

Generating stationary Gaussian random fields

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1 Fourier Transforms

Continuous Suppose we have a function $f : \mathbb{R} \rightarrow \mathbb{R}$ which is periodic, of period L_x . The Fourier transform of $f(x)$ is defined to be

$$\hat{f}(k_n) = \int_{x=0}^{L_x} f(x)e^{-ik_n x} dx, \quad k_n = n \frac{2\pi}{L_x} = n\Delta k, \quad n \in \mathbb{Z}.$$

with inverse transform given by

$$f(x) = \frac{\Delta k}{2\pi} \sum_{k_n=-\infty}^{\infty} \hat{f}(k_n)e^{ik_n x}.$$

In this form, Parseval's theorem on the conservation of energy becomes

$$\int_0^{L_x} f^2(x) dx = \frac{\Delta k}{2\pi} \sum_{k_n=-\infty}^{\infty} |\hat{f}(k_n)|^2.$$

Discrete If we only know $f(x)$ on a finite number of equally-spaced points, (such as is the case, for example, in a numerical representation of f), then a discrete transform can be defined so that it becomes the continuous transform in the limit as the number of points goes to infinity. Suppose we know the value of f at points $\{x_j\}$, with $x_j = j\Delta x = j \frac{L_x}{N_x}$, $j = 0, 1, \dots, (N_x - 1)$. The parameters are defined so that $\Delta x = \frac{L_x}{N_x}$ is the grid spacing and N_x is the number of points. Then $e^{ix_j(k_n + N_x \Delta k)} = e^{ix_j k_n}$, (where $\Delta k = \frac{2\pi}{L_x}$ as in the continuous case), so only the first N_x values of k_n give useful information. Let the discrete Fourier transform and its inverse be

$$\begin{aligned} \hat{f}(k_n) &= \Delta x \sum_{j=0}^{N_x-1} f(x_j)e^{-ix_j k_n} \\ f(x_j) &= \frac{\Delta k}{2\pi} \sum_{n=0}^{N_x-1} \hat{f}(k_n)e^{ix_j k_n} \end{aligned}$$

Parseval's theorem becomes

$$\frac{\Delta k}{2\pi} \sum_{n=0}^{N_x-1} |\hat{f}(k_n)|^2 = \Delta x \sum_{j=0}^{N_x-1} f^2(x_j).$$

Implementation in Matlab Matlab defines its transforms to be

$$\begin{aligned} \hat{f}(k_n) &= \sum_{j=0}^{N_x-1} f(x_j) e^{-ix_j k_n} \\ f(x_j) &= \frac{1}{N_x} \sum_{n=0}^{N_x-1} \hat{f}(k_n) e^{ix_j k_n} \end{aligned}$$

So the above transform can be implemented in Matlab as

$$\begin{aligned} FT(f) &= \Delta x \times \text{MatlabFT}(f) \\ FT^{-1}(\hat{f}) &= \frac{1}{\Delta x} \times \text{MatlabFT}^{-1}(\hat{f}) \end{aligned}$$

2 Dimensions The above transforms can be easily extended to 2 (or more) dimensions. In 2 dimensions, the transforms are

$$\begin{aligned} \hat{f}(k, m) &= \Delta x \Delta z \sum_{x,z} f(x, z) e^{-i(kx+mz)} \\ f(x, z) &= \frac{\Delta k \Delta m}{(2\pi)^2} \sum_{k,m} \hat{f}(k, m) e^{i(kx+mz)} \end{aligned}$$

with energies

$$\frac{\Delta k \Delta m}{(2\pi)^2} \sum_{k,m} |\hat{f}(k, m)|^2 = \Delta x \Delta z \sum_{x,z} f^2(x, z).$$

2 Generating Stationary Gaussian Random Fields

2.1 Scalar-Valued Fields

Complex-valued fields Suppose we would like to generate a complex-valued stationary scalar Gaussian random field $\phi(x) : [0, L_x] \rightarrow \mathbb{C}$ with covariance function $C(x)$, ie $E[\phi(x_0)\bar{\phi}(x_0+x)] = C(x)$, and expected mean value 0 (a field with non-zero mean can be constructed by adding the mean on afterwards.) This can be done most easily in Fourier space, using $\hat{C}(k) = FT(C)$. Suppose we are working with a finite set of points, $\{x_j\}$ and $\{k_n\}$, defined as before. (The x -continuous version follows by taking the limit as $N_x \rightarrow \infty$.) Let

$$\hat{\phi}(k) = \sqrt{\frac{L_x \hat{C}(k)}{2}} (A_k + iB_k),$$

where A_k, B_k are independent Gaussian random variables with mean 0, variance 1. Then

$$\phi(x) = FT^{-1}(\hat{\phi})$$

is a Gaussian random field satisfying the requirements (Yaglom 1962).

Parseval's identity holds for each realization. Taking expected values gives

$$\frac{\Delta k}{2\pi} \sum_{k_n} E[|\hat{\phi}(k_n)|^2] = \Delta x \sum_{x_j} E[\phi^2(x_j)] = L_x C(0), \quad (1)$$

which shows that $C(0)$ can be interpreted as the expected variance per unit length.

Real-valued fields A real-valued field can be generated from a complex one by taking real or imaginary parts. By definition, if $\phi = \phi_1 + i\phi_2$ is a stationary complex Gaussian random field with covariance function $C(x)$, then ϕ_1, ϕ_2 are independent, real-valued Gaussian random fields with identical covariance functions $C(x)/2$ (eg Hida& Hitsuda, 1990). This leads to nice way to generate samples of a real field on a computer, as one simply has to generate a complex-valued field with covariance function $2C(x)$ and take its real and imaginary parts, giving two realizations for the price of one.

Note that if ϕ is real-valued, then $\hat{\phi}$ will *not* be a Gaussian field, as its Fourier transform satisfies $\hat{\phi}(k) = \tilde{\hat{\phi}}(-k)$. This leads to an asymmetry in the equations if we try to generate $\hat{\phi}$ on its own, as the 0-mode, negative modes, and $N_x/2$ -mode (if N_x is even) must be given special treatment:

$$\begin{aligned} \hat{\phi}(k) &= \sqrt{L_x \hat{C}(k)} A_k & k = 0, N_x/2 \\ \hat{\phi}(k) &= \sqrt{\frac{L_x \hat{C}(k)}{2}} (A_k + iB_k) & 0 < k < N_x/2 \\ \hat{\phi}(k) &= \tilde{\hat{\phi}}(-k) & k < 0 \end{aligned}$$

A_k, B_k are independent $N(0, 1)$ random variables as before.