IB: Solution by Fourier transform

We've seen that the linear wave PDE $iu_t = h(-i\nabla_x)u$ admits plane wave solutions $u(x,t) = e^{i(\xi\cdot x-\omega t)}$ satisfying the dispersion relation $\omega = h(\xi)$. However, we are primarily interested in solutions with spatial decay: $u(x,t) \to 0$ as $|x| \to \infty$ – physically, because this represents localized disturbances from equilibrium, and mathematically, to permit analysis in classical function spaces – and plane waves do not decay.

Since the PDE is linear and homogeneous, linear combinations

$$u(x,t) = a_1 e^{i(\xi_1\cdot x-h(\xi_1)t)} + a_2 e^{i(\xi_2\cdot x-h(\xi_2)t)} + \cdots + a_N e^{i(\xi_N\cdot x-h(\xi_N)t)}, \quad a_1,\ldots,a_N \in \mathbb{C}$$

of plane wave solutions with different wave vectors ξ are also solutions, and (formally) so are continuous superpositions

$$u(x,t) = \int_{\mathbb{R}^n} a(\xi)e^{i(\xi\cdot x-h(\xi)t)}\,d\xi.$$

If now u solves the **Cauchy problem** for this PDE:

$$\begin{cases} iu_t = h(-i\nabla_x)u, & x\in\mathbb{R}^n,\ t>0 \\ u(x,0) = u_0(x) \end{cases}$$

with specified initial data $u_0(x)$, then

$$u_0(x) = u(x,0) = \int_{\mathbb{R}^n} a(\xi)e^{i\xi\cdot x}\,d\xi,$$

which you probably recognize as the **Fourier transform**. Recall its definition

$$\widehat{u_0}(\xi) := (2\pi)^{-n/2}\int_{\mathbb{R}^n} e^{-i\xi\cdot x}u_0(x)\,dx,$$

and that of the **inverse Fourier transform**

$$u_0(x) = (\widehat{u_0})^\vee(x) := (2\pi)^{-n/2}\int_{\mathbb{R}^n} e^{i\xi\cdot x}\widehat{u_0}(\xi)\,d\xi,$$

to conclude that we have $a(\xi) = (2\pi)^{-n/2}\widehat{u_0}(\xi)$ and so a (formal, at this stage) solution of our Cauchy problem:

$$\boxed{u(x,t) = e^{-ith(-i\nabla)}u_0 := (2\pi)^{-n/2}\int_{\mathbb{R}^n} e^{i(\xi\cdot x-h(\xi)t)}\widehat{u_0}(\xi)\,d\xi},$$

where we have introduced also the *semigroup notation* $e^{-ith(-i\nabla)}$ for the operator mapping intial data to the solution of the Cauchy problem at time t.

Let's work out a couple of examples.

1. Transport equation in $\mathbb{R}^n$: for $c \in \mathbb{R}^n$, $\left\{ \begin{array}{c} u_t + c \cdot \nabla u = 0 \\ u(x,0) = u_0(x) \end{array} \right\}$. Here $h(\xi) = c \cdot \xi$,

   $$u(x,t) = e^{-tc\cdot\nabla} u_0 = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{i(\xi\cdot(x-ct))} \widehat{u_0}(\xi)\, d\xi = (\widehat{u_0})^\vee(x-ct) = u_0(x-ct)$$

   (which we already knew anyway). Note in this simple example it is easy to express the solution directly in terms of the initial data $u_0(x)$, rather than its Fourier transform $\widehat{u_0}(\xi)$.

2. Schrödinger equation: $\left\{ \begin{array}{l} iu_t = -\Delta u \\ u(x,0) = u_0(x) \end{array} \right\}$. Here $h(\xi) = |\xi|^2$,

   $$u(x,t) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{i(\xi\cdot x - |\xi|^2 t)}\, \widehat{u_0}(\xi)\, d\xi.$$

To express our solution of the Cauchy problem directly in terms of the initial data $u_0(x)$, rather than its Fourier transform $\widehat{u_0}(\xi)$, we need to recall some properties of the Fourier transform:

- differentiation $-i\nabla_x$ corresponds to multiplication by ξ: $\widehat{-i\nabla_x f}(\xi) = \xi \widehat{f}(\xi)$
- as a consequence, $(h(-i\nabla_x)f)\widehat{\ }(\xi) = h(\xi)\widehat{f}(\xi)$
- convolution corresponds to multiplication: $\widehat{f * g}(\xi) = (2\pi)^{n/2}\widehat{f}(\xi)\widehat{g}(\xi)$, where $(f*g)(x) := \int_{\mathbb{R}^n} f(y)g(x-y)\, dx$

Using these, we can solve $iu_t = h(-i\nabla_x)u$ by taking the Fourier transform (in x) and solving an ODE in t:

$$i\widehat{u}_t = h(\xi)\widehat{u} \implies \widehat{u}(\xi,t) = e^{-ih(\xi)t}\widehat{u_0}(\xi) \implies u(x,t) = (2\pi)^{-n/2}\left((e^{-ih(\xi)t})^\vee * u_0\right)(x).$$

So (formally) defining the **fundamental solution** (or **Greens function**)

$$\boxed{\Phi_t(x) := (2\pi)^{-n/2}(e^{-ih(\xi)t})^\vee(x)},$$

we arrive at the (formal) expression for the solution of the Cauchy problem:

$$\boxed{u(x,t) = (\Phi_t * u_0)(x)}.$$

Important Remark: these expressions are all formal unless/until we can be sure the functions/integrals involved are well-defined. For example, $e^{-ih(\xi)t}$ is a bounded function of ξ (since h is real-valued), but does not decay as $|\xi| \to \infty$, and so the integral defining

$$\Phi_t(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i(\xi\cdot x - h(\xi)t)}\, d\xi$$

is not convergent (the integrand is not an integrable function), and so should be interpreted in the sense of *distributions* (generalized functions), and/or through a limiting procedure such as

$$\Phi_t(x) = (2\pi)^{-n} \lim_{\varepsilon \to 0+} \int_{\mathbb{R}^n} e^{i(\xi \cdot x - h(\xi)t)} e^{-\varepsilon|\xi|}\, d\xi.$$

Let's compute some fundamental solutions:

1. 1D transport (again!): $u_t + cu_x = 0$. Recall $h(\xi) = c\xi$, so (formally)

   $$\Phi_t(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i(\xi(x-ct))}\, d\xi.$$

   Exercise: show that $\Phi_t(x) = (2\pi)^{-1/2}(e^{-ic\xi t})^{\vee}(x) = \delta(x - ct)$ (delta function). More precisely: for any smooth function $\phi(x)$ with *compact support* (vanishing outside some interval),

   $$\lim_{\varepsilon \to 0+} \int_{-\infty}^{\infty} \phi(x) \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\xi(x-ct)} e^{-\varepsilon|\xi|}\, d\xi \right] dx = \phi(ct).$$

   Given this, we may solve (for the third time now!) the Cauchy problem for the 1D transport equation $\left\{ \begin{array}{c} u_t + cu_x = 0 \\ u(x,0) = u_0(x) \end{array} \right\}$ as

   $$\begin{aligned} u(x,t) &= \left(e^{-c\partial_x} u_0\right)(x) = (\Phi_t * u_0)(x) = (\delta(\cdot - ct) * u_0)(x) \\ &= \int_{-\infty}^{\infty} \delta(y - ct) u_0(x - y)\, dy = u_0(x - ct). \end{aligned}$$

   Remark: this last integral is just formal – we are really just using the property that the delta 'function' (interpreted as a distribution, or as a measure) is the identity for convolution: $(\delta * f)(x) = f(x)$.

2. Schrödinger equation: $iu_t = -\Delta u$. Recall $h(\xi) = |\xi|^2$, so (formally)

   $$\Phi_t(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i(\xi \cdot x - |\xi|^2 t)}\, d\xi.$$

   Exercise: show that

   $$\Phi_t(x) := (2\pi)^{-n} \lim_{\varepsilon \to 0+} \int_{\mathbb{R}^n} e^{i(\xi \cdot x - |\xi|^2 t) - \varepsilon|\xi|^2}\, d\xi = (4\pi i t)^{-n/2} e^{i\frac{|x|^2}{4t}}$$

   (it is convenient to regularize the integral with $e^{-\varepsilon|\xi|^2}$ since $|\xi|^2$ appears already in the exponential, and Gaussians behave nicely with respect to Fourier transform).

Given this, we obtain a beautiful formula for the solution of the Cauchy problem $\left\{\begin{array}{l} iu_t = -\Delta u, \quad x \in \mathbb{R}^n,\ t > 0 \\ u(x,0) = u_0(x) \end{array}\right\}$ for the free Schrödinger equation:

$$\boxed{\begin{aligned} u(x,t) &= \left(e^{it\Delta}u_0\right)(x) = \left(\Phi_t * u_0\right)(x) = \left((4\pi i t)^{-n/2} e^{i\frac{|x|^2}{4t}} * u_0\right)(x) \\ &= (4\pi i t)^{-n/2} \int_{\mathbb{R}^n} e^{i\frac{|x-y|^2}{4t}} u_0(y)\, dy. \end{aligned}}$$

Important Remarks (the mathematical niceties of this formula):

- Since $|e^{i\frac{|x-y|^2}{4t}}| = 1$, this is just a bounded function of y, with no decay as $|y| \to \infty$, and so our formula as written is only well-defined if the initial data is integrable, i.e.

  $$u_0 \in L^1(\mathbb{R}^n) := \{f : \mathbb{R}^n \to \mathbb{C} \mid \|f\|_{L^1} := \int_{\mathbb{R}^n} |f(x)|\, dx < \infty\},$$

  in which case the resulting solution $u(x,t)$ is a bounded function of x for each $t > 0$:

  $$|u(x,t)| \le |4\pi i t|^{-n/2} \int_{\mathbb{R}^n} |e^{i\frac{|x-y|^2}{4t}}||u_0(y)|\, dy = (4\pi t)^{-n/2} \int_{\mathbb{R}^n} |u_0(y)|\, dy < \infty.$$

- However, we also know that $\widehat{u}(\xi,t) = e^{-i|\xi|^2 t}\widehat{u_0}(\xi)$, so $|\widehat{u}(\xi,t)| = |\widehat{u_0}(\xi)|$, and in particular, if the initial data is square-integrable,

  $$u_0 \in L^2(R^n) := \{f : \mathbb{R}^n \to \mathbb{C} \mid \|f\|_{L^2} := \left(\int_{\mathbb{R}^n} |f(x)|^2\, dx\right)^{1/2} < \infty\},$$

  then by **Plancherel's theorem** that the Fourier transform preserves the L^2 norm of functions,

  $$\|u(\cdot,t)\|_{L^2}^2 = \|\widehat{u}(\cdot,t)\|_{L^2}^2 = \int_{\mathbb{R}^n} |\widehat{u}(\xi,t)|^2\, d\xi = \int_{\mathbb{R}^n} |\widehat{u_0}(\xi)|^2\, d\xi = \|\widehat{u_0}\|_{L^2}^2 = \|u_0\|_{L^2}^2,$$

  and so our solution makes sense as an L^2 function of x for each t, indeed with the same L^2 norm. (More precisely: for $u_0 \in L^2$, we can approximate it (in the L^2 norm sense) by functions in $L^1 \cap L^2$, for which the solution formula makes sense, producing solutions (with the approximate initial data) in L^2, and by passing to the limit, produce the true solution, which lies in L^2.)

- It is not immediately clear from the solution formula in what sense (if any) $u(x,t) \to u_0(x)$ as $t \to 0+$, and indeed this relies on the rapid oscillation of the fundamental solution as $t \to 0+$. If $u_0 \in L^2$, one has:

  $$\|u(\cdot,t) - u_0\|_{L^2} \to 0 \text{ as } t \to 0+\,.$$

3. Airy equation: $u_t = u_{xxx}$. Recall $h(\xi) = \xi^3$, so (formally)
$$\Phi_t(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i(\xi x - \xi^3 t)} \, d\xi,$$
but we do not have an explicit formula as in the Schrödinger case.

4. wave equation: since it is second-order in time, the natural Cauchy problem is $\left\{ \begin{array}{l} u_{tt} = c^2 \Delta u, \quad x \in \mathbb{R}^n, \; t > 0 \\ u(x,0) = u_0(x), \;\; u_t(x,0) = v_0(x) \end{array} \right\}$, which we can again solve by taking Fourier transform (in x) and solving the (second order) ODE in t:
$$\widehat{u}_{tt} = -c^2 |\xi|^2 \widehat{u} \implies \widehat{u}(\xi,t) = \cos(c|\xi|t)\widehat{u_0}(\xi) + \frac{\sin(c|\xi|t)}{c|\xi|}\widehat{v_0}(\xi),$$
and noticing that $\cos(c|\xi|t) = \frac{\partial}{\partial t}\frac{\sin(c|\xi|t)}{c|\xi|}$,
$$\widehat{u}(\xi,t) = \frac{\sin(c|\xi|t)}{c|\xi|}\widehat{v_0}(\xi) + \frac{\partial}{\partial t}\frac{\sin(c|\xi|t)}{c|\xi|}\widehat{u_0}(\xi)$$
and then using the convolution property of the Fourier transform,
$$u(x,t) = (K_t * v_0)(x) + \frac{\partial}{\partial t}(K_t * u_0)(x), \quad K_t := (2\pi)^{-n/2}\left[\frac{\sin(c|\xi|t)}{c|\xi|}\right]^{\vee}.$$

Exercise: for $n = 1$, show that $K_t(x) = \frac{1}{2c}\left\{ \begin{array}{ll} 1 & -ct \le x \le ct \\ 0 & |x| > ct \end{array} \right\} = \frac{1}{2c}\chi_{[-ct,\, ct]}(x)$ by verifying that $\widehat{K_t}(\xi) = \frac{1}{\sqrt{2\pi}}\frac{\sin(ct\xi)}{c\xi}$, and thus derive **d'Alembert's formula** for the solution of the Cauchy problem for the 1D wave equation:
$$\boxed{u(x,t) = \tfrac{1}{2}\left(u_0(x-ct) + u_0(x+ct)\right) + \tfrac{1}{2c}\int_{x-ct}^{x+ct} v_0(y)\, dy}.$$

One famous implication of this formula is **finite speed of propagation**: information cannot travel faster than speed c (because the solution at (x,t) depends only on the initial data in the interval $[x-ct, \; x+ct]$).

Exercise: for $n = 3$, show that $K_t(x) = \frac{1}{2ct}\delta_{|x|=ct}$ (surface measure on the sphere $|x| = ct$) by verifying $\widehat{K_t}(\xi) = (2\pi)^{-3/2}\frac{\sin(ct|\xi|)}{c|\xi|}$, and thus derive the **Kirchoff formula** for the solution of the Cauchy problem for the 3D wave equation:
$$\boxed{u(x,t) = \tfrac{1}{2ct}\int_{|y-x|=ct} v_0(y)\, dS(y) + \tfrac{\partial}{\partial t}\left[\tfrac{1}{2ct}\int_{|y-x|=ct} u_0(y)\, dS(y)\right]}.$$

For $n = 3$ (and indeed all higher odd dimensions) we have not only finite speed of propagation but **(sharp) Huygen's principle**: the solution at (x,t) depends only on the initial data on the sphere $|y-x| = ct$ of radius ct centred at x – i.e., information moves *exactly* at speed c!