Problem 1. Consider the following vectors in $\mathbb{R}^4$:

$$\mathbf{a_1} = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 2 \end{bmatrix}, \quad \mathbf{a_2} = \begin{bmatrix} 2 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{a_3} = \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}.$$

(a) Without calculating, explain why the vectors $\mathbf{a_1}, \mathbf{a_2}$ and $\mathbf{a_3}$ cannot span $\mathbb{R}^4$.

(b) Is the set $\{\mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3}\}$ linearly independent?

(c) Determine if the vectors $\mathbf{b_1} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ -1 \end{bmatrix}$ and $\mathbf{b_2} = \begin{bmatrix} -3 \\ -3 \\ -3 \\ 3 \end{bmatrix}$ are in the span of the vectors $\mathbf{a_1}$, $\mathbf{a_2}$ and $\mathbf{a_3}$. If yes, write the vector as a linear combination of the $\mathbf{a_i}$.

Problem 2. Suppose A is a 3×3 matrix and let $\mathbf{y}$ be a vector in $\mathbb{R}^3$ such that the equation $A\mathbf{x} = \mathbf{y}$ does **not** have a solution. Does there exist a vector $\mathbf{z}$ in $\mathbb{R}^3$ such that the equation $A\mathbf{x} = \mathbf{z}$ has a unique solution? Justify your answer.

Problem 3. True or False?

(1) Let A be a non-zero 2×3 matrix. Then the columns of A span $\mathbb{R}^2$.

(2) The equation $A\mathbf{x} = \mathbf{b}$ is consistent if the augmented matrix corresponding to this equation has a pivot in every row.

(3) If two vectors in $\mathbb{R}^2$ are linearly independent then they span $\mathbb{R}^2$.

(4) The equation $A\mathbf{x} = \mathbf{b}$ is homogeneous if the zero vector is a solution.

Problem 4. Bonus! When you think about the connections between spans, solution sets to systems of linear equations, solutions to matrix equations and linear dependence, what comes to mind? How do these concepts fit together for you? Can you make a dictionary between these different concepts?