frac{4}{k^2\pi^2}\cos(k\pi x/2)|_1^2 \\
&= \frac{4}{k^2\pi^2}\left[\cos(k\pi/2) - 1 - \cos(k\pi) + \cos(k\pi/2)\right] = \begin{cases} -\frac{16}{k^2\pi^2} & k = 4m+2, m \in \mathbb{Z} \\ 0 & \textit{else} \end{cases},
\end{aligned}$$

so

$$\text{F.C.S. of } f(x) = \frac{1}{2} - \frac{4}{\pi^2}\sum_{m=0}^{\infty}\frac{1}{(2m+1)^2}\cos((2m+1)\pi x) = \frac{1}{2} - \frac{4}{\pi^2}\cos(\pi x) - \frac{4}{9\pi^2}\cos(3\pi x) - \cdots$$

(c) by evaluating $f(1)$, use each of your results from (a) and (b) in turn, to find the value of the sum of the squares of the reciprocals of the odd integers:

$$1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} + \cdots.$$

By the Fourier convergence theorem, the F.S.S of f converges to $f(1) = 1$ at $x = 1$ (since f is continuous there), so

$$1 = \frac{8}{\pi^2}\sum_{k=1, k \ odd}^{\infty}\frac{1}{k^2}\sin^2\left(\frac{k\pi}{2}\right) = \frac{8}{\pi^2}\sum_{k=1, k \ odd}^{\infty}\frac{1}{k^2} \implies \sum_{k=1, k \ odd}^{\infty}\frac{1}{k^2} = \frac{\pi^2}{8}.$$

Also by the Fourier convergence theorem, the F.C.S of f converges to $f(1) = 1$ at $x = 1$ (since f is continuous there), so

$$1 = \frac{1}{2} - \frac{4}{\pi^2}\sum_{m=0}^{\infty}\frac{\cos((2m+1)\pi)}{(2m+1)^2} = \frac{1}{2} + \frac{4}{\pi^2}\sum_{m=0}^{\infty}\frac{1}{(2m+1)^2} \implies \sum_{m=0}^{\infty}\frac{1}{(2m+1)^2} = \frac{\pi^2}{8}.$$

2. Solve the heat conduction problem:

$$\begin{aligned}
&u_t = 2u_{xx}, \quad 0 < x < 2, \ \ t > 0, \\
&u(0,t) = 0 = u(2,t) \\
&u(x,0) = \begin{cases} x & 0 \leq x \leq 1 \\ 2-x & 1 \leq x \leq 2 \end{cases}, \quad 0 \leq x \leq 2.
\end{aligned}$$

By the now familiar formula from class/notes/text with $L = 2$, and $\alpha^2 = 2$,

$$u(x,t) = \sum_{k=1}^{\infty} b_k \sin(k\pi x/2) e^{-k^2\pi^2 t/2}$$

with b_k already computed in question 1(a), so that

$$u(x,t) = \frac{8}{\pi^2} \sum_{k=1, k \ odd}^{\infty} \frac{(-1)^{(k-1)/2}}{k^2} \sin\left(\frac{k\pi x}{2}\right) e^{-\frac{k^2\pi^2 t}{2}} = \frac{8}{\pi^2} \sin\left(\frac{\pi x}{2}\right) e^{-\frac{\pi^2 t}{2}} - \frac{8}{9\pi^2} \sin\left(\frac{3\pi x}{2}\right) e^{-\frac{9\pi^2 t}{2}} + \cdots$$

Note: if you solved the original (typo ridden) version of this problem where the $2 - x$ in $f(x)$ was mistakenly written as $1 - x$, you would have computed a different set of coefficients b_k.

3. Find the solution of the following problem describing the temperature in a wire with insulated ends:

$$\begin{aligned} &u_t = 3u_{xx}, \quad 0 < x < 2, \ \ t > 0, \\ &u_x(0,t) = 0 = u_x(2,t) \\ &u(x,0) = x, \quad 0 \leq x \leq 2. \end{aligned}$$

Again from class/notes/text,

$$u(x,t) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos(k\pi x/2) e^{-3k^2\pi^2 t/4}$$

where

$$a_0 = \frac{2}{2} \int_0^2 x dx = \frac{1}{2} 2^2 = 2,$$

and for $k \geq 1$,

$$\begin{aligned} a_k &= \frac{2}{2} \int_0^2 x \cos(k\pi x/2) dx = x \frac{2}{k\pi} \sin(k\pi x/2)|_0^2 - \frac{2}{k\pi} \int_0^2 \sin(k\pi x/2) dx \\ &= \frac{4}{k^2\pi^2} \cos(k\pi x/2)|_0^2 = \frac{4}{k^2\pi^2} \left[\cos(k\pi) - 1\right] = \begin{cases} -\frac{8}{\pi^2 k^2} & k \ odd \\ 0 & k \ even \end{cases}, \end{aligned}$$

so

$$u(x,t) = 1 - \frac{8}{\pi^2} \sum_{k=1, k \ odd}^{\infty} \frac{1}{k^2} \cos\left(\frac{k\pi x}{2}\right) e^{-\frac{3k^2\pi^2 t}{4}} = 1 - \frac{8}{\pi^2} \cos\left(\frac{\pi x}{2}\right) e^{-\frac{3\pi^2 t}{4}} - \cdots$$

4. Solve the following problem describing heat conduction in a closed thin circular wire:

$$\begin{cases} u_t = u_{xx}, & -1 < x < 1,\ t > 0, \\ u(-1,t) = u(1,t), & u_x(-1,t) = u_x(1,t), \\ u(x,0) = |x|, & -1 \le x \le 1 \end{cases}$$

If you didn't see this in class/text/notes, then just carry out the separation of variables, and you'll find the 'X problem' to be

$$X'' = -\lambda^2 X, \quad X(-1) = X(1), \ \ X'(-1) = X'(1),$$

(you can check the separation constant must be ≤ 0*, so we may write it as* $-\lambda^2$*), for which the general solution is* $X = A\cos(\lambda x) + B\sin(\lambda x)$*, so* $X' = \lambda(-A\sin(\lambda x) + B\cos(\lambda x))$*, and when we impose the periodic BCs we find*

$$A\cos(\lambda) - B\sin(\lambda) = A\cos(\lambda) + B\sin(\lambda) \quad \Longrightarrow \quad 2B\sin(\lambda) = 0,$$

$$\lambda(A\sin(\lambda) + B\cos(\lambda)) = \lambda(-A\sin(\lambda) + B\cos(\lambda)) \quad \Longrightarrow \quad 2A\lambda\sin(\lambda) = 0$$

so (unless $A = B = 0$*), sin*$(\lambda) = 0$*, so* $\lambda = k\pi$*,* $k \in \{0, 1, 2, 3, \ldots\}$*, with corresponding solution* $X_k(x) = A_k\cos(k\pi x) + B_k\sin(k\pi x)$ *(we include* $k = 0$ *as* $\lambda = 0$ *produces the constant solution). Thus the general solution to the PDE and BC is*

$$u(x,t) = \sum_{k=0}^{\infty} (A_k\cos(k\pi x) + B_k\sin(k\pi x))\, e^{-k^2\pi^2 t}.$$

To satisfy the IC, we need

$$|x| = u(x,0) = \sum_{k=0}^{\infty} (A_k\cos(k\pi x) + B_k\sin(k\pi x))$$

which of course is the (full range) Fourier series for $f(x) = |x|$ *on* $[-1,1]$*. Thus, since* $|x|$ *is an even function,* $B_k = 0$ *for all* k*, while*

$$A_0 = \frac{a_0}{2} = \frac{2}{2}\int_0^1 |x|dx = \frac{1}{2},$$

and for $k \ge 1$*,*

$$A_k = \frac{2}{2}\int_0^1 x\cos(k\pi x)dx = x\frac{1}{k\pi}\sin(k\pi x)|_0^1 - \frac{1}{k\pi}\int_0^1 \sin(k\pi x)dx$$
$$= \frac{1}{k^2\pi^2}\cos(k\pi k)|_0^1 = \begin{cases} 0 & k \text{ even} \\ -\frac{2}{k^2\pi^2} & k \text{ odd} \end{cases},$$

so

$$u(x,t) = \frac{1}{2} - \frac{2}{\pi^2}\sum_{k=1, k\ odd}^{\infty} \frac{1}{k^2}\cos(k\pi x)e^{-k^2\pi^2 t}.$$