Problem 1. For $n \geq 2$, we define the **Vandermonde matrix** of size $n \times n$ to be the matrix

$$V_n(x_1, x_2, \ldots, x_n) = \begin{bmatrix} 1 & x_1 & x_1^2 & \ldots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \ldots & x_2^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & x_n^2 & \ldots & x_n^{n-1} \end{bmatrix}$$

where $x_1, x_2, \ldots, x_n$ are real numbers. Our aim in this problem is to find out when the Vandermonde matrix is invertible, for small values of n.

(1) Write down the Vandermonde matrix for $n = 2$.

(2) What is its determinant?

(3) For what values of x_1 and x_2 is $V_2(x_1, x_2)$ invertible? In this case, what is its inverse?

(4) Write down the Vandermonde matrix for $n = 3$.

(5) Compute the determinant of the Vandermonde matrix for $n = 3$.

(6) For what values of x_1, x_2 and x_3 is the Vandermonde matrix V_3 invertible?

(7) **Conclusion:** For $n = 2, 3$, the Vandermonde matrix is invertible if and only if

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(8) **Bonus!** From our understanding of the $n = 2$ and $n = 3$ cases, when do you think the Vandermonde matrix is invertible for any size n?

(9) **Extra bonus!** Using column operations as well as row operations, express $\det V_3(x_1, x_2, x_3)$ as a product $V_2(x_2, x_3) \cdot p(x_1, x_2, x_3)$ where $p(x_1, x_2, x_3)$ is some polynomial in x_1, x_2, x_3. If you are familiar with induction, use this to prove our conclusion for all n!

Problem 2. Serge and Khadija are thinking about two linear transformations, $\mathcal{T} : \mathbb{R}^2 \to \mathbb{R}^3$ given by

$$\mathcal{T} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix}$$

and $\mathcal{S} : \mathbb{R}^3 \to \mathbb{R}^2$ given by

$$\mathcal{S} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}.$$

Serge argues as follows:

(1) The image of the unit square under the transformation $\mathcal{T}$ is a square, which has zero volume in $\mathbb{R}^3$.
(2) Therefore $\det(\mathcal{T}) = 0$.
(3) The image of the unit cube under the transformation $\mathcal{S}$ is the unit square, which has volume 1 in $\mathbb{R}^2$
(4) Therefore $\det(\mathcal{S}) = 1$.
(5) The determinant $\mathcal{S} \circ \mathcal{T}$ is the product $\det \mathcal{S} \det \mathcal{T}$, and 0 times any number is 0, so $\det(\mathcal{S} \circ \mathcal{T}) = 1 \cdot 0 = 0$.

Khadija says that

$$(\mathcal{S} \circ \mathcal{T}) \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix},$$

so $\mathcal{S} \circ \mathcal{T}$ is the identity map, so it must have determinant 1. Serge agrees, but they are not sure where Serge's argument is wrong.

Which of the following statements are true? Justify your answer.

(i) $\det(\mathcal{S} \circ \mathcal{T}) = 1$.
(ii) $\det(\mathcal{S} \circ \mathcal{T}) = 0$.
(iii) Serge is correct in parts 1 and 3.
(iv) Serge is correct in parts 2 and 4.
(v) Serge is correct in saying that if $\mathcal{A}$ and $\mathcal{B}$ are linear transformations such that $\mathcal{A} \circ \mathcal{B}$ is defined, we have

$$\det(\mathcal{A} \circ \mathcal{B}) = \det(\mathcal{A}) \det(\mathcal{B}).$$

Problem 3. Let S be the set of all $n \times n$ matrices with determinant 1. Show the following:

(1) The product of two matrices in S is also in S.

(2) There exists a matrix E in S such that for all matrices A in S, $A \cdot E = E \cdot A = A$ (where $\cdot$ denotes matrix multiplication).

(3) For every matrix A in S, there exists a matrix A^{-1} in S satisfying $A \cdot A^{-1} = A^{-1} \cdot A = E$.

(4) For all matrices A, B and C in S, $(A \cdot B) \cdot C = A \cdot (B \cdot C)$.

You have just shown that S, under matrix multiplication, is a **group**, *known as the* **special linear group**. *You can learn more about groups by taking a course in abstract algebra!*