Chip-Firing and Rotor-Routing on Directed Graphs

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Abstract. We give a rigorous and self-contained survey of the abelian sandpile model and rotor-router model on finite directed graphs, highlighting the connections between them. We present several intriguing open problems.

1. Introduction

The abelian sandpile and rotor-router models were discovered several times by researchers in different communities operating independently. The abelian sandpile model was invented by Dhar [Dha90] as a test-bed for the concept of self-organized criticality introduced in [BTW87]. Related ideas were explored earlier by Engel [Eng75, Eng76] in the form of a pedagogical tool (the "probabilistic abacus") and by Spencer [Spe87, pp. 32–35]. The rotor-router model was first introduced by Priezzhev *et al.* [PDDK96] (under the name "Eulerian walkers model") in connection with self-organized criticality. It was rediscovered several times: by Rabani, Sinclair and Wanka [RSW98] as an approach to load-balancing in multiprocessor systems, by Propp [Pro01] as a way to derandomize models such as internal diffusion-limited aggregation (IDLA) [DF91, LBG92], and by Dumitriu, Tetali, and Winkler as part of their analysis of a graph-based game [DTW03]. Articles on the chip-firing game in the mathematical literature include [Big99, Big97, BLS91, BL92]. Those on the rotor-router model include [Lev02, LP05, HP08, LP07a, LP07b]. Below we briefly describe the two models, deferring the formal definitions to later sections.

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The **abelian sandpile model** on a directed graph G, also called the **chip-firing game**, starts with a collection of chips at each vertex of G. If a vertex v has at least as many chips as outgoing edges, it can **fire**, sending one chip along each outgoing edge to a neighboring vertex. After firing a sequence of vertices in turn, the process stops when each vertex with positive out-degree has fewer chips than out-going edges. The order of firings does not affect the final configuration, a fact we shall discuss in more detail in Section 2.

To define the **rotor-router model** on a directed graph G, for each vertex of G, fix a cyclic ordering of the outgoing edges. To each vertex v we associate a **rotor** $\rho(v)$ chosen from among the outgoing edges from v. A chip performs a walk on G according to the **rotor-router rule**: if the chip is at v, we first increment the rotor $\rho(v)$ to its successor $e = (v, w)$ in the cyclic ordering of outgoing edges from v, and then route the chip along e to w. If the chip ever reaches a **sink**, i.e. a vertex of G with no outgoing edges, the chip will stop there; otherwise, the chip continues walking forever.

We develop the basic theory of the abelian sandpile model in section 2 and define the main algebraic object associated with it, the sandpile group of G [Dha90] (also called the "critical group" by some authors, e.g. [Big99, Wag00]). Furthermore, we establish the basic results about **recurrent** chip configurations, which play an important role in the theory. In Section 3 we define a notion of recurrent configurations for the rotor-router model on directed graphs and give a characterization for them in terms of oriented spanning trees of G. The sandpile group acts naturally on recurrent rotor configurations, and this action is both transitive and free. We deduce appealing proofs of two basic results of algebraic graph theory, namely the Matrix-Tree Theorem [Sta99, 5.6.8] and the enumeration of Eulerian tours in terms of oriented spanning trees [Sta99, Cor. 5.6.7]. We also derive a family of bijections between the recurrent chip configurations of G and the recurrent rotor configurations of G. Such bijections have been constructed before, for example in [BW97]; however, our presentation differs significantly from the previous ones. Section 4 establishes stronger results for both models on Eulerian digraphs and undirected graphs. In Section 5 we present an alternative view of the rotor-router model in terms of "cycle-popping," borrowing an idea from Wilson's work on loop-erased random walk; see [PW98]. We conclude in Section 6 by presenting some open questions.

2. Chip-Firing

In a finite directed graph (**digraph**) $G = (V, E)$, a directed edge $e \in E$ points from the vertex $\text{tail}(e)$ to the vertex $\text{head}(e)$. We allow self-loops ($\text{head}(e) = \text{tail}(e)$) as well as multiple edges ($\text{head}(e) = \text{head}(e')$ and $\text{tail}(e) = \text{tail}(e')$) in G. The **out-degree** $\text{outdeg}(v)$ of a vertex v (also denoted by d_v) is the number of edges e with $\text{tail}(e) = v$, and the **in-degree** $\text{indeg}(v)$ of v is the number of edges e with $\text{head}(e) = v$. A vertex is a **sink** if its out-degree is zero. A **global sink** is a sink s

such that from every other vertex there is a directed path leading to s. Note that if there is a global sink, then it is the unique sink.

If G has the same number of edges from v to w as from w to v for all vertices $v \neq w$ then we call G **bidirected**. In particular, a bidirected graph is obtained by replacing each edge of an undirected graph with a pair of directed edges, one in each direction.

Label the vertices of G as $v_1, v_2, \ldots, v_n$. The **adjacency matrix** A of G is the $n \times n$ matrix whose (i, j)-entry is the number of edges from v_i to v_j, which we denote by d_{v_i,v_j} or d_{ij}. The (graph) **Laplacian** of G is the $n \times n$ matrix $\Delta = D - A$, where D is the diagonal matrix whose (i, i)-entry is the out-degree of v_i. That is,

$$\Delta_{i,j} = \begin{cases} -d_{ij} & \text{for } i \neq j, \\ d_i - d_{ii} & \text{for } i = j. \end{cases}$$

Note that the entries in each row of Δ sum to zero. If the vertex v_i is a sink then the i^{th} row of Δ is zero.

A **chip configuration** σ on G, also called a **sandpile** on G, is a vector of non-negative integers indexed by the non-sink vertices of G, where $\sigma(v)$ represents the number of chips at vertex v. A chip configuration σ is **stable** if $\sigma(v) < d_v$ for every non-sink vertex v. We call a vertex v **active** in σ if v is not a sink and $\sigma(v) \geq d_v$. An active vertex v can **fire**, resulting in a new chip configuration σ' obtained by moving one chip along each of the d_v edges emanating from v; that is, $\sigma'(w) = \sigma(w) + d_{vw}$ for all $w \neq v$ and $\sigma'(v) = \sigma(v) - d_v + d_{vv}$. We call the configuration σ' a **successor** of σ.

By performing a sequence of firings, we may eventually arrive at a stable chip configuration, or we might continue firing forever, as the following examples show.

Example 2.1. Consider the complete directed graph on three vertices (without self-loops). Then placing three chips at a vertex gives a configuration that stabilizes in one move, while placing four chips at a vertex gives a configuration that never stabilizes (see Figure 1).

It might appear that the choice of the order in which we fire vertices could affect the long-term behavior of the system; however, this is not the case, as the following lemma shows (and Figure 2 illustrates).

Lemma 2.2 ([Dha90],[DF91])**.** *Let G be any digraph, let $\sigma_0, \sigma_1, \ldots, \sigma_n$ be a sequence of chip configurations on G, each of which is a successor of the one before, and let $\sigma'_0, \sigma'_1, \ldots, \sigma'_m$ be another such sequence with $\sigma'_0 = \sigma_0$.*

1. *If σ_n is stable, then $m \leq n$, and moreover, no vertex fires more times in $\sigma'_0, \ldots, \sigma'_m$ than in $\sigma_0, \ldots, \sigma_n$.*
2. *If σ_n and σ_m are both stable, then $m = n$, $\sigma_n = \sigma'_n$, and each vertex fires the same number of times in both histories.*

Proof. Part 2 is an immediate corollary of part 1, which we now prove. If part 1 fails, then consider a counterexample with $m+n$ minimal. Let v_i be the vertex that

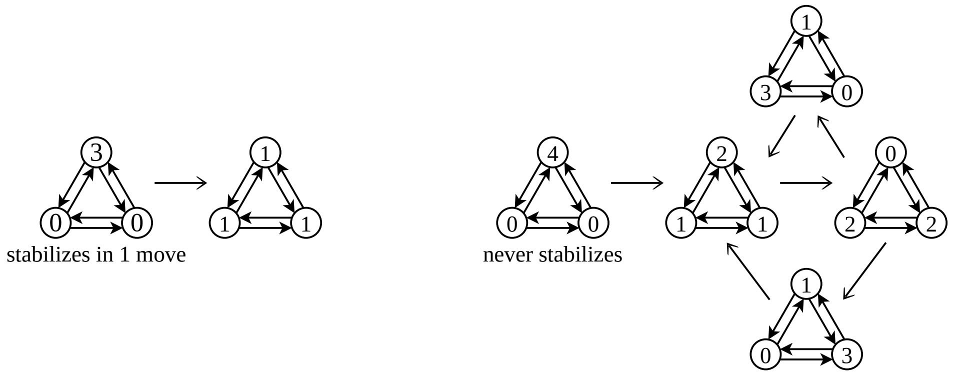


FIGURE 1. Some chip configurations eventually stabilize, while others never stabilize.

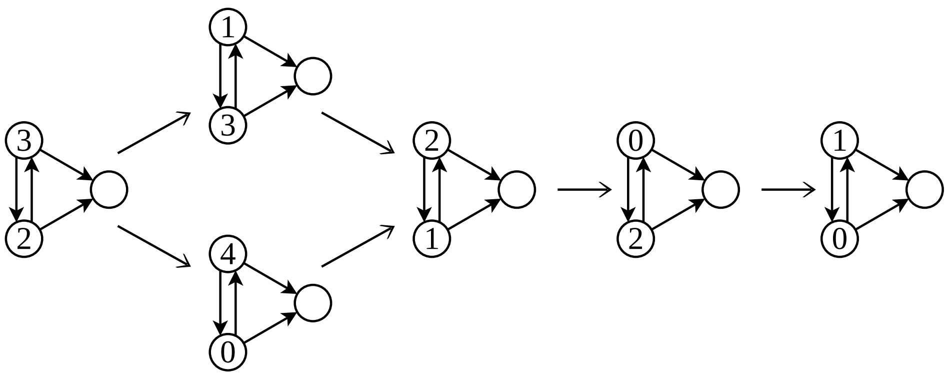


FIGURE 2. Commutation of the chip-firing operations.

fires when σ_{i-1} becomes σ_i, and v_i' be the vertex that fires when σ_{i-1}' becomes σ_i'. Vertex v_1' must be fired at some stage (say the i^{th}) in the sequence of configurations $\sigma_0, \ldots, \sigma_n$, since σ_n is stable; say $v_i = v_1'$. Then $v_i, v_1, v_2, \ldots, v_{i-1}, v_{i+1}, \ldots, v_n$ is a permissible firing sequence that turns σ_0 into σ_n with the same number of firings at each site as the unpermuted sequence. The firing sequences $v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_n$ and $v_2', v_3', \ldots, v_m'$ then constitute a smaller counterexample to the lemma (with initial configuration σ_1'), contradicting minimality. $\square$

Definition 2.3. Starting from a configuration σ, Lemma 2.2 shows that there is at most one stable configuration that can be reached by a finite sequence of firings (and that if such a configuration exists, then no infinite sequence of firings is possible). If such a stable configuration exists we denote it σ° and call it the **stabilization** of σ.

Thus far, the presence or absence of sinks was irrelevant for our claims. For the rest of this section, we assume that the digraph G has a global sink s.

Lemma 2.4. *If digraph G has a global sink, then every chip configuration on G stabilizes.*

Proof. Let N be the number of chips in the configuration. Given a vertex v of G, let $v_0, v_1, \ldots, v_{r-1}, v_r$ be a directed path from $v_0 = v$ to $v_r = s$. Every time v_{r-1} fires, it sends a chip to the sink which remains there forever. Thus v_{r-1} can fire at most N times. Every time v_{r-2} fires, it sends a chip to v_{r-1}, and $d_{v_{r-1}}$ such chips will cause v_{r-1} to fire once, so v_{r-2} fires at most $d_{v_{r-1}}N$ times. Iterating backward along the path, we see that v fires at most $d_{v_1} \cdots d_{v_{r-1}}N$ times. Thus each vertex can fire only finitely many times, so by Lemma 2.2 the configuration stabilizes. □

We remark that when G is connected and the sink is the only vertex with in-degree exceeding its out-degree, the bound one gets from the above argument on the total number of firings is far from optimal; see Theorem 4.8 for a better bound.

Define the **chip addition operator** E_v as the map on chip configurations that adds a single chip at vertex v and then lets the system stabilize. In symbols,

$$E_v\sigma = (\sigma + \mathbf{1}_v)^\circ$$

where $\mathbf{1}_v$ is the configuration consisting of a single chip at v.

Lemma 2.5. *On any digraph with a global sink, the chip addition operators commute.*

Proof. Given a chip configuration σ and two vertices v and w, whatever vertices are active in $\sigma + \mathbf{1}_v$ are also active in configuration $\sigma' = \sigma + \mathbf{1}_v + \mathbf{1}_w$. Applying to σ' a sequence of firings that stabilizes $\sigma + \mathbf{1}_v$, we obtain the configuration $E_v\sigma + \mathbf{1}_w$. Stabilizing this latter configuration yields $E_wE_v\sigma$. Thus $E_wE_v\sigma$ is a stabilization of σ'. Interchanging the roles of v and w, the configuration $E_vE_w\sigma$ is also a stabilization of σ'. From Lemma 2.2 we conclude that $E_wE_v\sigma = E_vE_w\sigma$. □

Lemma 2.5 is called the **abelian property**; it justifies Dhar's coinage "abelian sandpile model". From the above proof we also deduce the following.

Corollary 2.6. *Applying a sequence of chip addition operators to σ yields the same result as adding all the associated chips simultaneously and then stabilizing.*

Let G be a digraph on n vertices with global sink s. The **reduced Laplacian** Δ' of G is obtained by deleting from the Laplacian matrix Δ the row and column corresponding to the sink. Note that firing a non-sink vertex v transforms a chip configuration σ into the configuration $\sigma - \Delta'_v$, where Δ'_v is the row of the reduced Laplacian corresponding to v. Since we want to view the configurations before and after firing as equivalent, we are led to consider the group quotient $\mathbb{Z}^{n-1}/H$, where $H = \mathbb{Z}^{n-1}\Delta'$ is the integer row-span of Δ'.

Definition 2.7. Let G be a digraph on n vertices with global sink s. The **sandpile group** of G is the group quotient

$$\mathcal{S}(G) = \mathbb{Z}^{n-1}/\mathbb{Z}^{n-1}\Delta'(G).$$

The connection between the sandpile group and the dynamics of sandpiles on G is made explicit in Corollary 2.16. For the graph in Figure 3, the sandpile group is the cyclic group of order 3. The group structure of $\mathcal{S}(G)$ when G is a tree is investigated in [Lev07].

Lemma 2.8. *The order of $\mathcal{S}(G)$ is the determinant of the reduced Laplacian $\Delta'(G)$.*

Proof. The order of $\mathcal{S}(G)$ equals the index of the lattice $H = \mathbb{Z}^{n-1}\Delta'$ in $\mathbb{Z}^{n-1}$, and, recalling that the volume of a parallelepiped is the determinant of the matrix formed from its edge-vectors, we deduce that this in turn equals the determinant of Δ'. □

Lemma 2.9. *Let G be a digraph with a global sink. Every equivalence class of $\mathbb{Z}^{n-1}$ modulo $\Delta'(G)$ contains at least one stable chip configuration of G.*

Proof. Let δ be the configuration given by $\delta(v) = d_v$ for all v, and let δ° be its stabilization. Then $\delta^\circ(v) < d_v$ for all $v \neq s$, so $\delta - \delta^\circ$ is a positive vector equivalent to the zero configuration. Given any $\alpha \in \mathbb{Z}^{n-1}$, let m denote the minimum of all the coordinates of α together with 0 (so that $m \leq 0$). Then the vector

$$\beta = \alpha + (-m)(\delta - \delta^\circ)$$

is nonnegative and equivalent to α. Hence β° is a stable chip configuration in the same equivalence class as α. □

Example 2.10. An equivalence class may contain more than one stable chip configuration. For example, consider the complete directed graph on three vertices, with one of the vertices made into a sink by deletion of its two outgoing edges (see Figure 3). It is easy to see that there are two stable configurations in the equivalence class of the identity: the configuration in which each of the two non-sink vertices has 0 chips and the configuration in which each of the two vertices has 1 chip. It might seem natural that, if either of these two configurations is to be preferred as a representative of the identity element in the sandpile group, it should be the former. However, this instinct is misleading, as we now explain.

FIGURE 3. Two stable chip configurations in the equivalence class of the identity.

Definition 2.11. A chip configuration σ is **accessible** if from any other chip configuration it is possible to obtain σ by a combination of adding chips and selectively firing active vertices. A chip configuration that is both stable and accessible is called **recurrent**.

Remark 2.12. There are several definitions of “recurrent” that are used in the literature. Lemma 2.17 below shows that these definitions (including the one above) are equivalent for any digraph with a global sink.

We will see shortly (in Lemmas 2.13 and 2.15) that each equivalence class in $\mathbb{Z}^{n-1}/\mathbb{Z}^{n-1}\Delta'$ contains a unique recurrent chip configuration. It is customary to represent each element of the sandpile group by its unique recurrent element. In Example 2.10 the all-1 configuration is accessible, but the all-0 configuration is not. Therefore the all-1 configuration is taken as the canonical representative of the identity. (However, in the context of cluster-firing and superstabilization as described in Definition 4.3 and Lemma 4.6, the all-0 configuration will be the preferred representative.)

Lemma 2.13. *Let G be a digraph with a global sink. Every equivalence class of $\mathbb{Z}^{n-1}$ modulo $\Delta'(G)$ contains at least one recurrent chip configuration of G.*

Proof. Given $\alpha \in \mathbb{Z}^{n-1}$, let m denote the minimum of all the coordinates of α together with 0, so that $m \leq 0$. Write $d_{\max}$ for the maximum out-degree of a vertex in G. Then α is equivalent to the configuration

$$\beta = \alpha + [d_{\max} + (-m)](\delta - \delta^\circ)$$

with δ as in the proof of Lemma 2.9. Since $\delta - \delta^\circ$ has all entries positive, we have $\beta \geq [d_{\max} + (-m)](\delta - \delta^\circ) \geq d_{\max}(\delta - \delta^\circ) \geq d_{\max}$. (Inequalities between two vectors or between a vector and a scalar are interpreted componentwise.) In particular, β is accessible, since any chip configuration can first be stabilized, so that each vertex has fewer than $d_{\max}$ chips, and then supplemented with extra chips to obtain β. Therefore any configuration obtained from β by firing is also accessible. In particular the stabilization β° is thus recurrent and equivalent to α. □

Next we will show that every equivalence class modulo Δ' contains at most one recurrent configuration, making use of the following lemma.

Lemma 2.14. *Let $\epsilon = (2\delta) - (2\delta)^\circ$, where δ is given by $\delta(v) = d_v$ as before. If σ is recurrent, then $(\sigma + \epsilon)^\circ = \sigma$.*

Proof. If σ is recurrent then it is accessible, so it can be reached from δ by adding some (non-negative) configuration ζ and selectively firing. But since σ is also stable this implies that $(\zeta + \delta)^\circ = \sigma$. Consider the configuration

$$\gamma = (\zeta + \delta) + \epsilon = 2\delta + \zeta + \delta - (2\delta)^\circ.$$

Since $\epsilon \geq 0$, we may start from γ and fire a sequence of vertices that stabilizes $\zeta + \delta$, to obtain the configuration $\sigma + \epsilon$. On the other hand, since $\delta - (2\delta)^\circ \geq 0$

we may start from γ and fire a sequence of vertices that stabilizes 2δ, to obtain the configuration $(2\delta)^\circ + \zeta + \delta - (2\delta)^\circ = \zeta + \delta$, which in turn stabilizes to σ. By Lemma 2.2 it follows that $(\sigma + \epsilon)^\circ = \sigma$. □

Lemma 2.15. *Let G be a digraph with a global sink. Every equivalence class of $\mathbb{Z}^{n-1}$ modulo $\Delta'(G)$ contains at most one recurrent chip configuration of G.*

Proof. Let σ_1 and σ_2 be recurrent and equivalent mod Δ'. Label the non-sink vertices $v_1, \ldots, v_{n-1}$. Then $\sigma_1 = \sigma_2 + \sum_{i \in J} c_i \Delta'_i$, where the c_i are nonzero constants, Δ'_i is the row of the reduced Laplacian Δ' corresponding to v_i, and the index i runs over some subset J of the integers $1, \ldots, n-1$. Write $J = J_- \cup J_+$, where $J_- = \{i : c_i < 0\}$ and $J_+ = \{i : c_i > 0\}$, and let

$$\sigma = \sigma_1 + \sum_{i \in J_-} (-c_i)\Delta'_i = \sigma_2 + \sum_{i \in J_+} c_i \Delta'_i.$$

Let ϵ denote the everywhere-positive chip configuration defined in Lemma 2.14. Take k large enough so that $\sigma' = \sigma + k\epsilon$ satisfies $\sigma'(v_i) \geq |c_i| d_{v_i}$ for all i. Starting from σ', we may fire each vertex v_i for $i \in J_-$ a total of $-c_i$ times, and each of the intermediate configurations is a valid chip configuration because all the entries are nonnegative. The resulting configuration $\sigma_1 + k\epsilon$ then stabilizes to σ_1 by Lemma 2.14. Likewise, starting from σ' we may fire each vertex v_i for $i \in J_+$ a total of c_i times to obtain $\sigma_2 + k\epsilon$, which stabilizes to σ_2. By Lemma 2.2 it follows that $\sigma_1 = \sigma_2$. □

Corollary 2.16. *Let G be a digraph with a global sink. The set of all recurrent chip configurations on G is an abelian group under the operation $(\sigma, \sigma') \mapsto (\sigma + \sigma')^\circ$, and it is isomorphic via the inclusion map to the sandpile group $\mathcal{S}(G)$.*

Proof. Immediate from Lemmas 2.13 and 2.15. □

In view of this isomorphism, we will henceforth use the term "sandpile group" to refer to the group of recurrent configurations.

It is of interest to consider the **identity element** I of the sandpile group, i.e. the unique recurrent configuration equivalent to the all-0 configuration. Here is one method to compute I. Let σ be the configuration $2\delta - 2$. (Arithmetic combinations of vectors and scalars are to interpreted componentwise.) Since $\sigma^\circ \leq \delta - 1$ we have $\sigma - \sigma^\circ \geq \delta - 1$, so $\sigma - \sigma^\circ$ is accessible. Since $\sigma - \sigma^\circ$ is equivalent to 0, the identity element is given by $I = (\sigma - \sigma^\circ)^\circ$.

Figure 4 shows identity elements for the $L \times L$ square grid with "wired boundary," for several values of L. (To be more precise, the graph G is obtained by replacing each edge of the undirected square grid with a pair of directed edges, and adjoining a sink vertex s along with two edges from each of the four corner vertices to s and one edge from each of the other boundary vertices to s.) The identity element of this graph was studied by Le Borgne and Rossin [BR02], but most basic properties of this configuration, such as the existence of the large square in the center, remain unproved.

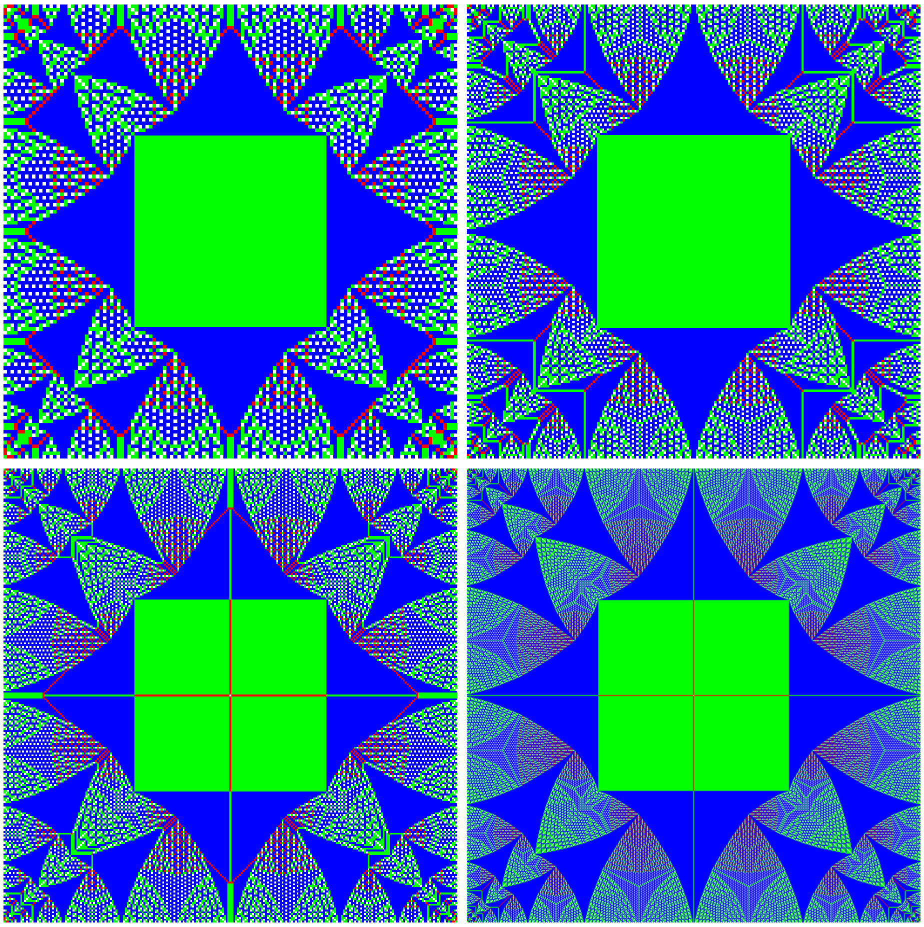


FIGURE 4. The identity element of the sandpile group of the $L \times L$ square grid for different values of L, namely $L = 128$ (upper left), 198 (upper right), 243 (lower left), and 521 (lower right). The color scheme is as follows: orange=0 chips, red=1 chip, green=2 chips, and blue=3 chips.

Figure 5 shows another example, the identity element for the 100×100 directed torus. (That is, for each vertex $(i, j) \in \mathbb{Z}/100\mathbb{Z} \times \mathbb{Z}/100\mathbb{Z}$, there are directed edges from (i, j) to $(i+1 \bmod 100, j)$ and to $(i, j+1 \bmod 100)$, and we make $(0, 0)$ (the lower-left vertex) into a sink by deleting its two outgoing edges.)

Figure 6 shows a third example, the identity element for a disk-shaped region of $\mathbb{Z}^2$ with wired boundary.

Also of interest is the **inverse** of a recurrent chip configuration σ (that is, the recurrent chip configuration $\bar{\sigma}$ such that $(\sigma + \bar{\sigma})^\circ = I$). One way to compute

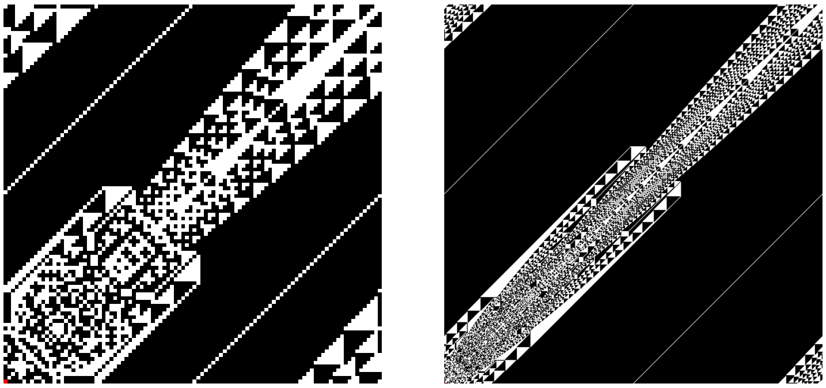


Figure 5. The identity element of the sandpile group of the 100×100 directed torus (left) and the 500×500 directed torus (right). The color scheme is as follows: white=0 chips, black=1 chip, and the sink, which is at the lower-left corner, is shown in red.

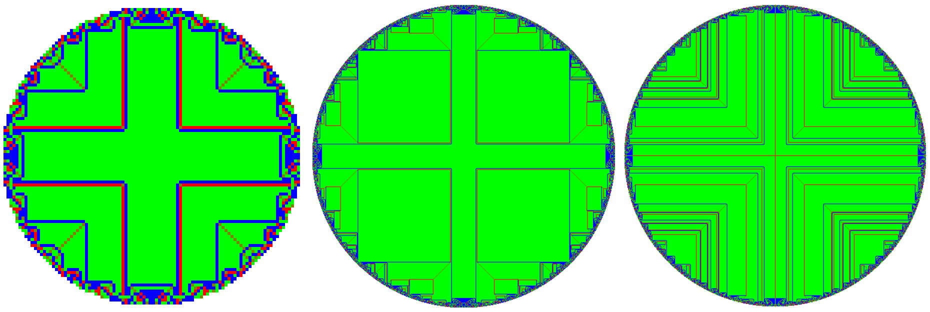


Figure 6. The identity element of the sandpile group of disk-shaped regions of diameter 100 (left), 512 (middle), and 521 (right). The color scheme is as follows: orange=0 chips, red=1 chip, green=2 chips, and blue=3 chips.

the inverse is via $\bar{\sigma} = (\zeta - \zeta^\circ - \sigma)^\circ$, where ζ is any chip configuration satisfying $\zeta \geq 3\delta - 3$. (Here $\zeta - \zeta^\circ - \sigma$ is accessible, since it has at least $d_v - 1$ chips at each vertex v.)

Given two chip configurations σ and ζ, we say that σ is **reachable** from ζ (via excitation-relaxation operations) if there exists a configuration β such that $\sigma = (\zeta + \beta)^\circ$. Note that this implies that σ is stable. A digraph is **strongly connected** if for any two distinct vertices v, w there are directed paths from v to w and from w to v. We write $G \setminus s$ for the graph obtained from G by deleting the vertex s along with all edges incident to s.

Lemma 2.17. *Let G be a digraph with a global sink s, and let σ be a chip configuration on G. The following are equivalent.*

(1) *σ is recurrent; that is, σ is reachable from any configuration ζ.*
(2) *If ζ is any configuration reachable from σ, then σ is reachable from ζ.*
(3) *σ is reachable from any configuration of the form $E_v\sigma$, where v is a non-sink vertex of G.*
(4) *Each strongly connected component of $G \setminus s$ contains a vertex v such that σ is reachable from $E_v\sigma$.*

Proof. Since trivially (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4), it suffices to show (4) $\Rightarrow$ (1).

If (4) holds, there is a chip configuration α such that $(\sigma + \alpha)^\circ = \sigma$ and α is nonzero on at least one vertex of each strongly connected component of G. There exists a positive integer k such that selective firing from $k\alpha$ results in a chip configuration β with at least one chip at each vertex. Moreover, $(\sigma + \beta)^\circ = \sigma$.

Now let ζ be any chip configuration. Since β has at least one chip at each vertex, we have $\zeta \leq \sigma + \ell\beta$ for some integer ℓ. Thus we may add chips to ζ and then stabilize to obtain the configuration $(\sigma + \ell\beta)^\circ = \sigma$. Hence σ is recurrent. $\square$

We also note that, for a digraph G with a global sink, the sandpile group is isomorphic to the additive group of harmonic functions modulo 1 on G that vanish on the sink [Sol99]. A function $f : V(G) \to [0, 1)$ is **harmonic modulo 1** if $d_v f(v) = \sum_w d_{v,w} f(w) \mod 1$ for all vertices v. For a sandpile configuration σ, the associated harmonic function f is the solution of

$$\sum_w \Delta'_{v,w} f(w) = \sigma(v).$$

For the graph in Figure 3, the three harmonic functions are $(f(v_1), f(v_2)) = (0, 0)$, $(f(v_1), f(v_2)) = (1/3, 2/3)$, and $(f(v_1), f(v_2)) = (2/3, 1/3)$.

We conclude this section by pointing out a link between the sandpile group and spanning trees. By Lemma 2.8 the order of the sandpile group of G equals the determinant of the reduced Laplacian Δ' of G. By the **matrix-tree theorem** [Sta99, 5.6.8], this determinant equals the number of **oriented spanning trees** of G rooted at the sink (that is, acyclic subgraphs of G in which every non-sink vertex has out-degree 1). Various bijections have been given for this correspondence; see, for example, [BW97]. In Section 3 we will use the rotor-router model to describe a particularly natural bijection, and deduce the matrix-tree theorem as a corollary.

3. Rotor-Routing

Chip-firing is a way of routing chips through a directed graph G in such a fashion that the chips emitted by any vertex v travel in equal numbers along each of the outgoing edges. In order to ensure this equality, however, chips must wait at a vertex v until sufficiently many additional chips have arrived to render v active. Rotor-routing is an alternative approach to distributing chips through G which dispenses with this waiting step. Since we cannot ensure exact equality without waiting, we settle for the condition that the chips emitted by any vertex v travel in *nearly* equal numbers along each of the edges emanating from v. We ensure that this near-equality holds by using a rotor mechanism to decide where each successive chip emitted from a vertex v should be routed.

Given a directed graph G, fix for each vertex v a cyclic ordering of the edges emanating from v. For an edge e with tail v we denote by e^+ the next edge after e in the prescribed cyclic ordering of the edges emanating from v.

Definition 3.1. A **rotor configuration** is a function ρ that assigns to each non-sink vertex v of G an edge $\rho(v)$ emanating from v. If there is a chip at a non-sink vertex v of G, **routing the chip** at v (for one step) consists of updating the rotor configuration so that $\rho(v)$ is replaced with $\rho(v)^+$, and then moving the chip to the head of $\rho(v)^+$. A **single-chip-and-rotor state** is a pair consisting of a vertex w (which represents the location of the chip) and a rotor configuration ρ. The **rotor-router operation** is the map that sends a single-chip-and-rotor state (w, ρ) (where w is not a sink) to the state (w^+, ρ^+) obtained by routing the chip at w for one step. (See Figure 7 for examples of the rotor-router operation.)

As we will see, there is an important link between chip-firing and rotor-routing. A hint at this link comes from a straightforward count of configurations. Recall that a stable chip configuration is a way of assigning some number of chips between 0 and $d_v - 1$ to each non-sink vertex v of G. Thus, the number of stable configurations is exactly $\prod_v d_v$, where the product runs over all non-sink vertices. This is also the number of rotor configurations on G. Other connections become apparent when one explores the appropriate notion of recurrent states for the rotor-router model. We will treat two cases separately: digraphs with no sink, and digraphs with a global sink (Lemma 3.6 applies to both settings).

Definition 3.2. Let G be a **sink-free** digraph, i.e. one in which each vertex has at least one outgoing edge. Starting from the state (w, ρ), if iterating the rotor-router operation eventually leads back to (w, ρ) we say that (w, ρ) is **recurrent**; otherwise, it is **transient**.

Our first goal is to give a combinatorial characterization of the recurrent states, Theorem 3.8. We define a **unicycle** to be a single-chip-and-rotor state (w, ρ) for which the set of edges $\{\rho(v)\}$ contains a unique directed cycle, and w lies on this cycle. (Equivalently, ρ is a connected functional digraph, and w is a vertex on the unique cycle in ρ.) The following lemma shows that the rotor-router operation takes unicycles to unicycles.

Lemma 3.3. *Let G be a sink-free digraph. If (w, ρ) is a unicycle on G, then (w^+, ρ^+) is also a unicycle.*

Proof. Since (w, ρ) is a unicycle, the set of edges $\{\rho(v)\}_{v \neq w} = \{\rho^+(v)\}_{v \neq w}$ contains no directed cycles. The set of edges $\{\rho^+(v)\}$ forms a subgraph of G in which every vertex has out-degree one, so it contains a directed cycle. Since any such cycle must contain the edge $\rho^+(w) = \rho(w)^+$, this cycle is unique, and w^+ lies on it. $\square$

Lemma 3.4. *Let G be a sink-free digraph. The rotor-router operation is a permutation on the set of unicycles of G.*

Proof. Since the set of unicycles is finite, by Lemma 3.3 it is enough to show surjectivity. Given a unicycle $U = (w, \rho)$, let $U^- = (w^-, \rho^-)$ be the state obtained by moving the chip from w to its predecessor w^- in the unique cycle through w, and replacing the rotor at w^- with its predecessor in the cyclic ordering of outgoing edges from w^-. Then the rotor-router operation applied to U^- yields U. It remains to show that U^- is a unicycle; for this it suffices to show that every directed cycle in ρ^- passes through w^-. Suppose that there is a directed cycle of rotors in ρ^- which avoids w^-. Since ρ^- agrees with ρ except at w^-, this same directed occurs within ρ and avoids w^-, a contradiction since w^- is on ρ's unique cycle. □

Corollary 3.5. *Let G be a sink-free digraph. If (w, ρ) is a unicycle on G, then (w, ρ) is recurrent.*

In Lemma 3.7, below, we show that the converse holds when G is strongly connected. We will need the following lemma, which is analogous to Lemma 2.4 for the abelian sandpile. A vertex w is **globally reachable** if for each other vertex v there is a directed path from v to w.

Lemma 3.6. *Let G be a digraph with a globally reachable vertex w. For any starting vertex and rotor configuration, iterating the rotor-router operation a suitable number of times yields a state in which the chip is at w.*

Proof. Since w is globally reachable, either G is sink-free or w is the unique sink. Thus either we can iterate the rotor-router operation indefinitely, or the chip eventually visits w. In the former case, since G is finite, the chip visits some vertex v infinitely often. But if x is a vertex that is visited infinitely often and there is an edge from x to y, then y is also visited infinitely often. Inducting along a path from v to w, we conclude that the chip eventually visits w. □

Lemma 3.7. *Let G be a strongly connected digraph. If (w, ρ) is a recurrent single-chip-and-rotor state on G, then it is a unicycle.*

Proof. Since G is strongly connected, every vertex is globally reachable. Hence by Lemma 3.6, if we start from any initial state and iterate the rotor-router rule sufficiently many times, the chip visits every vertex of G.

Suppose (w, ρ) is a recurrent state. Once every vertex has been visited and we return to the state (w, ρ), suppose the rotors at vertices $v_1 \ldots, v_k$ form a directed cycle. If w does not lie on this cycle, then for each i, the last time the chip was at v_i it moved to v_{i+1}, and hence the edge from v_i to v_{i+1} was traversed more recently than the edge from v_{i-1} to v_i. Carrying this argument around the cycle leads to a contradiction. Thus, every directed cycle in the rotor configuration must pass through w. But now if we start from w and follow the rotors, the first vertex we revisit must be w. Hence (w, ρ) is a unicycle. □

Combining Corollary 3.5 and Lemma 3.7, we have proved the following.

Theorem 3.8. *Let G be a strongly connected digraph. Then (w, ρ) is a recurrent single-chip-and-rotor state on G if and only if it is a unicycle.*

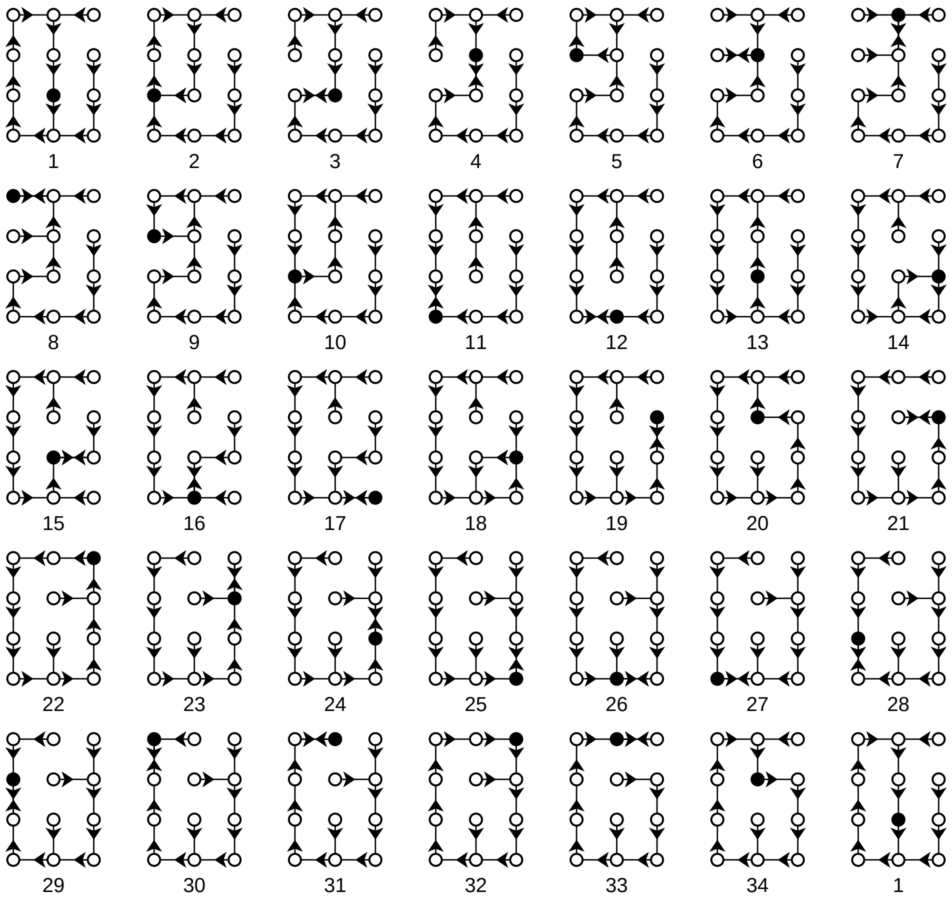


FIGURE 7. The unicycle configurations resulting from the evolution of a particular unicycle on the bidirected 3-by-4 rectangular grid. By Lemma 4.9, the chip traverses each directed edge exactly once before the original unicycle is obtained. Thus the number of distinct unicycle configurations equals the number of directed edges, in this case 34. From Lemma 4.11 it follows that from any given unicycle, after some number of steps, the state will be the same but with the cycle's direction reversed. This occurs, for example, with unicycles 1 and 13.

Next we consider the case when G is a digraph with a global sink. Note that we cannot apply the rotor-router operation to states in which the chip is at the sink. We call these **absorbing states**. For any starting state, if we iterate the rotor-router operation sufficiently many times, the chip must eventually arrive at the sink by Lemma 3.6.

A **chip-and-rotor state** is a pair $\tau = (\sigma, \rho)$ consisting of a chip configuration σ and rotor configuration ρ on G. A non-sink vertex is **active** in τ if it has at least one chip. If v is active, then **firing** v results in a new chip-and-rotor state given by replacing the rotor $\rho(v)$ with $\rho(v)^+$ and moving a single chip from v to the head of $\rho(v)^+$ (and removing the chip if $\rho(v)^+$ is a sink). We say that τ' is a **successor** of τ if it is obtained from τ by firing an active vertex. We say that τ is **stable** if no vertex can fire, i.e., all chips have moved to a sink and disappeared. The rotor-router operation has the following abelian property analogous to Lemma 2.2.

Lemma 3.9. *Let G be a digraph with a global sink. Let τ_0, $\tau_1, \dots, \tau_n$ be a sequence of chip-and-rotor states of G, each of which is a successor of the one before. If $\tau_0, \tau_1', \dots, \tau_m'$ is another such sequence, and τ_n is stable, then $m \leq n$. If in addition τ_m' is stable, then $m = n$ and $\tau_n = \tau_n'$, and for each vertex w, the number of times w fires is the same for both histories.*

Proof. Let v_i and v_i' be the vertices that are fired in τ_{i-1} and τ_{i-1}' to obtain τ_i and τ_i', respectively. We will show that if τ_n is stable and the sequences v and v' agree in the first $i-1$ terms for some $i \leq m$, then some permutation of v agrees with v' in the first i terms. Since v_i' is active in $\tau_{i-1} = \tau_{i-1}'$, it must be active in $\tau_i, \tau_{i+1}, \dots$, until it is fired. Since τ_n is stable, it follows that $v_j = v_i'$ for some $j > i$. Let j be the minimal such index. Starting from τ_0, the vertices $v_1, v_2, \dots, v_{i-1}, v_j, v_i, v_{i+1}, \dots, v_{j-1}, v_{j+1}, \dots, v_n$ can be fired in that order, resulting in the same stable configuration τ_n. Moreover, this sequence agrees with v' in the first i terms.

By induction, it follows that the sequence v' is an initial subsequence of a permutation of v. In particular, $m \leq n$. If τ_m' is also stable, by interchanging the roles of τ and τ', we obtain that v' is a permutation of v. □

Given a non-sink vertex v in G, the **chip addition operator** E_v is the map on rotor configurations given by adding a chip at vertex v and iterating the rotor-router operation until the chip moves to the sink. By Lemma 3.9 and the reasoning used in the proof of Lemma 2.5, the operators E_v commute. This is the **abelian property** of the rotor-router model.

If, rather than running the chips until they reach the sink, each chip is run for a fixed number of steps, then the abelian property fails, as the example in Figure 8 illustrates. (The proof of Lemma 3.9 requires that chips be indistinguishable, and it is not possible to run each chip for a fixed number of steps without distinguishing between them.) Despite the failure of commutativity, this way of routing chips has some interesting properties, similar to the bound given in Proposition 3.21; see work of Cooper and Spencer [CS06].

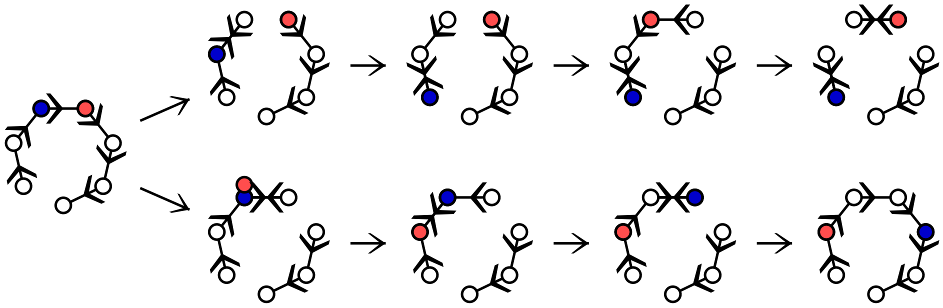


FIGURE 8. Failure of the abelian property for rotor-router walk stopped after two steps.

A rotor configuration ρ on G is **acyclic** if the rotors do not form any directed cycles. If G has a global sink, then ρ is acyclic if and only if the rotors form an oriented spanning tree rooted at the sink.

Lemma 3.10. *Let G be a digraph with a global sink, and let v be a vertex of G. The chip addition operator E_v is a permutation on the set of acyclic rotor configurations on G.*

Proof. We first argue that applying E_v to an acyclic rotor configuration yields an acyclic rotor configuration: this is proved by induction on the number of rotor-routing steps, where the induction hypothesis states that every path leads to either the sink or to the chip.

Since the set of acyclic rotor configurations is finite, it suffices to show surjectivity. Let ρ be an acyclic rotor configuration on G, add an edge e from the sink s to v to form a sink-free digraph G', and assign the rotor $\rho(s) = e$. Then $U = (s, \rho)$ is a unicycle on G'. Starting from U, we can iterate the *inverse* of the rotor-router operation (which by Lemma 3.4 is well-defined for unicycles) until the next time we reach a state $U' = (s, \rho')$ with the chip at s. Because U' is a unicycle with chip at s, deleting the edge e leaves an acyclic rotor configuration ρ'. Observe that running the rotor-router operation from U' to U, upon ignoring the edge e, is equivalent to applying E_v to ρ' and obtaining ρ. ☐

Definition 3.11. We next describe an action of the sandpile group on acyclic rotor configurations. Given a chip configuration σ and a rotor configuration ρ on G, write $\sigma(\rho)$ for the rotor configuration obtained by adding $\sigma(v)$ chips at each vertex v and routing them all to the sink. By Lemma 3.9 the order of these routings is immaterial. Thus we may write $\sigma(\rho)$ as

$$\sigma(\rho) = \left(\prod_{v \in V(G)} E_v^{\sigma(v)} \right) \rho,$$

where the product symbol represents composition of the operators.

It is trivial that $\sigma_2(\sigma_1(\rho)) = (\sigma_1 + \sigma_2)(\rho)$.

Since acyclic rotor configurations on G can be identified with oriented spanning trees rooted at the sink, Lemma 3.10 implies that every chip configuration σ acts as a permutation on the set of oriented spanning trees of G rooted at the sink.

Lemma 3.12. *Let G be a digraph with a global sink, and let ρ be an acyclic rotor configuration on G. If the chip configurations σ_1 and σ_2 are equivalent modulo the reduced Laplacian Δ' of G, then $\sigma_1(\rho) = \sigma_2(\rho)$.*

Proof. If $\sigma(v) \geq d_v$, and we route (for one step) d_v of the chips at v, then the rotor at v makes one full turn and one chip is sent along each outgoing edge from v. By Lemma 3.9, it follows that if σ' is a successor to σ (that is, σ' is obtained from σ by firing some active vertex v), then $\sigma(\rho) = \sigma'(\rho)$ for any rotor configuration ρ. Inducting, we obtain $\sigma(\rho) = \sigma^\circ(\rho)$ for any rotor configuration ρ.

In particular, if I is the recurrent chip configuration that represents the identity element of the sandpile group, we have

$$I(I(\rho)) = (I + I)(\rho) = (I + I)^\circ(\rho) = I(\rho)$$

for any rotor configuration ρ. By Lemma 3.10 the map $\rho \mapsto I(\rho)$ is a permutation on the set of acyclic rotor configurations, so it must be the identity permutation. Now if σ_1, σ_2 are equivalent modulo Δ', then $(\sigma_1 + I)^\circ$ and $(\sigma_2 + I)^\circ$ are recurrent and equivalent modulo Δ', hence equal by Lemma 2.15. Since

$$\sigma_i(\rho) = \sigma_i(I(\rho)) = (\sigma_i + I)(\rho) = (\sigma_i + I)^\circ(\rho), \qquad i = 1, 2,$$

we conclude that $\sigma_1(\rho) = \sigma_2(\rho)$. ☐

It follows from Lemma 3.12 that the sandpile group of G acts on the set of oriented spanning trees of G rooted at the sink. Our next lemma shows that this action is transitive.

Lemma 3.13. *Let G be a digraph with a global sink. For any two acyclic rotor configurations ρ and ρ' on G, there exists a chip configuration σ on G such that $\sigma(\rho) = \rho'$.*

Proof. For a non-sink vertex v, let $\alpha(v)$ be the number of edges e such that $\rho(v) < e \leq \rho'(v)$ in the cyclic ordering of outgoing edges from v. Starting with rotor configuration ρ, and with $\alpha(v)$ chips at each vertex v, allow each chip to take just one step. The resulting rotor configuration is ρ'; let β be the resulting chip configuration, so that $\alpha(\rho) = \beta(\rho')$, and let γ be the inverse of the corresponding element $(\beta + I)^\circ$ of the sandpile group. By Lemma 3.12 and the fact that $\beta + \gamma$ is equivalent to 0 modulo Δ', we have

$$(\alpha + \gamma)(\rho) = (\beta + \gamma)(\rho') = \rho'.$$

☐

Next we define recurrent rotor configurations on a digraph with a global sink, and show they are in bijection with oriented spanning trees.

Definition 3.14. Let G be a digraph with a global sink. Given rotor configurations ρ and ρ' on G, we say that ρ is **reachable** from ρ' if there is a chip configuration σ such that $\sigma(\rho') = \rho$. We say that ρ is **recurrent** if it is reachable from any other configuration ρ'.

Note that in contrast to Definition 3.2, the location of the chip plays no role in the notion of recurrent states on a digraph with global sink.

Lemma 3.15. *Let G be a digraph with a global sink. A rotor configuration ρ on G is recurrent if and only if it is acyclic.*

Proof. By Lemma 3.10, any configuration reachable from an acyclic configuration must be acyclic, so recurrent implies acyclic. Conversely, if ρ is acyclic and ρ' is any rotor configuration, the configuration $\mathbf{1}(\rho')$ (where $\mathbf{1}$ denotes the configuration with one chip at each vertex) is acyclic, since the rotor at each vertex points along the edge by which a chip last exited. By Lemma 3.13 there is a chip configuration σ such that $\sigma(\mathbf{1}(\rho')) = \rho$, so ρ is reachable from ρ' and hence recurrent. $\square$

Just as for the sandpile model, there are several equivalent definitions of recurrence for the rotor-router model.

Lemma 3.16. *Let G be a digraph with a global sink s, and let ρ be a rotor configuration on G. The following are equivalent.*

(1) *ρ is acyclic.*
(2) *ρ is recurrent; that is, ρ is reachable from any rotor configuration ρ'.*
(3) *If ρ' is reachable from ρ, then ρ is reachable from ρ'.*
(4) *ρ is reachable from any rotor configuration of the form $E_v\rho$, where v is a vertex of G.*
(5) *Each strongly connected component of $G \setminus s$ contains a vertex v such that ρ is reachable from $E_v\rho$.*

Proof. By Lemma 3.15 we have (1) $\Rightarrow$ (2), and trivially (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5).

If property (5) holds, let $C_1, \dots, C_\ell$ be the strongly connected components of $G \setminus s$, and for each i, let $v_i \in C_i$ be such that ρ is reachable from $E_{v_i}\rho$. Choose an integer k large enough so that if we start k chips at any v_i and route them to the sink, every vertex in C_i is visited at least once. Let

$$\rho' = \left(\prod_i E_{v_i}\right)^k \rho.$$

Then in ρ', the rotor at each vertex points along the edge by which a chip last exited, so ρ' is acyclic. Since ρ is reachable from ρ', by Lemma 3.10 it follows that ρ is acyclic. Thus (5) $\Rightarrow$ (1), completing the proof. $\square$

Next we show that the action of the sandpile group on the set of oriented spanning trees of G is free.

Lemma 3.17. *Let G be a digraph with a global sink, and let σ_1 and σ_2 be recurrent chip configurations on G. If there is an acyclic rotor configuration ρ of G such that $\sigma_1(\rho) = \sigma_2(\rho)$, then $\sigma_1 = \sigma_2$.*

Proof. Let $\sigma = \sigma_1 + \overline{\sigma_2}$ (recall that $\overline{\sigma_2}$ is the inverse of σ_2.) By Lemma 3.12, $\sigma(\rho) = \overline{\sigma_2}(\sigma_1(\rho)) = \overline{\sigma_2}(\sigma_2(\rho)) = (\sigma_2 + \overline{\sigma_2})(\rho) = \rho$. Since $\sigma(\rho) = \rho$, after adding σ to ρ, for each vertex v, the rotor at v makes some integer number c_v of full rotations. Each full rotation results in d_v chips leaving v, one along each outgoing edge. Hence $\sigma = \sum_v c_v \Delta'_v$, which is in the row span of the reduced Laplacian, so σ is equivalent to 0 modulo $\Delta'(G)$, and hence $\sigma_1 = \sigma_2$ by Lemma 2.15. □

Corollary 3.18 (Matrix Tree Theorem). *Let G be a digraph and v a vertex of G. The number of oriented spanning trees of G rooted at v is equal to the determinant of the reduced Laplacian $\Delta'(G)$ obtained by deleting from $\Delta(G)$ the row and column corresponding to v.*

Proof. Without loss of generality we may assume the graph is loopless, since loops affect neither the graph Laplacian nor the number of spanning trees.

If v is not globally reachable, then there are no spanning trees rooted at v, and there is a set of vertices S not containing v, such that there are no edges in G from S to S^c. The rows of $\Delta'(G)$ corresponding to vertices in S sum to zero, so $\Delta'(G)$ has determinant zero.

If v is globally reachable, delete all outgoing edges from v to obtain a digraph G' with global sink v. Note that G and G' have the same reduced Laplacian, and the same set of oriented spanning trees rooted at v.

Fix an oriented spanning tree ρ of G'. The mapping $\sigma \mapsto \sigma(\rho)$ from $\mathcal{S}(G')$ to the set of oriented spanning trees of G' is a surjection by Lemma 3.13 and is one-to-one by Lemma 3.17, and by Lemma 2.8, $|\mathcal{S}(G')| = \det \Delta'(G')$. □

Given a digraph G with global sink, define its **rotor-router group** as the subgroup of permutations of oriented spanning trees of G generated by the chip addition operators E_v.

Lemma 3.19. [LL07] *The rotor-router group for a digraph G with a global sink is isomorphic to the sandpile group $\mathcal{S}(G)$.*

Proof. The action of the sandpile group on oriented spanning trees is a homomorphism from the sandpile group $\mathcal{S}(G)$ onto the rotor-router group. For any two distinct sandpile group elements σ_1 and σ_2, for any oriented spanning tree ρ (there is at least one), by Lemma 3.17 we have $\sigma_1(\rho) \neq \sigma_2(\rho)$, so the associated rotor-router group elements are distinct, i.e., the group homomorphism is an isomorphism. □

Since the number of recurrent chip configurations of G equals the number of oriented spanning trees, it is natural to ask for a bijection. Although there is no truly "natural" bijection, since in general there is no canonical spanning tree to correspond to the identity configuration, we can use the rotor-router model to define a family of bijections. Fix any oriented spanning tree ρ rooted at the sink, and associate it with the identity configuration I. For any other oriented spanning tree ρ', by Lemma 3.13 there exists $\sigma \in \mathcal{S}(G)$ with $\sigma(\rho) = \rho'$; moreover, σ is unique by Lemma 3.17. Associate σ with ρ'. Since this defines a surjective map from recurrent configurations to oriented spanning trees, it must be a bijection.

Remark 3.20. A variant of the rotor-router rule relaxes the cyclic ordering of edges emanating from a vertex, and merely requires one to choose *some* edge emanating from the current location of the chip as the new rotor-setting and move the chip along this edge. This is the **branching operation** introduced by Propp and studied by Athanasiadis [Ath97]. Alternatively, one can put a probability distribution on the edges emanating from each vertex, and stipulate that the new edge is to be chosen at random. This gives the tree-walk introduced by Anantharam and Tsoucas [AT89] in their proof of the Markov chain tree theorem of Leighton and Rivest [LR86].

We conclude this section with the following result of Holroyd and Propp [HP08], which illustrates another area of application of the rotor-router model.

Proposition 3.21. *Let $G = (V, E)$ be a digraph and let $Y \subseteq Z$ be sets of vertices. Assume that from each vertex there is a directed path to Z. Let σ be a chip configuration on G. If the chips perform independent simple random walks on G stopped on first hitting Z, let $H(\sigma, Y)$ be the expected number of chips that stop in Y. If the chips perform rotor-router walks starting at rotor configuration ρ and stopped on first hitting Z, let $H_\rho(\sigma, Y)$ be the number of chips that stop in Y. Then*

$$|H_\rho(\sigma, Y) - H(\sigma, Y)| \le \sum_{\text{edges } e} |h(\text{head of } e) - h(\text{tail of } e)| \tag{1}$$

where $h(v) := H(1_v, Y)$.

Note that the bound on the right side does not depend on ρ or σ.

Proof. To each edge $e = (u, v)$ we assign a *weight*

$$\mathrm{wt}(e) = \begin{cases} 0 & \text{if } e = \rho(u), \\ h(u) - h(v) + \mathrm{wt}(e^-) & \text{otherwise.} \end{cases}$$

Here e^- is the edge preceding e in the cyclic ordering of edges emanating from u. Since h is a harmonic function on G, the sum of $h(u) - h(v)$ over all edges $e = (u, v)$ emanating from u is zero, so the formula $\mathrm{wt}(e) = h(u) - h(v) + \mathrm{wt}(e^-)$ remains valid even when $e = \rho(u)$. We assign weight $\sum_v \mathrm{wt}(\rho(v))$ to a rotor configuration ρ, and weight $h(v)$ to a chip located at v. By construction, the sum of rotor and chip weights in any configuration is invariant under the operation of rotating the rotor at a chip and then routing the chip. Initially, the sum of all chip weights is $H(\sigma, Y)$. After all chips have stopped, the sum of the chip weights is $H_\rho(\sigma, Y)$. Their difference is thus at most the change in rotor weights, which is bounded above by the sum in (1). $\square$

Similar bounds hold even for some infinite directed graphs in which the right side of (1) is not finite. Thus rotor-routing can give estimates of hitting probabilities with very small error. See [HP08] for more details.

4. Eulerian Graphs

A digraph $G = (E, V)$ is **Eulerian** if it is strongly connected, and for each vertex $v \in V$ the in-degree and the out-degree of v are equal. We call G an **Eulerian digraph with sink** if it is obtained from an Eulerian digraph by deleting all the outgoing edges from one vertex; equivalently, G has a sink and every other vertex has out-degree that is at least as large as its in-degree. An **Eulerian tour** of a digraph G is a cycle in G that uses each edge exactly once. Such a tour exists if and only if G is Eulerian. Note that for any connected undirected graph, the corresponding bidirected graph is Eulerian. In this section we show some results that do not hold for general digraphs, but are true for Eulerian ones. We first treat the sandpile model, and then the rotor-router model.

Lemma 4.1 (Burning algorithm [Dha90]**).** *Let G be an Eulerian digraph with sink. A chip configuration σ is recurrent if and only if $(\sigma + \beta)^\circ = \sigma$, where*

$$\beta(v) = \mathrm{outdeg}(v) - \mathrm{indeg}(v) \geq 0.$$

If σ is recurrent, each vertex fires exactly once during the stabilization of $\sigma + \beta$.

Proof. By the "(4) $\Rightarrow$ (1)" part of Lemma 2.17, if $(\sigma+\beta)^\circ = \sigma$, then σ is recurrent. Conversely, suppose σ is recurrent. Label the non-sink vertices $v_1, \ldots, v_{n-1}$. Since

$$\beta = \sum_{i=1}^{n-1} \Delta'_i, \tag{2}$$

the configurations σ and $(\sigma + \beta)^\circ$ are both recurrent and equivalent modulo Δ'. By Lemma 2.15 it follows that they are equal.

Let c_i be the number of times vertex v_i fires during the stabilization of $\sigma+\beta$. Then

$$\sigma = (\sigma + \beta)^\circ = \sigma + \beta - \sum_{i=1}^{n-1} c_i \Delta'_i.$$

The rows of Δ' are linearly independent, so from (2) we deduce $c_i = 1$ for all i. $\square$

Informally, the burning algorithm can be described as follows: to determine whether σ is recurrent, first "fire the sink" to obtain the configuration $\sigma+\beta$. Then σ is recurrent if and only if every non-sink vertex fires in the stabilization of $\sigma+\beta$. In the non-Eulerian case, there is a generalization of the burning algorithm known as the **script algorithm**, due to Eugene Speer [Spe93].

Let H be an induced subgraph of G not containing the sink. We say that H is **ample** for a chip configuration σ on G if there is a vertex v of H that has at least as many chips as the in-degree of v in H.

Lemma 4.2. *Let G be an Eulerian digraph with sink s. A stable chip configuration σ on G is recurrent if and only if every nonempty induced subgraph of $G \setminus s$ is ample for σ.*

Proof. If σ is recurrent, there is a chip configuration α such that $(\delta + \alpha)^\circ = \sigma$, where $\delta(v) = d_v$. Each vertex of G fires at least once in the process of stabilizing $\delta + \alpha$. Given a nonempty induced subgraph H of G, let v be the vertex of H which first finishes firing. After v finishes firing, it must receive at least as many chips from its neighbors as its in-degree in H, so $\sigma(v)$ is at least the in-degree of v in H. Thus H is ample for σ.

Conversely, suppose that every nonempty induced subgraph of G is ample for σ. Let β be the chip configuration defined in Lemma 4.1. Starting from $\sigma + \beta$, fire as many vertices as possible under the condition that each vertex be allowed to fire only once. Let H be the induced subgraph on the set of vertices that do not fire. Since each vertex v of H is unable to fire even after receiving one chip from each incoming edge whose other endpoint lies outside H, we have

$$\sigma(v) + d_v - \mathrm{indeg}_H(v) \leq d_v - 1.$$

Thus H is not ample and consequently must be empty. So every vertex fires once, after which we obtain the configuration $\sigma + \beta - \sum_{i=1}^{n-1} \Delta'_i = \sigma$. Hence $(\sigma + \beta)^\circ = \sigma$, which implies σ is recurrent by Lemma 2.17. ☐

Next we define a variant of chip-firing called cluster-firing (see Figure 9), and we use Lemma 4.2 to characterize the stable states for cluster-firing. This gives rise to a notion of "superstable states" which are in some sense dual to the recurrent states.

Definition 4.3. Let G be a digraph with a global sink. Let σ be a chip configuration on G, and let A be a nonempty subset of the non-sink vertices of G. The **cluster-firing** of A yields the configuration

$$\sigma' = \sigma - \sum_{i \in A} \Delta'_i.$$

If σ' is nonnegative we say that the cluster A is **allowed to fire**. We say that σ is **superstable** if no cluster is allowed to fire.

Note that a cluster A may be allowed to fire even if no subset of A is allowed to fire. For example, in Figure 9, a cluster of two vertices is allowed to fire even though neither of its vertices alone is allowed to fire.

One could consider an even more general operation, "multicluster-firing," in which different vertices can be fired different numbers of times, so long as at the end of the firings, each vertex has a nonnegative number of chips. However, this further-generalized firing operation does not yield anything new for Eulerian digraphs, since any multicluster-firing can be expressed as a sequence of cluster-firings: Let m denote the maximal number of times that a vertex fires in the multicluster-firing, and C_j denote the set of vertices that fire at least j times in

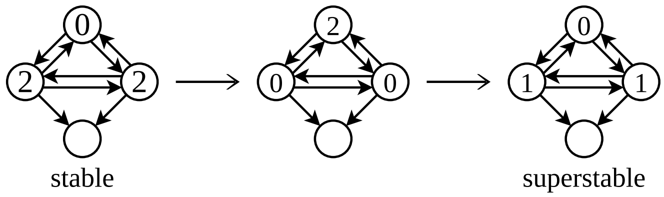


FIGURE 9. An example of cluster-firing, resulting in a superstable chip configuration. The bottom vertex is the sink.

the multicluster-firing. Since the digraph is Eulerian, C_m may be cluster-fired, and so by induction the sets $C_m, C_{m-1}, \ldots, C_1$ can be cluster-fired in that order.

Denote by δ the chip configuration $\delta(v) = d_v$ in which each vertex has as many chips as outgoing edges, and by $\mathbf{1}$ the configuration with a single chip at each vertex.

Theorem 4.4. *Let G be an Eulerian digraph with sink. A chip configuration σ on G is superstable if and only if $\delta - \mathbf{1} - \sigma$ is recurrent.*

Proof. A cluster A is allowed to fire if and only if for each vertex $v \in A$ we have

$$\sigma(v) - d_v + \mathrm{indeg}_A(v) \geq 0.$$

This is equivalent to $d_v - 1 - \sigma(v) < \mathrm{indeg}_A(v)$, i.e., the induced subgraph on A is not ample for $\delta - \mathbf{1} - \sigma$. By Lemma 4.2 the proof is complete. $\square$

By Lemmas 2.13 and 2.15, every equivalence class modulo Δ' contains a unique recurrent configuration, so we obtain the following.

Corollary 4.5. *Let G be an Eulerian digraph with sink. Every equivalence class modulo $\Delta'(G)$ contains a unique superstable configuration.*

As a consequence, we obtain that the cluster-firing model on Eulerian digraphs is abelian; this was proved by Paoletti [Pao07] in the bidirected case.

Corollary 4.6. *Let G be an Eulerian digraph with sink. Let $\sigma_0, \sigma_1, \ldots, \sigma_n$ be a sequence of chip configurations on G, each of which is obtained from the one before by a cluster-firing, with σ_n superstable. Then any sequence of cluster-firings that starts from σ_0 and ends in a superstable configuration ends in σ_n.*

We call the configuration σ_n in Corollary 4.6 the **superstabilization** of σ_0. The following result provides a way to compute the superstabilization.

Proposition 4.7. *Let σ be a chip configuration on an Eulerian digraph with sink. The superstabilization of σ is given by*

$$\sigma^* = \delta - \mathbf{1} - (\delta - \mathbf{1} - \sigma^\circ + I)^\circ$$

where I is the identity element of the sandpile group.

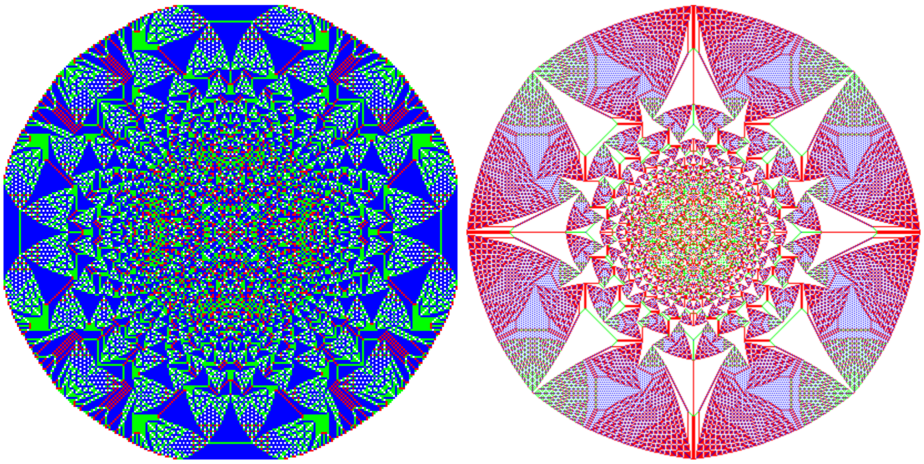


FIGURE 10. Stable sandpile (on left) and superstable sandpile (on right) of 100,000 chips, obtained by placing 100,000 chips at the origin of the integer lattice $\mathbb{Z}^2$ and (super)stabilizing. The color scheme is as follows: white=0 chips, red=1 chip, green=2 chips, and blue=3 chips.

Proof. Since the configuration $\zeta = (\delta - \mathbf{1} - \sigma^\circ + I)^\circ$ is reachable from the identity element, it is recurrent, hence $\sigma^* = \delta - \mathbf{1} - \zeta$ is superstable by Theorem 4.4. Since σ and σ^* are equivalent modulo Δ', it follows from Corollary 4.5 that σ^* is the superstabilization of σ. ☐

Our final result concerning the sandpile model on Eulerian digraphs is a theorem of Van den Heuvel [vdH01]; see also [Tar88]. We give a shorter and more direct proof than that presented in [vdH01]. By an **bidirected graph with sink** s, we will mean the digraph obtained from an undirected graph by first replacing each edge by a pair of directed edges in opposite directions, and then deleting all outgoing edges from s. The **effective resistance** between two vertices of G is an important quantity in electrical network theory; see, e.g., [DS84]. In particular, the quantity $R_{\max}$ appearing in the proposition below is always bounded above by the diameter of G, but for many graphs it is substantially smaller than the diameter.

Proposition 4.8. *Let G be a bidirected graph with sink, and let σ be a chip configuration on G. The total number of chip moves needed to stabilize σ is at most*

$$2m\,|\sigma|\,R_{\max}$$

where m is the number of edges, $|\sigma|$ is the total number of chips, and $R_{\max}$ is the maximum effective resistance between any vertex of G and the sink.

Note that firing a vertex v consists of d_v chip moves.

Proof. Let $\sigma = \sigma_0, \sigma_1, \ldots, \sigma_k = \sigma^\circ$ be a sequence of chip configurations with σ_{i+1} obtained from σ_i by firing a single active vertex x_i. Define the weight of σ_i to be

$$\mathrm{wt}(\sigma_i) = \sum_x \sigma_i(x)\, \mathrm{wt}(x),$$

where

$$\mathrm{wt}(x) = \mathbb{E}_x T_s$$

is the expected time for a simple random walk started at x to hit the sink. By conditioning on the first step X_1 of the walk, we compute

$$\Delta\, \mathrm{wt}(x) = \mathbb{E}_x(\mathbb{E}_{X_1} T_s - T_s) = -1,$$

so firing the vertex x_i decreases the total weight by d_{x_i}. Thus

$$\mathrm{wt}(\sigma_i) - \mathrm{wt}(\sigma_{i+1}) = d_{x_i}. \tag{3}$$

By [CRR$^+$97], the function wt is bounded by $2mR_{\max}$. Since the final weight $\mathrm{wt}(\sigma^\circ)$ is nonnegative, summing (3) over i we obtain that the total number of chip moves N needed to stabilize σ is at most

$$N = \sum_{i=0}^{k-1} d_{x_i} = \mathrm{wt}(\sigma) - \mathrm{wt}(\sigma^\circ) \le 2m\,|\sigma|\, R_{\max}. \qquad \square$$

Next we present results about the rotor-router model specific to the Eulerian case. An example of the next lemma is illustrated in Figure 7.

Lemma 4.9. *Let G be an Eulerian digraph with m edges. Let $U = (w, \rho)$ be a unicycle in G. If we iterate the rotor-router operation m times starting from U, the chip traverses an Eulerian tour of G, each rotor makes one full turn, and the state of the system returns to U.*

Proof. Iterate the rotor-router operation starting from U until some rotor makes more than a full turn. Let it be the rotor at vertex v. During this process, v must emit the chip more than d_v times. Hence if $v \neq w$, then v must also receive the chip more than d_v times. Since G is Eulerian, this means that some neighboring vertex u must send the chip to v more than once. However, when the chip goes from u to v for the second time, the rotor at u has executed more than a full turn, contradicting our choice of v. Thus when the rotor at w has made a full turn, the rotors at the other sites have made at most a full turn.

We can now repeat this argument starting from the configuration obtained after the rotor at w has made a full turn. In this way, the future history of the system is divided up into segments, each of length at most m, where the chip is at w at the start of each segment. It follows that over the course of the future history of the system, the chip is at w at least d_w/m of the time.

Since G is strongly connected, we may apply this same argument to every state in the future history of the system, with every vertex of G playing the role of w. As the system evolves, the chip is at v at least d_v/m of the time. Since $\sum_v d_v/m = 1$, the chip is at v exactly d_v/m of the time. Hence, as the rotor at w

executes a full turn, the rotors at the other sites also execute a full turn. Since every rotor makes a full turn, every edge is traversed exactly once, so the chip traverses an Eulerian tour. □

We can use Lemma 4.9 to give a bijective proof of a classical result in enumerative combinatorics relating the number of Eulerian tours of an Eulerian graph G to the number of oriented spanning trees of G (see, e.g., [Sta99, Cor. 5.6.7]).

Corollary 4.10. *Let $G = (V, E)$ be an Eulerian digraph. Fix an edge $e \in E$ and let $\mathrm{tail}(e) = w$. Let $\mathcal{T}(G, v)$ denote the number of oriented spanning trees in G rooted at w, and let $\epsilon(G, e)$ be the number of Eulerian tours in G starting with the edge e. Then*

$$\epsilon(G, e) = \mathcal{T}(G, w) \prod_{v \in V} (d_v - 1)!.$$

Proof. There are $\prod_{u \in V}(d_v - 1)!$ ways to fix cyclic orderings of the outgoing edges from each vertex. There are $\mathcal{T}(G, w)$ ways to choose a unicycle $U = (w, \rho)$ with the chip at w and the rotor $\rho(w) = e^-$, where e^- is the edge preceding e in the cyclic ordering of outgoing edges from w. Given these data, we obtain from Lemma 4.9 an Eulerian tour of G starting with the edge e, namely the path traversed by the rotor-router walk in m steps.

To show that this correspondence is bijective, given an Eulerian tour starting with the edge e, cyclically order the outgoing edges from each vertex v in the order they appear in the tour. Let $\rho(w) = e^-$ and for $v \neq w$ let $\rho(v)$ be the outgoing edge from v that occurs last in the tour. Then $U = (w, \rho)$ is a unicycle. □

The following result was first announced in [PPS98], in the case of rotor-router walk on a square lattice.

Corollary 4.11. *Let G be a bidirected planar graph with the outgoing edges at each vertex ordered clockwise. Let (w, ρ) be a unicycle on G with the cycle $\mathcal{C}$ oriented clockwise. After the rotor-router operation is iterated some number of times, each rotor internal to $\mathcal{C}$ has performed a full rotation, each rotor external to $\mathcal{C}$ has not moved, and each rotor on $\mathcal{C}$ has performed a partial rotation so that $\mathcal{C}$ is now oriented counter-clockwise.*

Proof. Let G' be the graph obtained from G by deleting all vertices and edges external to $\mathcal{C}$. Note that G', like G, is Eulerian. Let ρ^- be the rotor configuration on G' obtained from ρ by "regressing" each rotor whose vertex lies on $\mathcal{C}$; that is, if v lies on $\mathcal{C}$ and $\rho(v)$ is the edge e, let $\rho(v) = e^-$, the edge immediately preceding e in the cyclic ordering of the edges emanating from v. Consider the unicycle $U^- = (w, \rho^-)$ on G' and note that it has $\mathcal{C}$ oriented counter-clockwise. Starting from U^- and applying the rotor-router operation $\#\mathcal{C}$ times, the chip will traverse the cycle $\mathcal{C}'$, resulting in the state $U = (w, \rho|_{G'})$. By Lemma 4.9, further iteration of the rotor-router operation on G' returns the system to the state U^-. Since the outgoing edges at each vertex are ordered clockwise, it is straightforward to see that applying the rotor-router rule to U on G' and to (w, ρ) on G results in the same evolution up until the time that state U^- on G' is reached. □

Lemma 4.12. *Let G be an Eulerian digraph, and let G_v be the Eulerian digraph with sink obtained by deleting the outgoing edges from vertex v. Then the abelian sandpile groups $\mathcal{S}(G_v)$ and $\mathcal{S}(G_w)$ corresponding to different choices of sink are isomorphic.*

Proof. Recall that the sandpile group $\mathcal{S}(G_v)$ is isomorphic to $\mathbb{Z}^{n-1}/\mathbb{Z}^{n-1}\Delta'(G)$; we argue that for Eulerian digraphs G it is also isomorphic to $\mathbb{Z}^n/\mathbb{Z}^n\Delta(G)$. Vectors in $\mathbb{Z}^{n-1}$ are isomorphic to vectors in $\mathbb{Z}^n$ whose coordinates sum to 0, and modding out a vector in $\mathbb{Z}^{n-1}$ by a row of the reduced Laplacian Δ' corresponds to modding out the corresponding vector in $\mathbb{Z}^n$ by the corresponding row of the full Laplacian Δ. For Eulerian digraphs G, the last row of the full Laplacian Δ is the negative of the sum of the remaining rows, so modding out by this extra row has no effect. □

We mention one other result that applies to undirected planar graphs, due to Corri and Rossin [CR00].

Theorem 4.13 ([CR00]). *If G and G^* are dual undirected planar graphs, then the sandpile groups of G and G^* are isomorphic. (By Lemma 4.12, the locations of the sink are irrelevant.)*

5. Stacks and Cycle-Popping

Let G be a digraph with a global sink. In this section we describe a more general way to define rotor-router walk on G, using arbitrary stacks of rotors at each vertex in place of periodic rotor sequences. To each non-sink vertex v of G we assign a bi-infinite **stack** $\rho(v) = (\rho_k(v))_{k\in\mathbb{Z}}$ of outgoing edges from v. To **pop** the stack, we shift it to obtain $(\rho_{k+1}(v))_{k\in\mathbb{Z}}$. To **reverse pop** the stack, we shift it in the other direction to obtain $(\rho_{k-1}(v))_{k\in\mathbb{Z}}$. The rotor-router walk can be defined in terms of stacks as follows: if the chip is at vertex v, pop the stack $\rho(v)$, and then move the chip along the edge $\rho_1(v)$. We recover the ordinary rotor-router model in the case when each stack $\rho(v)$ is a periodic sequence of period d_v in which each outgoing edge from v appears once in each period.

The collection of stacks $\rho = (\rho(v))$, where v ranges over the non-sink vertices of G, is called a **stack configuration** on G. We say that ρ is **infinitive** if for each edge $e = (v, w)$, and each positive integer K, there exist stack elements

$$\rho_k(v) = \rho_{k'}(v) = e$$

with $k \geq K$ and $k' \leq -K$. This condition guarantees that rotor-router walk eventually reaches the sink.

Given a stack configuration ρ, the stack elements $\rho_0(v)$ define a rotor configuration on G. We say that ρ is **acyclic** if ρ_0 contains no directed cycles. If $\mathcal{C} = \{v_1, \ldots, v_m\}$ is a directed cycle in ρ_0, define $\mathcal{C}\rho$ to be the stack configuration obtained by reverse popping each of the stacks $\rho(v_i)$; we call this reverse popping the cycle $\mathcal{C}$. (If $\mathcal{C}$ is not a directed cycle in ρ_0, set $\mathcal{C}\rho = \rho$.)

Theorem 5.1. [PW98] *Let G be a digraph with a global sink, and let ρ^0 be an infinitive stack configuration on G. There exist finitely many cycles $\mathcal{C}_1, \ldots, \mathcal{C}_m$ such that the stack configuration*

$$\rho = \mathcal{C}_m \cdots \mathcal{C}_1 \rho^0$$

is acyclic. Moreover, if $\mathcal{C}'_1, \ldots, \mathcal{C}'_n$ is any sequence of cycles such that the stack configuration $\rho' = \mathcal{C}'_n \cdots \mathcal{C}'_1 \rho^0$ is acyclic, then $\rho' = \rho$.

If v is a non-sink vertex of G, the **chip addition operator** E_v applied to the infinitive stack configuration ρ is the stack configuration ρ' obtained by adding a chip at v and performing rotor-router walk until the chip reaches the sink. The next lemma shows that these operators commute with cycle-popping.

Lemma 5.2. *Let G be a digraph with a global sink, let ρ be an infinitive stack configuration on G, and let $\mathcal{C}$ be a directed cycle in G. Then*

$$E_v(\mathcal{C}\rho) = \mathcal{C}(E_v\rho).$$

Proof. Write $\rho' = E_v\rho$. Let $v = v_0, v_1, \ldots, v_n = s$ be the path taken by a chip performing rotor-router walk from v to the sink starting with stack configuration ρ. If this path is disjoint from $\mathcal{C}$, then the chip performs the same walk starting with stack configuration $\mathcal{C}\rho$, and the cycle $\mathcal{C}$ is present in ρ'_0 if and only if it is present in ρ_0, so the proof is complete.

Otherwise, choose k minimal and ℓ maximal with $v_k, v_\ell \in \mathcal{C}$. The rotor $\rho'_0(v_\ell)$ points to a vertex not in $\mathcal{C}$, so the cycle $\mathcal{C}$ is not present in ρ'_0. Thus we must show $E_v(\mathcal{C}\rho) = \rho'$. With stack configuration $C\rho$, the chip will first travel the path $v_0, \ldots, v_k$, next traverse the cycle $\mathcal{C}$, and finally continue along the remainder of the path $v_k, \ldots, v_n$. Thus the stack at each vertex $w \in \mathcal{C}$ is popped one more time in going from $\mathcal{C}\rho$ to $E_v(\mathcal{C}\rho)$ than in going from ρ to $E_v\rho$; the stack at each vertex $w \notin \mathcal{C}$ is popped the same number of times in both cases. $\square$

The next lemma uses cycle-popping to give a constructive proof of the injectivity of the chip addition operators E_v on acyclic stack configurations. In the case of periodic rotor stacks, we gave a non-constructive proof in Lemma 3.10.

Lemma 5.3. *Let G be a digraph with a global sink. Given an acyclic infinitive stack configuration ρ on G and a non-sink vertex v, there exists an acyclic infinitive stack configuration ρ' such that $E_v\rho' = \rho$.*

Proof. Let ρ^0 be the stack configuration obtained from ρ by reverse popping the stack at each of the vertices on the unique path in ρ_0 from v to the sink. A rotor-router walk started at v with stack configuration ρ^0 will travel directly along this path to the sink, so $E_v\rho^0 = \rho$. If ρ^0 is acyclic, the proof is complete. Otherwise, by Theorem 5.1 there are cycles $\mathcal{C}_1, \ldots, \mathcal{C}_m$ such that $\rho' = \mathcal{C}_m \cdots \mathcal{C}_1 \rho^0$ is acyclic. By Lemma 5.2, we have

$$E_v\rho' = \mathcal{C}_m \cdots \mathcal{C}_1(E_v\rho^0) = \mathcal{C}_m \cdots \mathcal{C}_1\rho = \rho$$

where in the last equality we have used that ρ is acyclic. □

Note that the proof shows the following: if ρ, ρ' are acyclic infinitive stack configurations and $E_v\rho' = \rho$, then the unique path in ρ_0 from v to the sink is the **loop-erasure** of the path taken by rotor-router walk started at v with initial configuration ρ'.

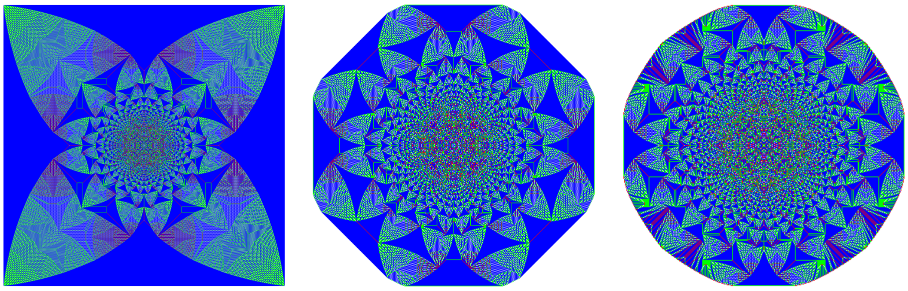
6. Conjectures and Open Problems

In this section we discuss some natural questions about chip-firing and rotor-routing that remain unanswered.

Fey-den Boer and Redig [FR07] consider **aggregation** in the sandpile model on $\mathbb{Z}^d$. In their setup, the underlying graph for the chip-firing game is the infinite undirected d-dimensional cubic lattice $\mathbb{Z}^d$. Start with each site containing $h \leq 2d-2$ chips. Here h may be even be taken negative, corresponding to starting with a "hole" of depth $H = -h$ at each lattice site; that is, each site absorbs the first H chips it receives, and thereafter fires every time it receives an additional four chips. Now add n chips to the origin and stabilize. Denote by $S_{n,H}$ the set of sites in $\mathbb{Z}^d$ which fired in the process of stabilization.

Theorem 6.1 ([FR07])**.** *Let $\mathcal{C}(r)$ denote the cube of side length $2r+1$ centered at the origin in $\mathbb{Z}^d$. For each n there exists an integer r_n such that $S_{n,2-2d} = \mathcal{C}(r_n)$.*

In two dimensions, Theorem 6.1 states that $S_{n,-2}$ is a square. Simulations indicate that for general $H \geq -2$, the limiting shape of $S_{n,H}$ in $\mathbb{Z}^2$ may be a polygon with $4H+12$ sides.



$H = -2$ $H = -1$ $H = 0$

FIGURE 11. Sandpile aggregates of 250,000 chips in $\mathbb{Z}^2$ at hole depths $H = -2$ (left), $H = -1$ (center), and $H = 0$.

Question 6.2. *In $\mathbb{Z}^2$, is the limiting shape of $S_{n,H}$ as $n \to \infty$ a regular $(4H+12)$-gon? Simulations indicate a regular $(4H+12)$-gon with some "rounding" at the corners; it remains unclear if the rounded portions of the boundary become* *negligible in the limit. Even if the limiting shape is not a polygon, it would still be very interesting to establish the weaker statement that it has the dihedral symmetry* D_{4H+12}.

The square, octagon and dodecagon corresponding to the cases $H = -2, -1, 0$ are illustrated in Figure 11. Regarding Question 6.2, we should note that even the existence of a limiting shape for $S_{n,H}$ has not been proved in the case $H > -2$. On the other hand, as $H \to \infty$ the limiting shape is a ball in all dimensions, as shown by Fey and Redig [FR07] and strengthened in [LP07b]. In the theorem below, ω_d denotes the volume of the unit ball in $\mathbb{R}^d$, and B_r denotes the discrete ball

$$B_r = \{x \in \mathbb{Z}^d \mid x_1^2 + \ldots + x_d^2 < r^2\}.$$

Theorem 6.3. [LP07b]. *Fix an integer $H \geq 2 - 2d$. Let $S_{n,H}$ be the set of sites in $\mathbb{Z}^d$ that fire in the process of stabilizing n particles at the origin, if every lattice site begins with a hole of depth H. Write $n = \omega_d r^d$. Then*

$$B_{c_1 r - c_2} \subset S_{n,H}$$

where

$$c_1 = (2d - 1 + H)^{-1/d}$$

and c_2 is a constant depending only on d. Moreover if $H \geq 1 - d$, then for any $\epsilon > 0$ we have

$$S_{n,H} \subset B_{c_1' r + c_2'}$$

where

$$c_1' = (d - \epsilon + H)^{-1/d}$$

and c_2' is independent of n but may depend on d, H and ϵ.

In particular, note that the ratio $c_1/c_1' \uparrow 1$ as $H \uparrow \infty$.

For many classes of graphs, the identity element of the sandpile group has remarkable properties that are not well understood. Let I_n be the identity element of the $n \times n$ grid graph G_n with wired boundary; the states I_n for four different values of n are pictured in Figure 4. Comparing the pictures of I_n for different values of n, one is struck by their extreme similarity. In particular, we conjecture that as $n \to \infty$ the pictures converge in the following sense to a limiting picture on the unit square $[0, 1] \times [0, 1]$.

Conjecture 6.4. *Let a_n be a sequence of integers such that $a_n \uparrow \infty$ and $\frac{a_n}{n} \downarrow 0$. For $x \in [0, 1] \times [0, 1]$ let*

$$f_n(x) = \frac{1}{a_n^2} \sum_{\substack{y \in G_n \\ ||y - nx|| \leq a_n}} I_n(y).$$

There is a sequence a_n and a function $f : [0, 1] \times [0, 1] \to \mathbb{R}_{\geq 0}$ which is locally constant almost everywhere, such that $f_n \to f$ at all continuity points of f.

Most intriguing is the apparent fractal structure in the conjectural f. Recent progress has been made toward understanding the fractal structure of the identity element of a certain orientation of the square grid; see [CPS07].

By Lemma 4.9, the recurrent orbits of the rotor-router operation on an Eulerian digraph are extremely short: although the number of unicycles is typically exponential in the number of vertices, the orbits are all of size equal to the number of edges. One would expect that such short orbits are not the norm for general digraphs.

Question 6.5. *Does there exist an infinite family of non-Eulerian strongly connected digraphs G_n, such that for each n, all the unicycles of G_n lie in a single orbit of the rotor-router operation?*

Another question stemming from Lemma 4.9 is the following. Fix two edges e_0 and e_1 of a digraph G. Starting from a unicycle on G, record a 0 each time the chip traverses the edge e_0, and record a 1 each time it traverses e_1. If G is Eulerian, then Lemma 4.9 implies that the resulting sequence will simply alternate $0, 1, 0, 1, \ldots$. For a general digraph, the sequence is periodic, since the initial unicycle must recur; what can be said about the period?

Lastly, the articles [PDDK96] and [PPS98] contain several conjectures that are supported by both credible heuristics and computer experiments, but that have not been rigorously proved. For instance, it appears that, with random initial rotor orientations, the set of sites visited by a rotor-router walk of length n in the plane typically has diameter on the order of $n^{1/3}$ [PDDK96] (compare this with the corresponding growth rate for random walk in the plane, which is $n^{1/2}$).

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