A Vancouver carshare company, Mo-vo, has three car drop-off/ pick-up locations (they're just getting started, ok? These things take time) around the city: 1 (UBC), 2 (Downtown) and 3 (East Van). Cars can be picked up and dropped off at any of the three locations, and Mo-vo's administration studies the locations of their fleet of cars at the start of each day. For each day t, they denote by $\mathbf{v_t} = \begin{bmatrix} x_t \\ y_t \\ z_t \end{bmatrix}$ the vector whose entries x_t, y_t and z_t are the proportions of cars in each of the locations 1, 2 and 3, respectively. This means that $x_t + y_t + z_t = 1$ and we call $\mathbf{v_t}$ a **state vector** of the system. From their studies, they construct a matrix A, called the **transition matrix** whose (i, j)-th entry denotes the proportion of cars at the location j on a given day that end up at location i the next day. The matrix they compute is:

$$A = \begin{bmatrix} 0.3 & 0.4 & 0.5 \\ 0.3 & 0.4 & 0.3 \\ 0.4 & 0.2 & 0.2 \end{bmatrix}$$

1. **Understanding the transition matrix**
   (a) If a car is at location 2, what is the probability that it will be at location 3 tomorrow?

   (b) What does the $(2, 1)$-entry of A tell you about cars being picked up and dropped off?

   (c) What do the diagonal entries of A tell you about cars being picked up and dropped off?

   (d) What do you notice about the entries down each column of A? Why does this make sense?

2. **Applying the transition matrix**
   (a) What does the third row tell you about the system?

   (b) If an equal proportion of the cars are at each location today, what will the proportions of the cars at each location be tomorrow?

(c) If $\mathbf{v_t}$ is today's state vector, what will tomorrow's state vector be?

3. **Long-term behaviour**

(a) If 100% of the cars are at UBC today, what will the proportions be tomorrow? The day after tomorrow? In a week? In 100 days? (you don't have to compute these last couple, just write down what you need to compute)

(b) Show that A is diagonalisable by computing the matrix product CDC^{-1}, where

$$D = \begin{bmatrix} 0.1 & 0 & 0 \\ 0 & -0.2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 1 & 7 \\ -3 & 0 & 6 \\ 2 & -1 & 5 \end{bmatrix}$$

(c) Using this diagonalisation, write an expression for $\mathbf{v_{t+100}}$ in terms of $\mathbf{v_t}$

(d) A **steady state** of A is a vector $\mathbf{w}$ such that $A\mathbf{w} = \mathbf{w}$. From your diagonalisation above, compute a steady state vector for A. Normalise it by dividing the entries by the sum of the entries.