Math 100 Midterm 2 Solutions

(1)(a) Using the chain rule, $\boxed{-\sin(e^{4x^2})e^{4x^2}8x}$

(b) Put $y = x^{e^x}$, so $\ln y = e^x \ln x$; differentiating with respect to x gives $\dfrac{1}{y}\dfrac{dy}{dx} = e^x \ln x + e^x/x$, so $\dfrac{dy}{dx} = ye^x(\ln x + 1/x) = \boxed{x^{e^x}e^x(\ln x + 1/x)}$

(c) Put $y = \dfrac{2^x}{1-2^x}$ and solve for x: $y - y2^x = 2^x$ so $\dfrac{y}{1+y} = 2^x$ so $f^{-1}(y) = x = \log_2 \dfrac{y}{1+y}$. So $f^{-1}(x) = \boxed{\log_2 \dfrac{x}{1+x}}$

(d) $\dfrac{dP}{dt} = kP$ so $P(t) = P(0)e^{kt}$. At $t = 2$ we get $600 = 400e^{k2}$ so $k = \frac{1}{2}\ln\frac{3}{2}$. Then $\left.\dfrac{dP}{dt}\right|_{t=0} = kP(0) = \boxed{200 \ln \dfrac{3}{2} \text{ individuals per hour}}$

(e) Put $f(x) = x^{1/3}$, so $f'(x) = \frac{1}{3}x^{-2/3}$ and the linear approximation about $a = 1000$ is $f(x) \approx f(1000) + f'(1000)(x - 1000)$, so $994^{1/3} \approx 1000^{1/3} + \frac{1}{3}1000^{-2/3}(-6) = 10 - \frac{2}{100} = \boxed{9.98}$

(f) $\sin(x^3) = x^3 - \dfrac{x^9}{3!} + \dfrac{x^{15}}{5!} - \cdots$, but the term in x^{15} is $\dfrac{f^{(15)}(0)}{15!}x^{15}$, so $f^{(15)}(0) = \boxed{\dfrac{15!}{5!}}$

(2) (a) Differentiating implicitly, $2x + 4y\dfrac{dy}{dx} - y - x\dfrac{dy}{dx} = 0$, so $\dfrac{dy}{dx} = \boxed{\dfrac{y-2x}{4y-x}}$

(b) $\frac{dy}{dx} = 0$ gives $y - 2x = 0$ so $y = 2x$. Substituting into the original equation gives $x^2 + 8x^2 - 2x^2 = 7$ so $x = \pm 1$, so $y = \pm 2$. Points are $\boxed{(1, 2), (-1, -2)}$

(3) Let r be the distance from the person to the car, and let x be the distance from the car to the point on the road closest to the person. Thus $r^2 = x^2 + 30^2$. Differentiating with respect to time t gives $2r\dfrac{dr}{dt} = 2x\dfrac{dx}{dt}$. We have $r = 50$ so $x = \sqrt{50^2 - 30^2} = 40$ and $\dfrac{dr}{dt} = -12$, so the speed is $\left|\dfrac{dx}{dt}\right| = \dfrac{50 \times 12}{40} = \boxed{15}$

(4) (a) By computing the first few derivatives, we see $f^{(n)}(x) = (-1)^{n-1}(n-1)!(1+x)^{-n}$. Hence $|R_n(x)| = \dfrac{|f^{n+1}(c)|}{(n+1)!}x^{n+1} = \dfrac{x^{n+1}}{(n+1)(1+c)^{n+1}}$ for some c between 0 and 1. This is largest when $c = 0$, so we get the given bound.

(b) (i) We need $\frac{1}{n+1} \leq 0.02$ so $n \geq \boxed{49}$ (ii) We need $\frac{(1/2)^{n+1}}{n+1} \leq 0.02$, so (after trying the first few values), $n \geq \boxed{3}$

(c) $f(\frac{1}{2}) = \ln(3/2) = \ln 3 - \ln 2$, and $f(\frac{1}{3}) = \ln(4/3) = 2\ln 2 - \ln 3$, so $\boxed{f(\tfrac{1}{2}) + f(\tfrac{1}{3})} = \ln 2$. If we are computing f using the Maclaurin series then this way is much more efficient because (b) shows that we need far fewer terms to get a given accuracy (this applies similarly to $f(\frac{1}{3})$ as well as $f(\frac{1}{2})$).