Self-organized criticality in a nutshell

J. Nagler, C. Hauert, and H. G. Schuster
Institut für Theoretische Physik und Astrophysik, Christian-Albrechts-Universität, Olshausenstraße 40, 24118 Kiel, Germany
(Received 26 February 1999; revised manuscript received 25 May 1999)

In order to gain insight into the nature of self-organized criticality (SOC), we present a minimal model exhibiting this phenomenon. In this analytically solvable model, the state of the system is fully described by a single-integer variable. The system organizes in its critical state without external tuning. We derive analytically the probability distribution of durations of disturbances propagating through the system. As required by SOC, this distribution is scale invariant and follows a power law over several orders of magnitude. Our solution also reproduces the exponential tail of the distribution due to finite size effects. Moreover, we show that large avalanches are suppressed when stabilizing the system in its critical state. Interestingly, avalanches are affected in a similar way when driving the system away from the critical state. With this model, we have reduced SOC dynamics to a leveling process as described by Ehrenfest's famous flea model. [S1063-651X(99)09209-0]

PACS number(s): 05.65.+b, 05.40.Fb, 05.10.Gg

I. INTRODUCTION

In 1987, Bak, Tang, and Wiesenfeld (BTW) [1] introduced *self-organized criticality* (SOC) by way of a so-called sandpile model. The work of Bak *et al.* has become the leading paradigm of SOC and triggered various theoretical and experimental works on the subject. They argued that SOC provides a natural framework in which to describe phenomena as diverse as $1/f$ noise in resistors, fluctuations of the river Nile, earthquakes, extinction of species, traffic jams, etc. For an excellent overview, see [2] and references therein.

In 1996, a random neighbor approximation of the original BTW model, together with a suitable definition of SOC, was presented by Flyvbjerg [3], which reads: A self-organizing critical system is a driven, dissipative system consisting of

(1) *a medium* that has
(2) *disturbances* propagating through it, causing
(3) a *modification* of the medium, such that eventually
(4) the medium is in a *critical state*, and
(5) the medium is *modified no more*.

This approach conserves the intriguing dynamics of SOC without the requirement of an extended system. We now present a simple model that is neither extended nor dissipative with regard to the amount of sand in the system, but still exhibits SOC behavior. Although, other grain-conserving models have been presented earlier [6], these systems did not display SOC, since they had to be tuned externally to the critical state. Moreover, our model allows for an analytical solution. In sharp contrast to the large number of models displaying SOC, only a few exact results are known at present (see, e.g., [7]).

II. MODEL

Roughly speaking, our model represents a conservative variant of Flyvbjerg's random neighbor model. The system consists of N dynamical sites and a large reservoir with N_{res} grains of sand ($N_{\mathrm{res}}>N$). Each dynamical site may contain one grain of sand such that we have N_0 empty and N_1 filled sites with $N_0+N_1=N$. In every time step, a grain from the reservoir is dropped on a random site. With probability N_0/N the chosen site is empty and becomes filled. Thus, we have $N_0\to N_0-1$, $N_1\to N_1+1$. Otherwise, with a probability of N_1/N, the site is already filled and is said to *topple*. This means that both grains are put back into the reservoir leading to $N_0\to N_0+1$, $N_1\to N_1-1$. Since we consider a large reservoir, changes in N_{res} are irrelevant, i.e., $N_{\mathrm{res}}>0$ all the time. Starting with all sites empty and repeating the above process over many time steps τ, the system will eventually reach a state with a characteristic number of filled sites specified by the mean up to time τ: $\langle N_1\rangle^\tau$. For $\tau\to\infty$, it follows from the symmetry of the model that the mean number of filled sites converges to the equilibrium value $\langle N_1\rangle^\tau=N/2$. As we will see, this state is critical in the sence that fluctuations around it have no inherent time or length scale, i.e., scale with a power law. Comparing Flyvbjerg's and our models, we see that both are driven by an external energy source. Our model satisfies Flyvbjerg's definition of SOC for random neighbor systems with the difference that it is (a) grain-conserving and (b) the driving force is realized in a somewhat different way: Whenever an avalanche is over, Flyvbjerg's driving mechanism throws another grain of sand into the system. As a consequence his model has two decoupled time scales: a large one on which the system is filled by adding grains from an external reservoir and a short one on which the avalanches run. In our model, the system approaches the critical state and we observe natural fluctuations around it. Hence, we have only one time scale relevant to the fluctuations and the equilibration or relaxation process.

III. DYNAMICS AND DISTURBANCES

The equation of motion for the probability $P^\tau(N_1)$ of finding N_1 grains of sand in the system at time τ reads

$$P^{\tau+1}(N_1)=\frac{N_1+1}{N}P^\tau(N_1+1)+\frac{N-N_1+1}{N}P^\tau(N_1-1). \tag{1}$$

Equation (1) describes the approach to equilibrium as well as fluctuations around it in the famous flea model by Ehrenfest [8]. It corresponds to Flyvbjerg's model when summing over the avalanche variable and by neglecting the absorbing sites. The solution of Eq. (1) is known [4] for the initial condition $N_1(0)=N_1^0$ and reads

$$P^{\tau}_{N_1^0}(N_1)=(-1)^{N_1^0}2^{-N}\sum_{j=-N/2}^{N/2}\left(\frac{2j}{N}\right)^{\tau}C^j_{N_1}C^{N/2-N_1^0}_{j+N/2}, \quad (2)$$

where the C^j_k's are defined by the identity

$$(1-z)^{N/2-j}(1+z)^{N/2+j}=\sum_{k=0}^{N}C^j_k z^k. \quad (3)$$

We define disturbances (*avalanches*) of the system as one-sided deviations from $\langle N_1\rangle^\tau$, denoting the mean number of filled sites up to time τ. An avalanche is said to start if $N_1<\langle N_1\rangle^\tau$ or $N_1>\langle N_1\rangle^\tau$ holds for the first time. The end is specified by the first return to the state $N_1\geqslant\langle N_1\rangle^\tau$ or $N_1\leqslant\langle N_1\rangle^\tau$, respectively. The end of one avalanche corresponds to the beginning of the subsequent one. The duration or lifetime λ of an avalanche is specified as the number of time steps τ required to return to $N_1\geqslant\langle N_1\rangle^\tau$ or $N_1\leqslant\langle N_1\rangle^\tau$, respectively.

Now we calculate avalanches at the critical point where the mean number of filled sites $\langle N_1\rangle^\tau$ reaches the equilibrium value $N/2$. Let $D(\lambda)$ denote the probability of an avalanche of length λ at the critical point. Thus, $D(\lambda)$ stands for the probability of all paths starting in $N_1=N/2$ at time τ and returning to $N_1=N/2$ for the first time at $\tau+\lambda$.

For $N/2$ even, this probability distribution of fluctuations in the Ehrenfest model was derived by Kac in 1947 [4] while discussing matters of irreversibility and Poincaré cycles in statistical mechanics:

$$D(\lambda)=-2^{1-N}\left(\frac{1}{2}+\frac{1}{N}\right)\sum{}'\left(\frac{2j}{N}\right)^{\lambda-2}C^{-1}_{N/2-j}C^{j}_{N/2-1}, \quad (4)$$

where the summation is extended over all odd integers j between $-N/2$ and $N/2$. The discrete solutions (1) and (4) are very hard to interpret. For this reason we consider the continuum limit $N\to\infty$ leading to intuitive solutions.

Let us introduce the scaling variables $x=(N_1-N/2)/\sqrt{N}$, $t=\tau/N$ and the probability density $f(x,t)=NP^\tau(N_1)$. Inserting these definitions into Eq. (1) and ignoring all subdominant powers $O(1/N)$, we obtain a Fokker-Planck equation:

$$\partial_t f(x,t)=\left(\frac{1}{2}\partial_x^2+2\partial_x x\right)f(x,t). \quad (5)$$

The stochastic nature of the process is reflected in the diffusive first term on the right-hand side of Eq. (5). The second term is convective, resulting in a force $F=-2x$ driving the system back to the critical value $x=0$. Equation 5 describes a random walker in a parabolic potential. In this picture, the lifetime of an avalanche corresponds to the first return time of the random walker starting and ending at $x=0$.

The analytical solution of Eq. (5) is known for the initial condition $f(x,0)=\delta(x-x_0)$, where $\delta(x)$ stands for Dirac's delta function, specifying that at time $t=0$ the system is in state $x=x_0$:

$$f(x,t)=\frac{1}{\sqrt{\pi\sigma(t)^2}}\exp\left(-\frac{(x-x_0e^{-2t})^2}{\sigma(t)^2}\right) \quad (6)$$

with

$$\sigma(t)^2=\frac{1}{2}(1-e^{-4t}). \quad (7)$$

Because of the diffusive term in Eq. (5), the information about the initial state x_0 vanishes exponentially fast with increasing time. In the long time limit $t\to\infty$, the stationary solution is the Gaussian:

$$f(x)=\sqrt{\frac{2}{\pi}}e^{-2x^2}. \quad (8)$$

Let $D(l)$ denote the probability of an avalanche of length l, where l scales as t with $l=\lambda/N$. Thus, $D(l)$ stands for the probability of all paths starting in $x=0$ at time t and returning to $x=0$ for the first time at $t+l$. This corresponds to the flux out of the system at an absorbing boundary in $x=0$ as a function of time.

A solution $g(x,t)$ of Eq. (5) with an absorbing boundary in $x=0$ is generated by differentiating Eq. (6) with respect to x_0. This yields indeed another solution, with initial condition $g(x,0)=\partial_{x_0}\delta(x-x_0)$, because ∂_{x_0} commutes with the operators of the Fokker-Planck equation (5):

$$g(x,t)=\partial_{x_0}f(x,t)=\frac{2}{\sqrt{\pi}\sigma(t)^3}\exp\left(-\frac{(x-x_0e^{-2t})^2}{\sigma(t)^2}\right)\times(x-x_0e^{-2t})e^{-2t}. \quad (9)$$

For $x_0=0$ and $x>0$, Eq. (9) is the solution for a random walker in a parabolic potential with initial condition $x_0=0$ and an absorbing boundary in $x=0$. The latter requires that $g(0,t)=0$ for all $t>0$.

In order to obtain the avalanche distribution $D(l)$, we have to calculate the flux $j(x,t)$ at $x=0$. Here the continuity equation together with Eq. (5) gives us a relation between the flux $j(x,t)$ and the solution $g(x,t)$ of Eq. (5):

$$j(x,t)=-\int\partial_t g(x,t)\mathrm{d}x. \quad (10)$$

For avalanches starting at $t=0$ and lasting until $t=l$, this leads to

$$D(l)=j(x,t=l)|_{x=0}=\sqrt{\frac{8}{\pi}}(1-e^{-4l})^{-3/2}e^{-2l}. \quad (11)$$

Note that Eq. (11) is exact in the continuum limit; however, it is an approximation to the avalanche distribution derived numerically from Eq. (1) and calculated below. For small avalanche lengths l, the discrete nature of our model plays an important role and thus the accuracy of Eq. (11) is expected to decrease. Figure 1 displays the perfect agreement of $D(l)$ up to small l, calculated according to Eq. (11) and numerically derived from Eq. (1). Up to avalanche lengths of the system size, $l\lesssim 1$ $(\lambda\lesssim N)$, the distribution follows a power law $\sim l^{-3/2}$, but for $l>1$ $(\lambda>N)$, the power law is exponentially suppressed with e^{-2l} due to finite size effects.

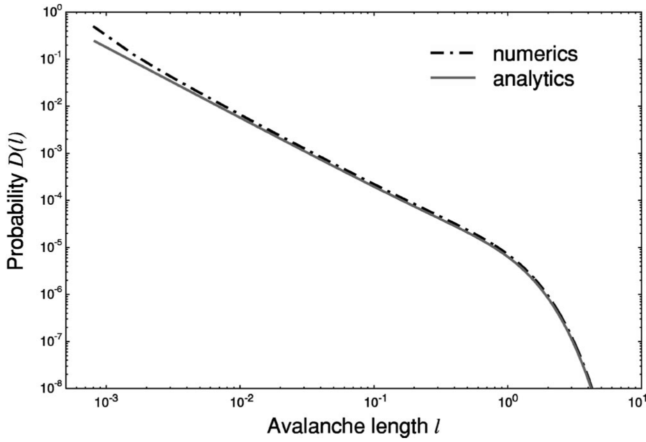

FIG. 1. Comparison of the approximate analytical (solid line) and numerical (dash-dotted line) solutions for the distribution of the avalanche lengths. The two curves are in perfect agreement in both the power law regime as well as the exponential tail. Only probabilities of very short avalanches are underestimated (see text). The numerical solution is calculated according to Eq. (1) with $N = 2500$. The avalanche lengths are scaled in terms of the system size N, such that $l=1$ corresponds to a length of $\lambda = N$.

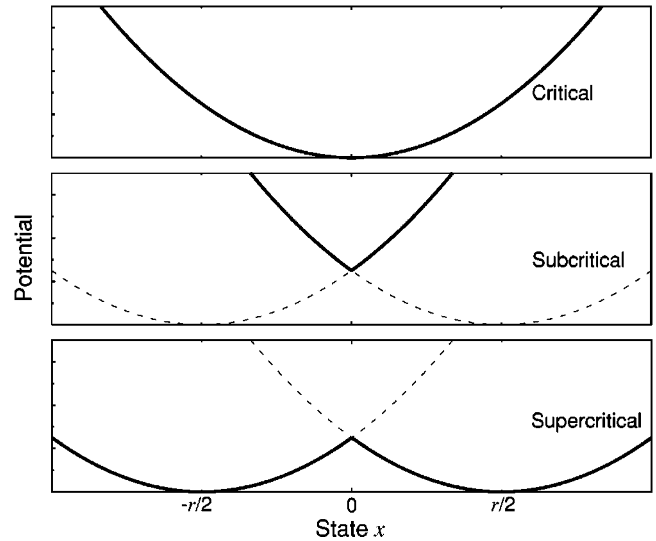

FIG. 2. Potential felt by the random walker in the undisturbed critical case (top graph), as well as in the subcritical (middle graph, solid line) and supercritical (bottom graph, solid line) regimes. The subcritical and supercritical regimes are obtained by a shift of $\pm r/2$ of the original parabolic potential. Note that in the supercritical regime, the two minima are separated by a potential barrier at $x=0$ and the walker may get trapped for long times around $x = \pm r/2$.

Unfortunately we could not show the discrete solution [Eq. (4)] because of numerical evaluation difficulties for $N = 2500$. However, we verified that for smaller N the analytical and numerical solutions of Eq. (1) are indentical for all avalanche sizes (not shown).

IV. SUPPRESSING DISTURBANCES

Let us now consider the problem of controlling avalanches. To suppress large avalanches, we stabilize the system in its critical state. The simplest method by which to achieve this is to rewrite Eq. (1) such that in every time step τ another grain of sand is added if $N_1 < \langle N_1 \rangle^\tau$ or removed if $N_1 > \langle N_1 \rangle^\tau$ with a constant probability ρ. This mechanism drives the system systematically back to its critical state. In the continuum limit $N \to \infty$, using the same scaling variables as before and setting $\rho = r/\sqrt{N}$, we obtain the Fokker-Planck equation for the disturbed system:

$$\partial_t f(x,t) = \left[\frac{1}{2} \partial_x^2 + 2\partial_x \left(x - \text{sgn}(x) \frac{r}{2} \right) \right] f(x,t), \qquad (12)$$

where sgn(x) denotes the signum function. The only difference between this equation and the undisturbed Eq. (5) is a shift of the parabolic potential by sgn(x)$r/2$. In analogy, we may also drive the system away from the critical state, resulting in a potential shift of $-\text{sgn}(x)r/2$ in the other direction. This divides the dynamics of the system into a subcritical and supercritical regime in terms of the avalanche distributions. In order to estimate these distributions, we consider as before a random walker in the respective potential. The different potentials felt by the walker are indicated in Fig. 2 for the undisturbed system as well as the subcritical and supercritical regimes.

Returning to the picture of the flea model, this controlling mechanism simply means that in every time step we count the number of fleas on both dogs and force, with a certain probability, another flea to jump from the dog bearing more fleas to the other one or vice versa.

Subcritical regime

The avalanche distribution $D(l,r)$ in the subcritical regime is estimated by considering all paths of a random walk starting and ending at $x=0$. Note, however, that now an increased force of $F = 2(x + r/2)$ drives the walker back to the origin. An avalanche with $x<0$ corresponds to a random walk starting and ending at $x = -r/2$ for $x \leqslant r/2$ in the undisturbed potential. Due to the symmetry of the system, the same argument holds for avalanches with $x>0$. Therefore, it is sufficient to consider only one case. In such an asymmetric situation, we are not able to calculate the exact avalanche distribution as done in Sec. III. For this reason, we estimate the avalanche distribution first by calculating the probability of all paths starting and ending at $x = -r/2$ [see Eq. (6)] in the undisturbed potential. Second, the distribution is weighted by the probability that the walker has not reached the forbidden area. This is specified by the integral of Eq. (11) from l to ∞. This leads to:

$$D(l,r) \sim \left(\int_l^\infty \sigma_{l'}^{-3} e^{-2l'} dl' \right) \sigma_l^{-1} \exp\left(-\frac{r^2(1-e^{-2l})^2}{2(1-e^{-4l})} \right)$$
$$= 2(1-e^{-4l})^{-1} e^{-2l} \exp\left(-\frac{r^2(1-e^{-2l})^2}{2(1-e^{-4l})} \right). \qquad (13)$$

Numerical solutions of the subcritical regime are shown in Fig. 3 for different values of ρ.

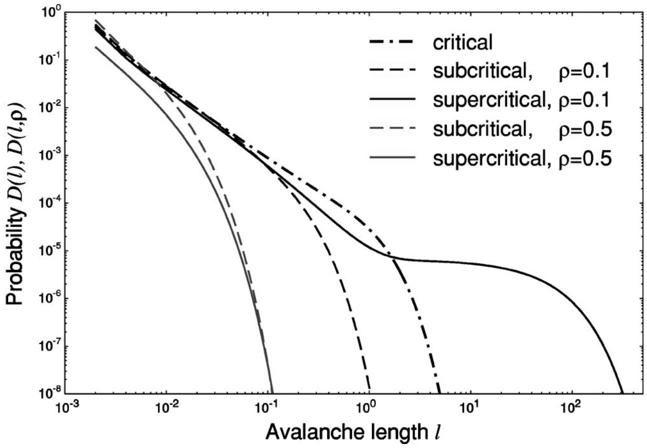

FIG. 3. Numerical solution of the subcritical and supercritical avalanche distributions for different degrees of controlling with $N=1000$. All distributions are normalized to 1. For avalanche lengths $l \lesssim 1$, the subcritical and supercritical distributions are very similar. However, in the supercritical regime, avalanches with $l>1$ are far more likely and occur over a certain range with almost constant probability. This corresponds to the analytical estimation in Eq. (14).

Supercritical regime

In the supercritical regime, the random walker feels the drift force $F=2(x+r/2)$ that restrains the walker from returning to the origin $x=0$. This is readily seen in Fig. 2, noting the potential barrier at $x=0$ separating the two energy minima at $x \pm r/2$. Thus, for large r the walker may become trapped around $x \pm r/2$ for very long times. However, for small avalanche lengths, the avalanche distribution is estimated using analogous arguments as in the subcritical regime. Estimations for the two regimes differ only for large avalanches $(l>1)$. Here, the distribution is estimated by the time it takes for a random walker starting at $x \pm r/2$ to reach $x=0$. This corresponds to a random walker starting at $x=0$ of the undisturbed potential, trying to reach $x \pm r/2$:

$$D(l,r) \sim (1-e^{-4l})^{-1/2} \exp\left(-\frac{r^2}{2(1-e^{-4l})}\right). \quad (14)$$

For lengths $l \gg 1$ $(\lambda \gg N)$, this leads to $D(l,r)=\text{const}$. Numerical solutions of the subcritical and supercritical regimes are shown in Fig. 3 for different values of the external disturbance ρ. The results for the supercritical regime resemble those obtained by Bundschuh and Lässig [5] for Flyvbjerg's model with very small absorption rates. In this limit, the two models, Flyvbjerg's and ours, become quite similar, the sole difference being that we do not distinguish between grains in the reservoir and those in the avalanche. They found an exponential distribution dominating the power law and drastically *increasing* the frequency of large avalanches. Together with Flyvbjerg, they point out that SOC systems must be sufficiently damped. In our model, this refers to the critical case with $r=0$. For $r \neq 0$, the power law and thus SOC is suppressed with an efficiency increasing with $|r|$. In particular, for $r<0$ we also observe an increase of large avalanches $(l>1)$ over the power law.

The numerical solution displayed in Fig. 3 shows that, with regard to the undisturbed system, avalanches up to a length of $l \lesssim 1$ are suppressed in both regimes. This suggests that the dynamics of a sandpile having an angle slightly above and slightly below the critical value are very similar (see Fig. 3).

V. CONCLUSIONS

We have shown that our simple model is SOC in the sense that it reaches a critical state without external tuning. In the critical state, the probability distribution of disturbances propagating through the system is scale invariant and follows a power law over many orders of magnitude. Moreover, the analytical solution in the continuum approximation [Eq. (11)] is in perfect agreement with numerical solutions of the discrete model [Eq. (1)] except for $t \sim O(1/N)$. It reproduces not only the power law regime, but also the exponential tail of the distribution. In addition, we have shown that a suitable drift, stabilizing the system in its critical state, suppresses large avalanches. Interestingly, the same effect is achieved when driving the system away from the critical state. This results in very similar avalanche distributions up to avalanche lengths corresponding to the system size.

We introduced a grain-conservating variant of sandpile models. However, an equivalently valid interpretation of our model is Ehrenfest's flea model [8]. Therefore, our model describes the process of approaching an equilibrium state in a large set of uncoupled two-state systems together with fluctuations (avalanches) around this state. Two-state systems where this effect is observable are abundant in nature (see, e.g., [9]). We just mention Ising magnets in the paramagnetic state. One-sided deviations of the magnetization from zero will display avalanches whose durations are distributed with a power law. Thus, we suggest that SOC is an inherent phenomenon of slow leveling processes.

[1] P. Bak, C. Tang, and K. Wiesenfeld, Phys. Rev. Lett. **59**, 381 (1987); Phys. Rev. A **38**, 364 (1988); C. Tang and P. Bak, J. Stat. Phys. **51**, 797 (1988); Phys. Rev. Lett. **60**, 2347 (1988).
[2] P. Bak, *How Nature Works* (Springer, New York, 1996).
[3] H. Flyvbjerg, Phys. Rev. Lett. **76**, 940 (1996).
[4] M. Kac, Am. Math. Monthly **54**, 7 (1947); e.g., in *Selected Papers on Noise and Stochastic Processes* (Dover Publications, New York, 1954).
[5] R. Bundschuh and M. Lässig, Phys. Rev. Lett. **77**, 4273 (1996); H. Flyvbjerg, *ibid.* **77**, 4274 (1996).
[6] R. Dickman, A. Vespignani, and S. Zapperi, Phys. Rev. E **57**, 5095 (1998); V. Priezzhev, J. Stat. Phys. **74**, 955 (1994).
[7] S. A. Janowsky and C. A. Laberge, J. Phys. A **26**, L973 (1993); D. Dhar, Phys. Rev. Lett. **64**, 1613 (1990).
[8] P. and T. Ehrenfest, Phys. Z. **8**, 311 (1907); W. Feller, *An Introduction to Probability Theory and its Applications*, 2nd ed. (Wiley, New York, 1971).
[9] L. Gammaitoni, P. Hänggi, P. Jung, and F. Marchesoni, Rev. Mod. Phys. **70**, 223 (1998).