Math 2210 Name: ______________________

Workshop *** NetID: ______________________

Due ***

Please answer these exercises, you may print this handout, annotate the PDF or write your answer on paper. When submitting your workshop assign problems to your pages, see ***`http://gradescope-static-assets.s3-us-west-2.amazonaws.com/help/submitting_hw_guide.pdf`*** *. It is important to be able to explain your reasoning to someone else in writing.* ***Please include full explanations and write your answers using complete sentences*** *(not just a bunch of mathematical symbols!). Please upload all the pages of your workshop to Gradescope.com by *** on ***. Make your grader's life easier by writing neatly and legibly!!*

In this workshop, we'll be playing a version of "Lights Out", an electronic game from 1995 in which a random subset of lights in a 5×5 grid lights up, and the aim of the game is to figure out how to turn all the lights out by clicking on a preferably minimal subset of the lights. The trick is that by clicking on a light, not only is the light toggled, but so are the (up to) four adjacent lights.

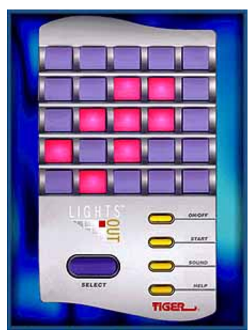
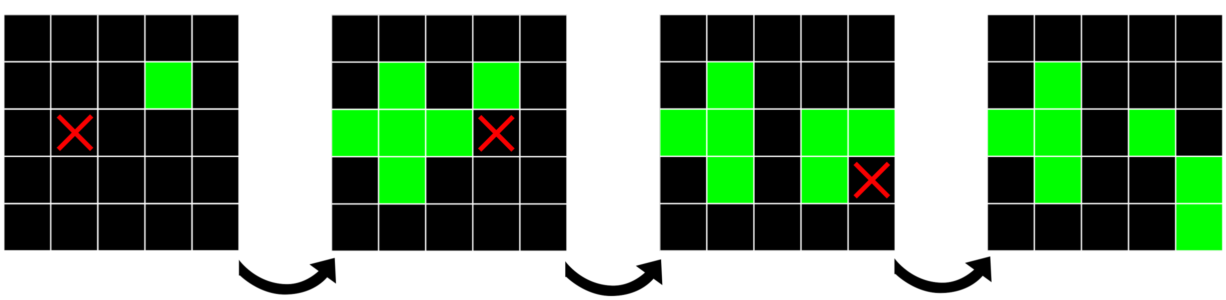


Figure 1: The electronic lights out game (Picture source:`http://matroidunion.org/?p=2160`)

We'll use green and black to denote lights that are on and off respectively. Some examples of game-play are shown below:



In the example above, the initial configuration consists of the light at position (2,4) being on and the rest of the lights being off. The red crosses indicate the light being clicked to reach the next configuration. Note that in the last step, since the light is on the right edge, it only has three adjacent lights to be toggled.

While this game might seem simple, it is in fact somewhat tricky to solve most puzzles, and our aim today is to figure out how to solve the puzzle using linear algebra.

Warm-up

Question 1. Before we start using any linear algebra, try playing an online version of the game, that you can find here. Try solving a puzzle with no help, then have a look at the solution given in their tutorial. However, part of our aim is to solve the puzzle *optimally*, i.e. with the smallest number of clicks.

You might have found that determining a solution to a given puzzle was not always obvious and often quite difficult. Our aim now is to model the game using mathematics and use our knowledge of linear algebra to find solutions to the puzzles.
First, we need to model the different states a single light can be in, namely on or off. We will write "1" for "on" and "0" for "off". So our collection of all possible states for a single light is the set $\{0, 1\}$, which we'll denote by $\mathbb{F}_2$.

Question 2. What do the states $\{0, 1\}$ change to when we click on a light?

Question 3. What happens if you click on the light twice?

Question 4. The idea here is that clicking on a light in our $\mathbb{F}_2$ model is "adding 1", except that we want to impose the condition "$1 + 1 = 0$", or in words "clicking twice is the same as not clicking at all." Note that $\mathbb{F}_2$ is *closed* under multiplication: products of elements in $\mathbb{F}_2$ are still in $\mathbb{F}_2$. With this in mind, fill out the following addition and multiplication tables for $\mathbb{F}_2$:

$+$	0	1
0		
1		

$\times$	0	1
0		
1		

Fact: Using $\mathbb{F}_2$ with multiplication and addition as in the tables above, we are able to row reduce and invert matrices just as we normally do.[1]

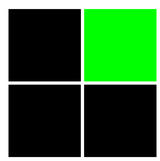
[1] We will not prove this fact, but the technical reason is that $\mathbb{F}_2$ is a *field*.

Lights out on a 2×2 grid

We will mainly work with a 2×2 version of the lights out game, so that calculations are easier. The initial state of a game consists of a 2×2 grid of lights, some of which are on and the remainder being off. We will use our $\mathbb{F}_2$ model for lights being on or off, and describe this configuration of lights by a size 4 vector with entries in $\mathbb{F}_2$, called the *configuration vector*, where the order of entries in the vector is given as follows:

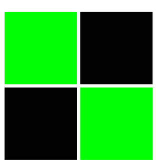
1	2
3	4

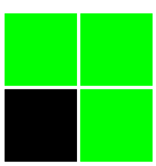
For example, the initial state:



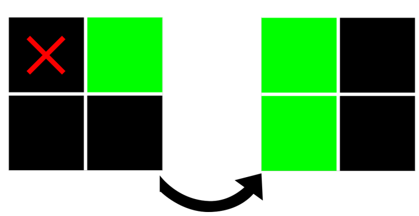
corresponds to the configuration vector $\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$

Question 5. Write down the configuration vectors corresponding to the following initial states:

a) 

b) 

Next, we need to model the *toggle vectors*: the vectors, denoted T_i, that correspond to clicking on a particular square in the grid. Recall that clicking on a light toggles both that light and all the edge-adjacent lights. In the 2×2 case, all lights are adjacent to exactly 2 other lights. For example, clicking on light 1 toggles lights 1,2 and 3. The following picture demonstrates how the state where only light 2 is on changes when we click on light 1:



The corresponding toggle vector is:

$$T_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

indicating that we should change (i.e. add 1 to the states of) lights 1,2 and 3.

Question 6. Write down the remaining toggle vectors T_2, T_3 and T_4.

Question 7. The puzzle has been solved when all the lights in the grid are off. What is the corresponding configuration vector?

Using this model, we can now see that finding a solution to the lights out puzzle in the initial configuration C consists of finding coefficients x_i for $i = 1, 2, 3, 4$ such that the following vector equation holds:

$$x_1T_1 + x_2T_2 + x_3T_3 + x_4T_4 + C = \mathbf{0}. \tag{1}$$

Question 8.

a) Let B be an $m \times n$ matrix with entries in $\mathbb{F}_2$. Show that $B + B = 0$, where 0 here is the $m \times n$ matrix with all zero entries.

b) Use this to explain why (1) is equivalent to the following equation:

$$x_1T_1 + x_2T_2 + x_3T_3 + x_4T_4 = C \tag{2}$$

c) Furthermore, explain why the coefficients x_1, x_2, x_3 and x_4 should just be elements of $\mathbb{F}_2$.

Question 9. Let B_1 and B_2 be two $m \times n$ matrices with entries in $\mathbb{F}_2$. Since we add matrices entry-wise, we know that

$$B_1 + B_2 = B_2 + B_1.$$

Explain why this tells you that the order that you press the buttons in the lights out game doesn't matter.

We can reinterpret (2) as the following matrix equation:

$$T\mathbf{x} = C \tag{3}$$

where T is the matrix whose columns are the toggle vectors t_i and $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$.

Now finding a solution for the puzzle with initial state C is equivalent to finding a solution $\mathbf{x}$ to the matrix equation (3).

Question 10.

a) Write down the matrix T and compute its determinant. What does this tell you about the rank of the matrix T?

b) Given an initial configuration C, does there always exist a solution?

c) Given a solution to the matrix equation (3), is this solution unique?

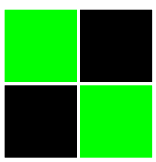
d) Given a solution to the puzzle with initial configuration C, is the solution to the puzzle given by the vector $\mathbf{x}$ unique? Is it minimal?

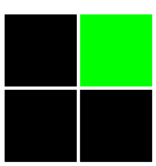
The following questions are optional

e) Compute the inverse of the matrix T in $\mathbb{F}_2$. Note that this means that the entries of T^{-1} should all be elements of $\mathbb{F}_2$ and that all row operations should take place in $\mathbb{F}_2$ (the addition and multiplication tables in the warm-up could be useful here).

f) Rewrite the matrix equation (3) using the inverse matrix T^{-1}.

g) Using this new equation, find what buttons you need to push to solve the lights out puzzle with the following initial states (and check them by hand):

i 

ii 

The following questions are optional, and simply show how to extend our previous work to the original game. They do not need to be completed or handed in.

The original lights out game

Let's return to the original lights out game on the 5×5 grid. Something you discovered for the 2×2 case is that the toggle matrix T was invertible, so that all initial configurations are solvable, with a unique solution (up to additionally pressing all the buttons an even number of times). You might be hopeful that this is true for the original game (or even for all grid sizes). Let's check if this is indeed true!

In this case, we use a similar way of numbering the entries to make our vectors: $\begin{bmatrix} 1 & 2 & 3 & \dots & 25 \\ 26 & 27 & 28 & \dots & 50 \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & 25^2 \end{bmatrix}$

The toggle matrix in the 5×5 case is the 25×25 matrix

$$T = \begin{bmatrix} M & I_5 & 0 & 0 & 0 \\ I_5 & M & I_5 & 0 & 0 \\ 0 & I_5 & M & I_5 & 0 \\ 0 & 0 & I_5 & M & I_5 \\ 0 & 0 & 0 & I_5 & M \end{bmatrix}$$

where each entry in T above is a 5×5 matrix, I_5 is the 5×5 identity matrix, 0 denotes the 5×5 matrix whose entries are all zero, and M is the matrix

$$M = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}.$$

Question 11.

a) Consider the vector $\mathbf{x} = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}^T$. What do you notice about $T\mathbf{x}$?

b) What does this tell you about the null space of T?

c) In fact, the null space of T is two-dimensional. What does this tell you about the rank of T?

d) Do all initial configurations in the original 5×5 lights out game have a solution?

e) Does a solution to the matrix equation (3) in this case necessarily correspond to a minimal solution to a puzzle?