Random Sorting Networks

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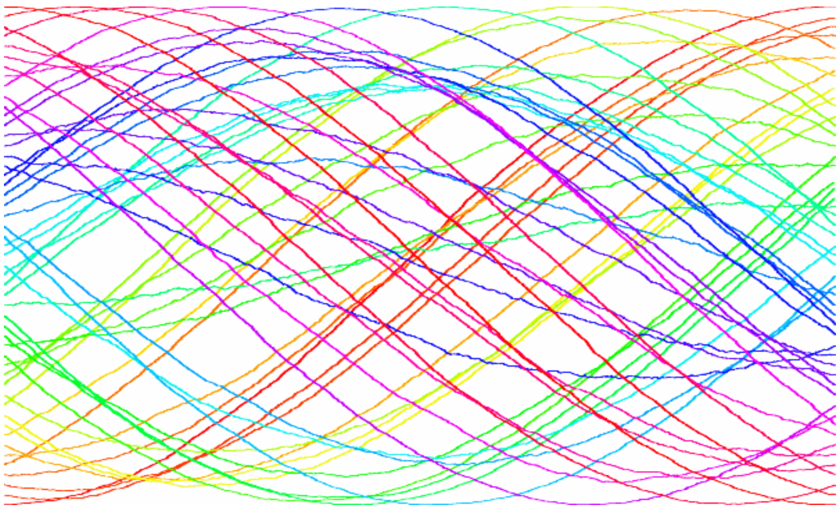
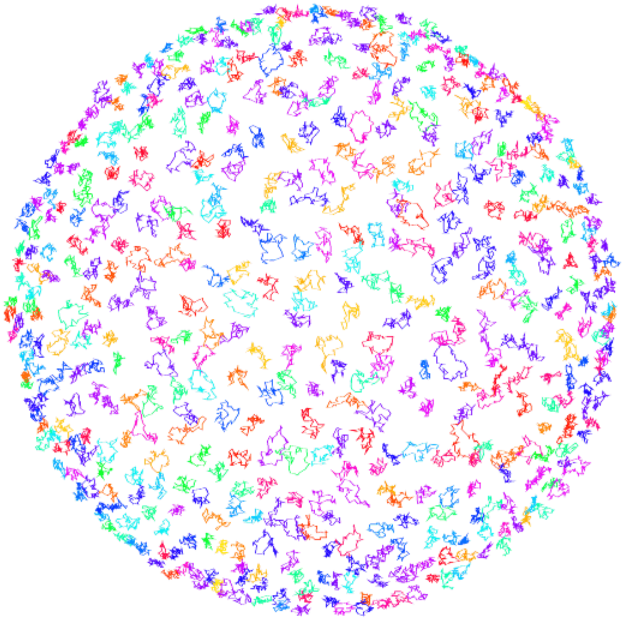
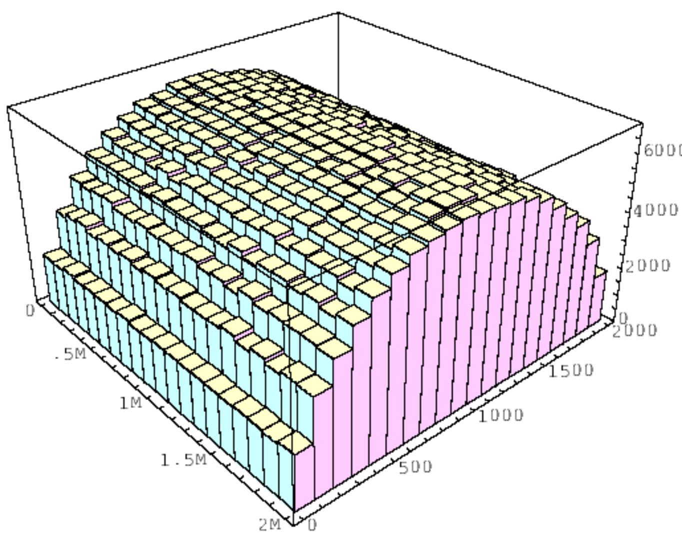
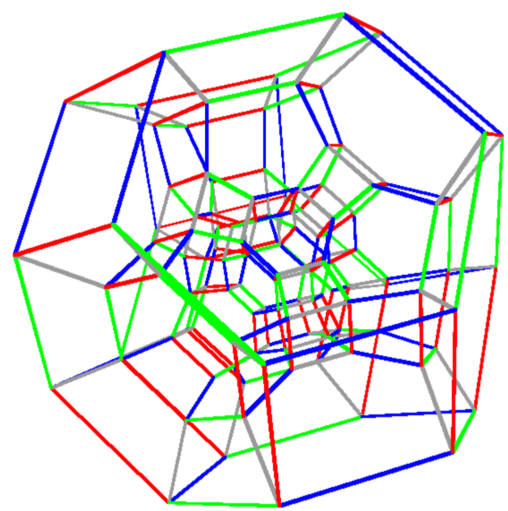
with: Omer Angel, Dan Romik, Balint Virag, Vadim Gorin





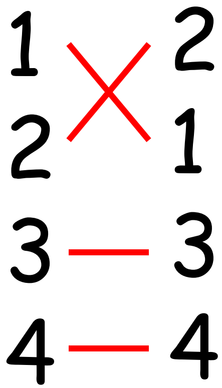




1
2
3
4

1
2
3
4

4
3
2
1



4
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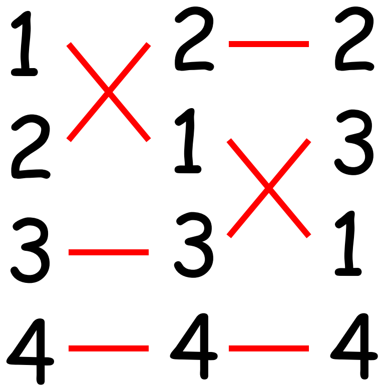
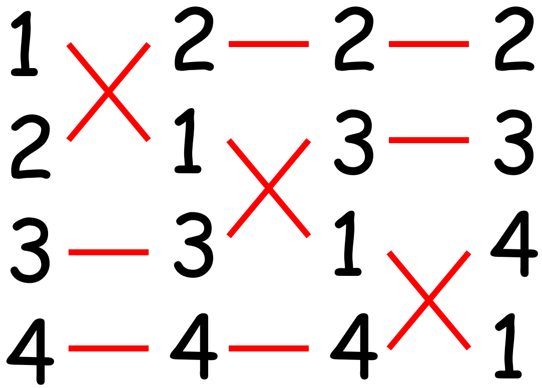


4

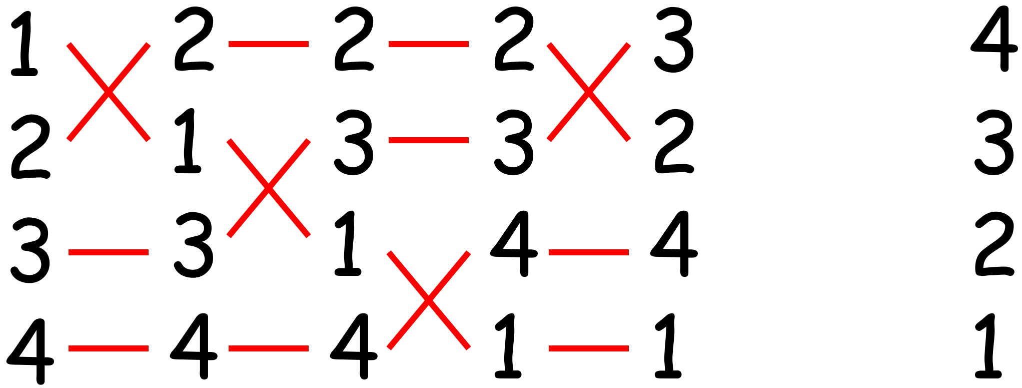
3

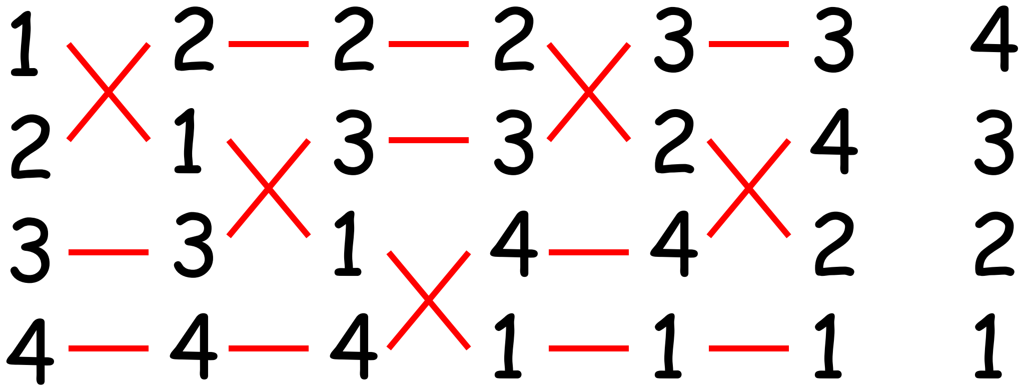
2

1

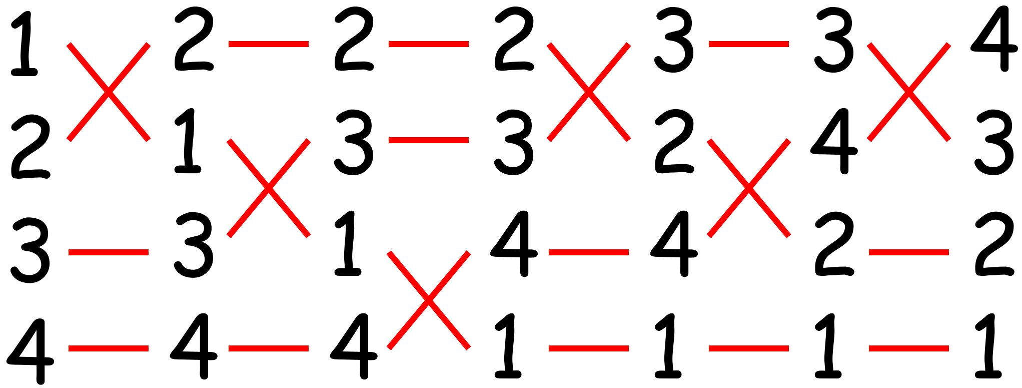


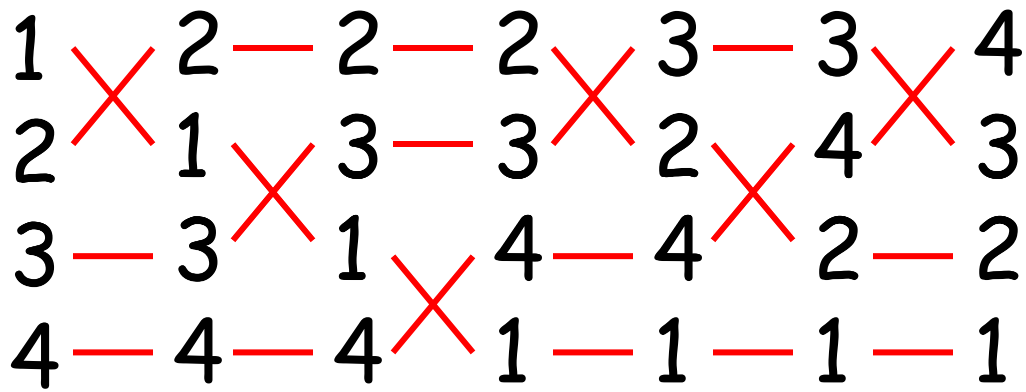
4
3
2
1


1 2 2 2 3 4
2 1 3 3 2 3
3 3 1 4 4 2
4 4 4 1 1 1



1 2 2 2 3 3 4
2 1 3 3 2 4 3
3 3 1 4 4 2 2
4 4 4 1 1 1 1


1 2 2 2 3 3 4
2 1 3 3 2 4 3
3 3 1 4 4 2 2
4 4 4 1 1 1 1

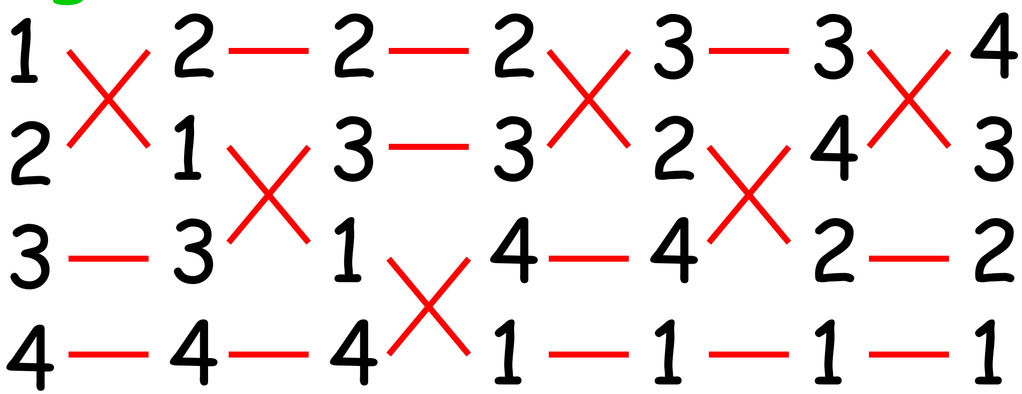


To get from $1\cdots n$ to $n\cdots 1$ requires

$$N := \binom{n}{2}$$

nearest-neighbour swaps

E.g. n=4:



<u>A Sorting Network</u> =

any route from $1 \cdots n$ to $n \cdots 1$

in exactly

$$N := \binom{n}{2}$$

nearest-neighbour swaps

Theorem (Stanley 1984).

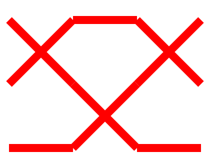
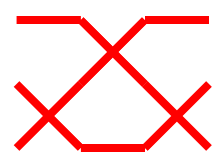
# of n-particle sorting networks =

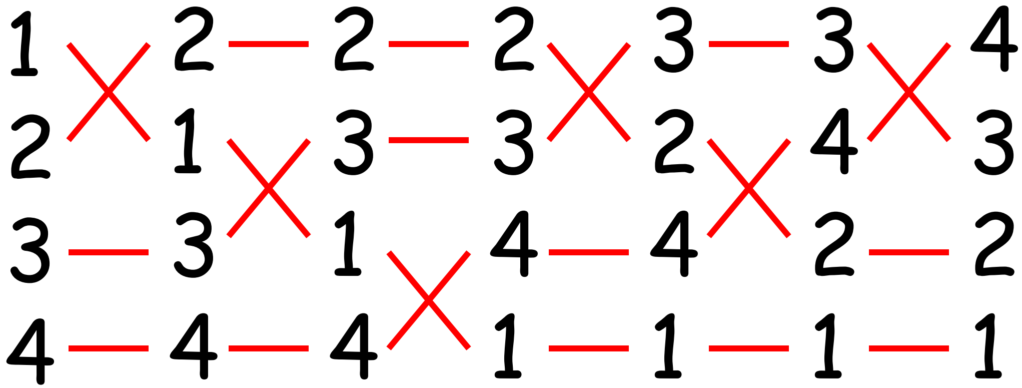
$$\frac{\binom{n}{2}!}{1^{n-1}3^{n-2}5^{n-3}\cdots(2n-3)^1}$$

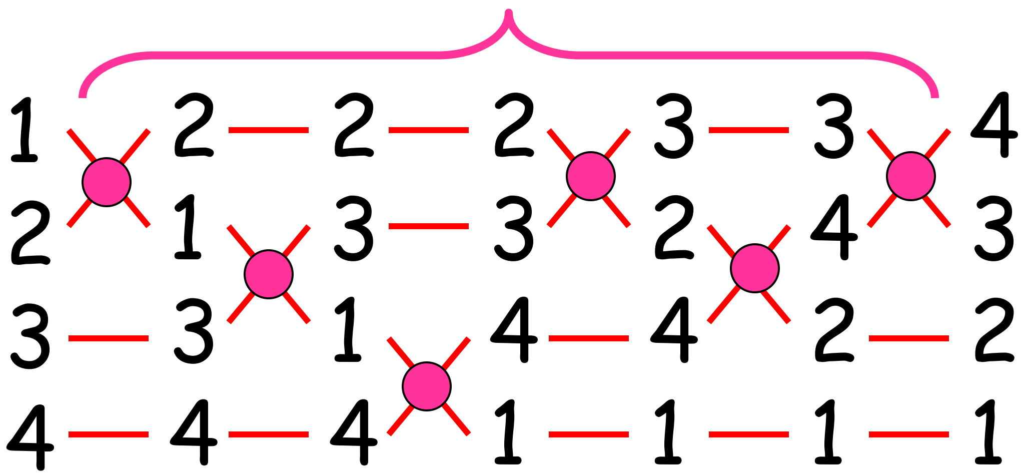
Uniform Sorting Network (USN):

choose an n-particle sorting network uniformly at random.

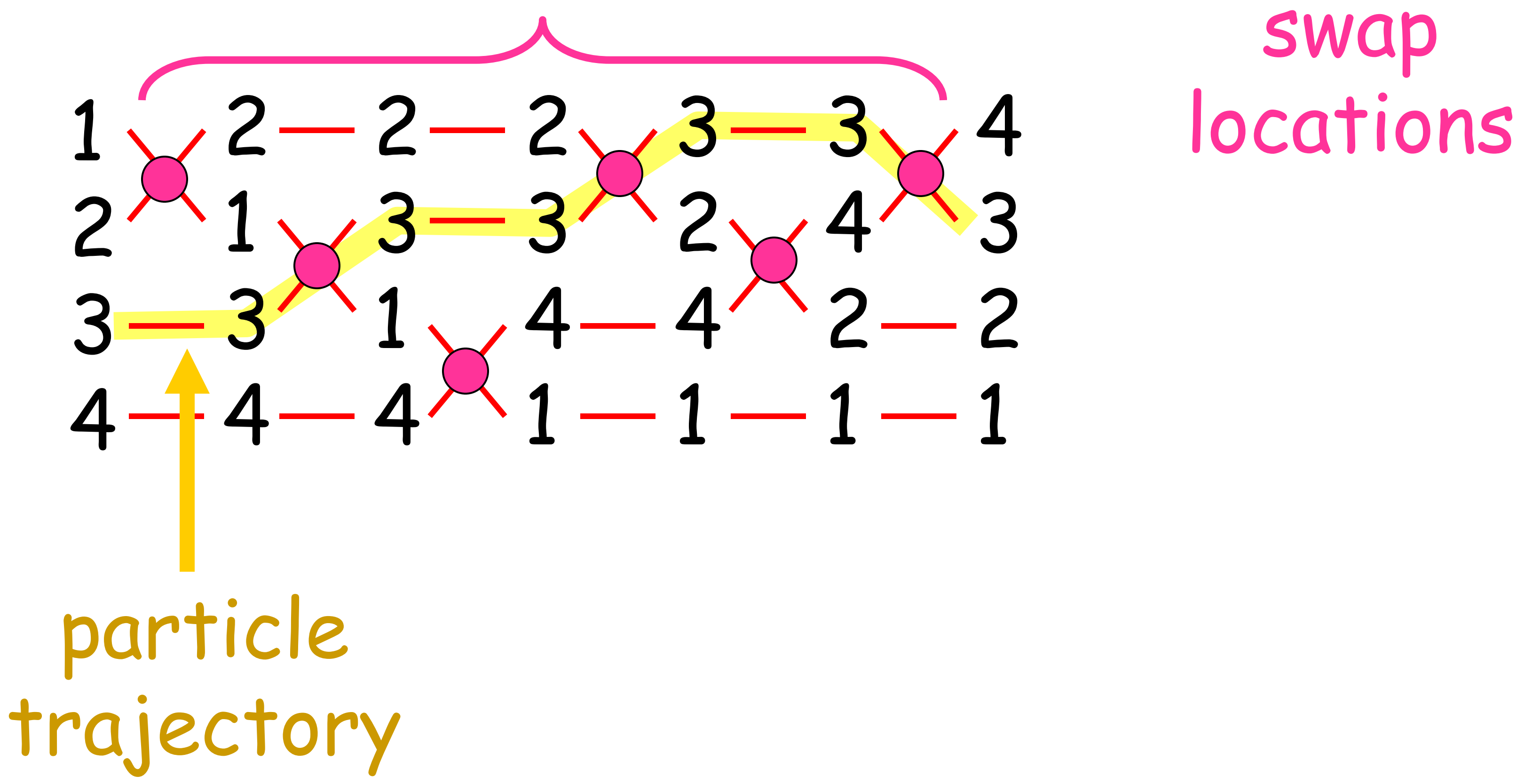
E.g. n=3:

$P\Big($  $\Big) = P\Big($  $\Big) = \frac{1}{2}$

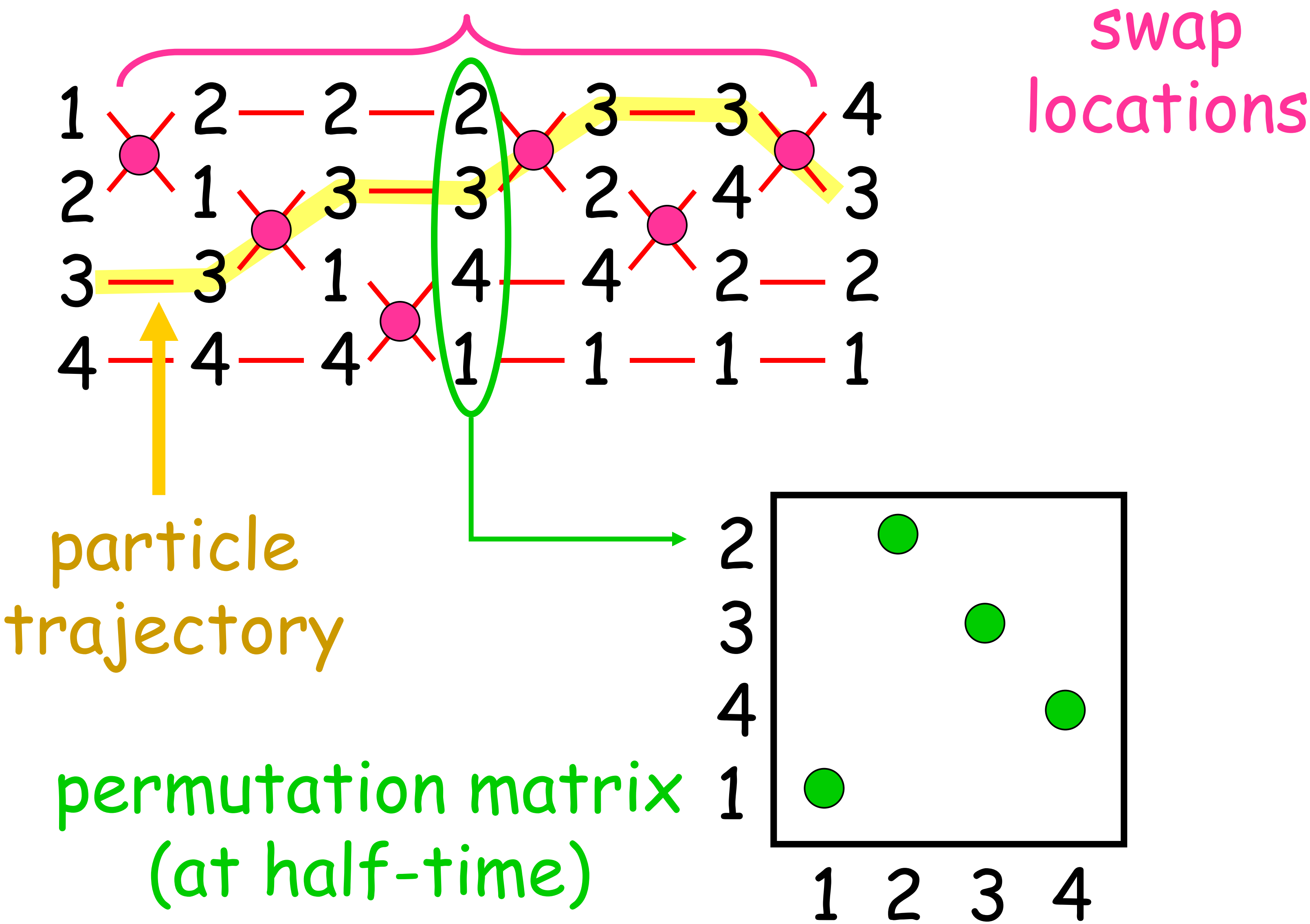

1 2 2 2 3 3 4
2 1 3 3 2 4 3
3 3 1 4 4 2 2
4 4 4 1 1 1 1



swap locations

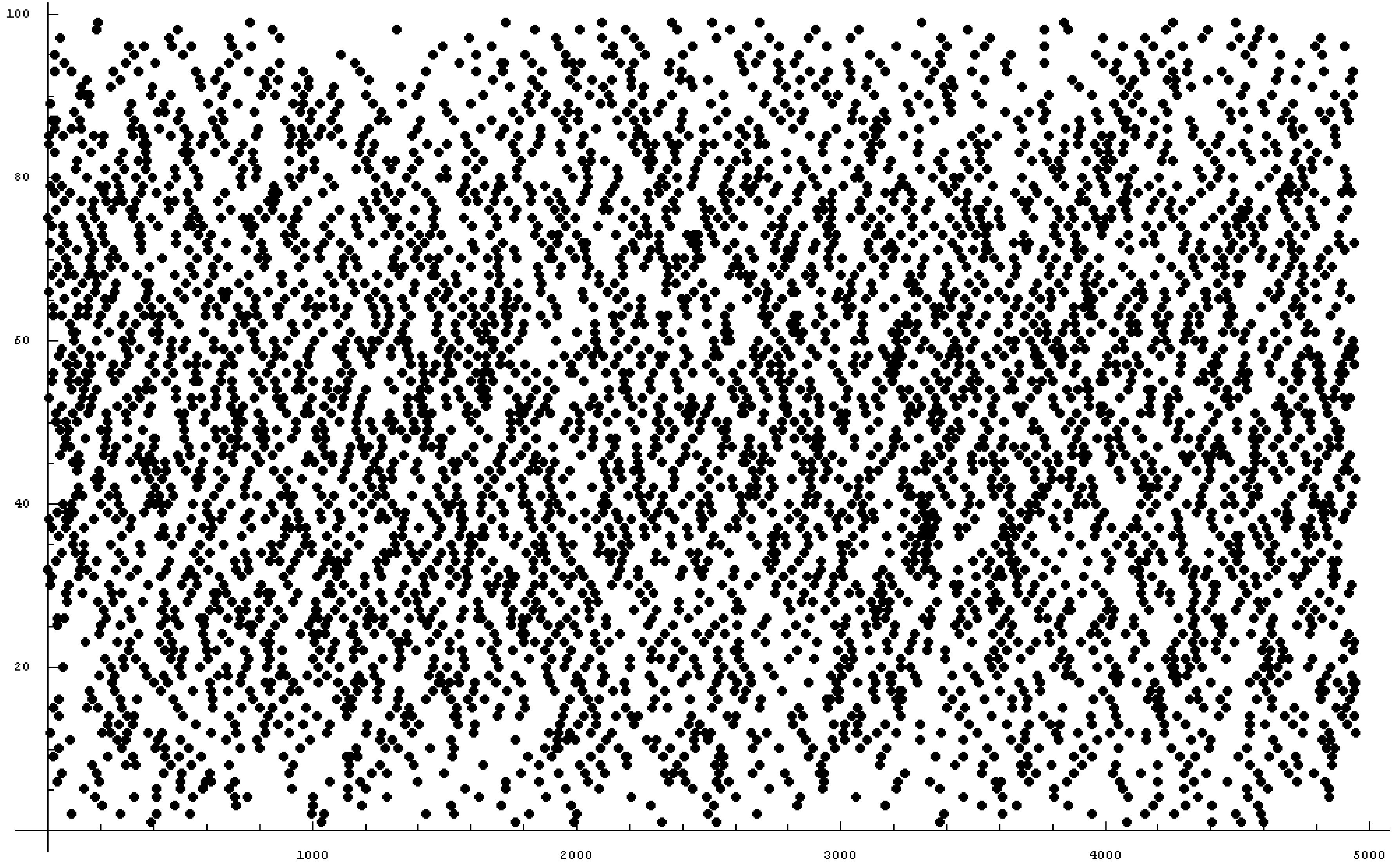


swap locations
1 2 2 2 3 3 4
2 1 3 3 2 4 3
3 3 1 4 4 2 2
4 4 4 1 1 1 1
particle trajectory

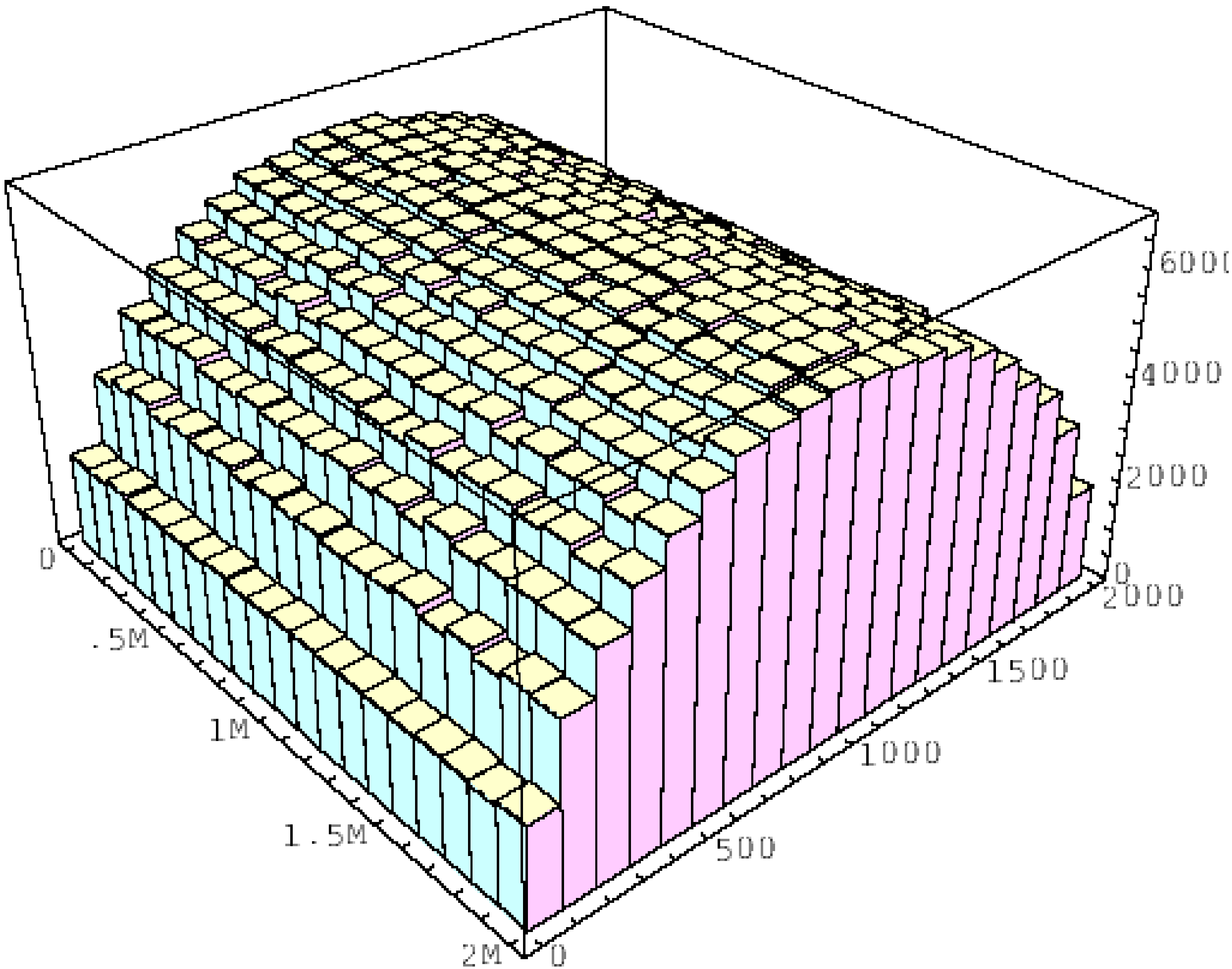


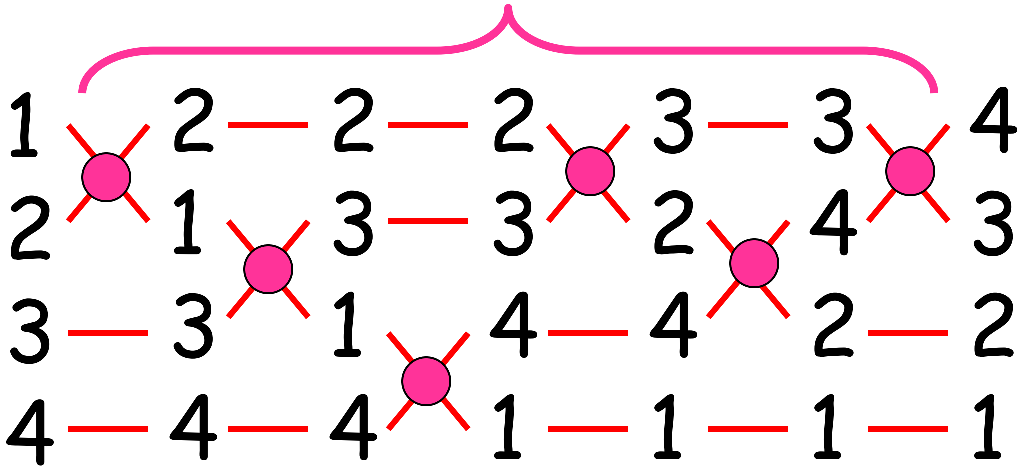
∃ efficient simulation algorithm for USN...

Swap locations, n=100



Swap locations, n=2000





swap locations

s_1	s_2	s_3	s_4	s_5	s_6
=	=	=	=	=	=
1	2	3	1	2	1

Theorem(Angel,H,Romik,Virag,2007) For USN:

1. Sequence of swap locations $(s_1,\ldots,s_N)$ is stationary $\forall$ n

2. Scaled first swap location

$$\frac{s_1}{n} \xrightarrow{dist} \text{semicircle random variable}$$

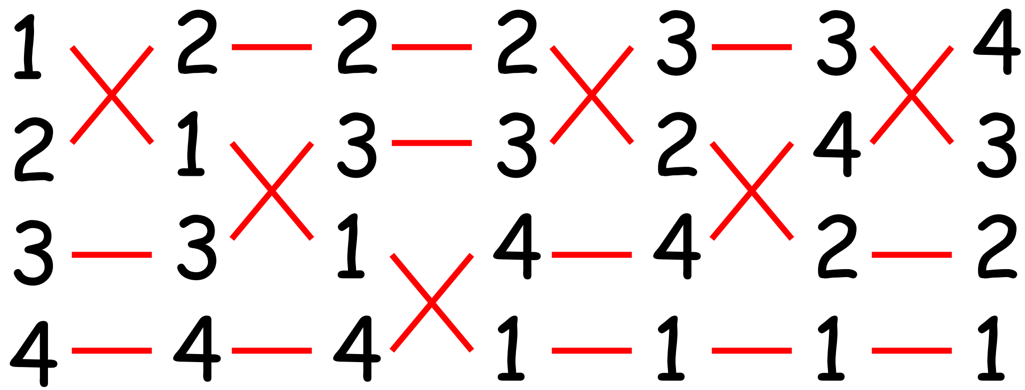
as $n \to \infty$

3. Scaled swap process

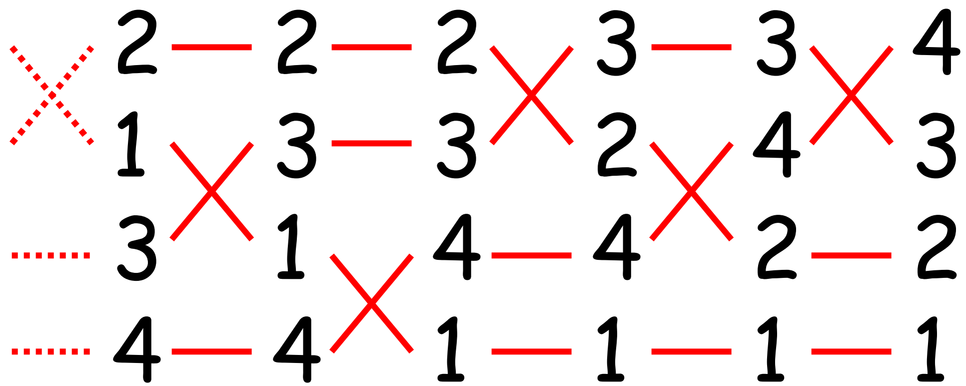
$$\overset{dist}{\Rightarrow} \text{semicircle} \times \text{Lebesgue}$$ as $n \to \infty$

(Note: not true for *all* sorting networks, e.g. bubble sort)

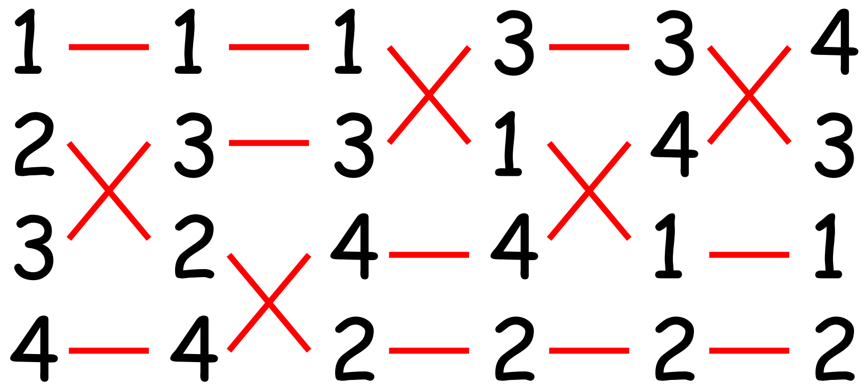
Proof of stationarity:



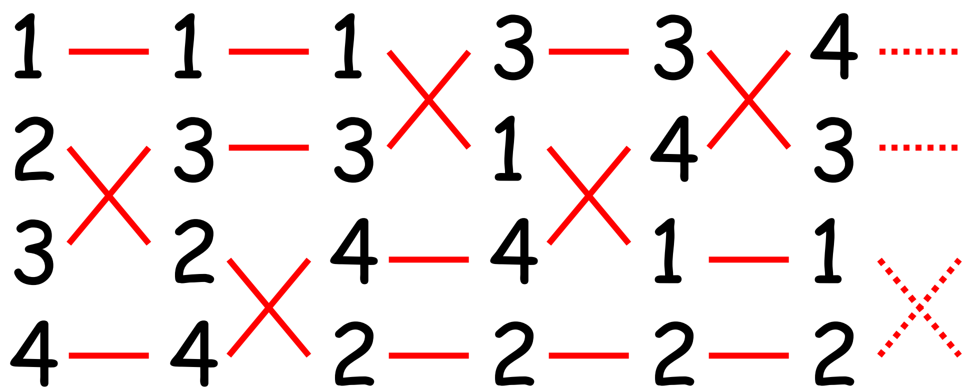
Proof of stationarity:



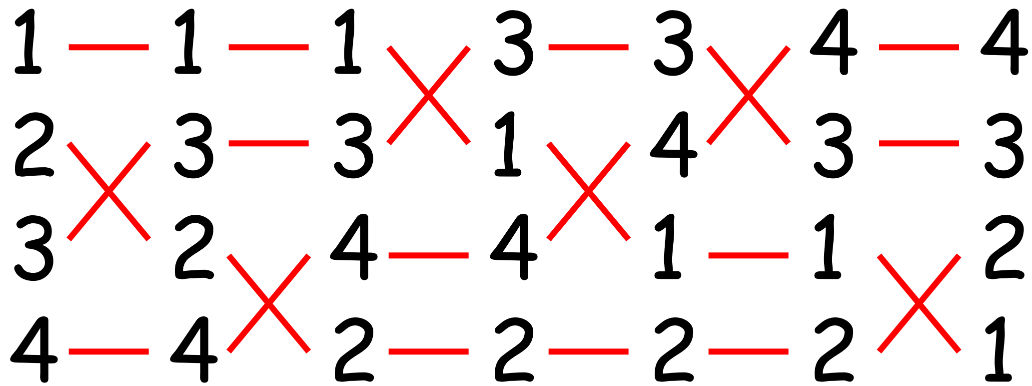
Proof of stationarity:



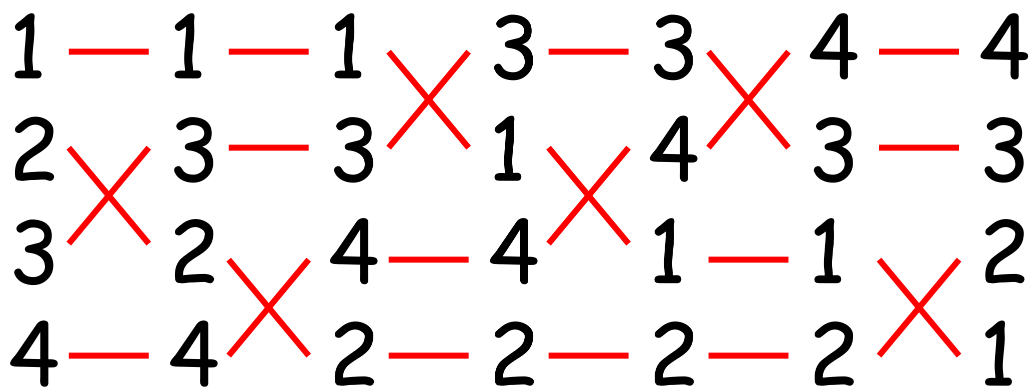
Proof of stationarity:



<u>Proof</u> of stationarity:



Proof of stationarity:



$(s_1,\dots,s_N) \mapsto (s_2,\dots,s_N,n-s_1)$ is a bijection from {sorting networks} to itself.

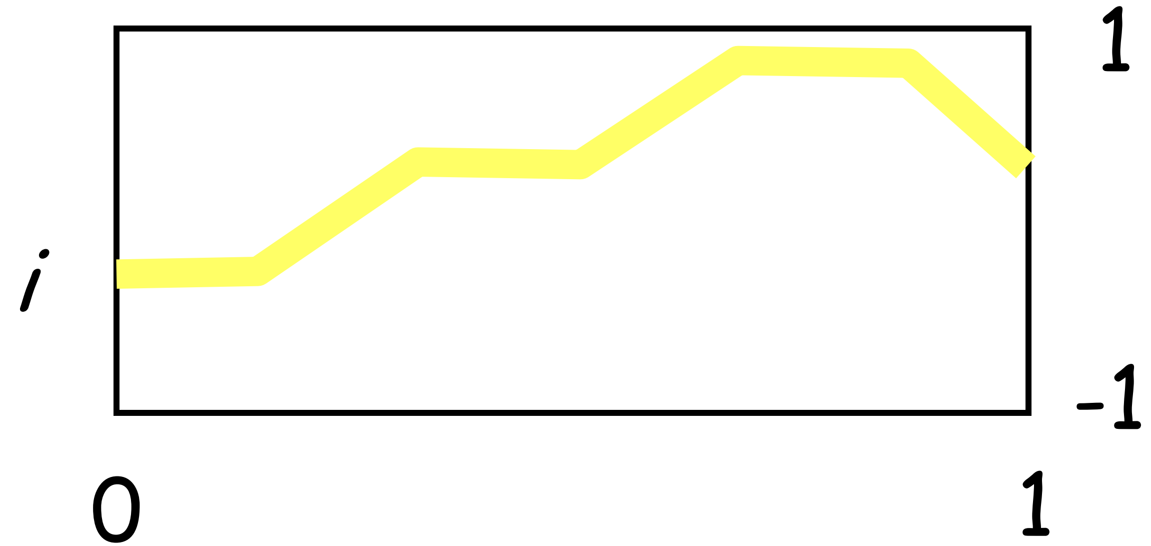
So for USN:

$$(s_2,\dots,s_N) \overset{d}{=} (s_1,\dots,s_{N-1}) \qquad \square$$

Selected trajectories, n=2000

Scaled trajectory of particle *i*:

$T_i:[0,1]\to[-1,1]$



Conjecture (AHRV)

trajectories $\to$ random Sine curves:

$$\max_{i,t} |T_i(t) - A_i^n \sin(\pi t + \Theta_i^n)| \xrightarrow{Prob} 0$$

(random)

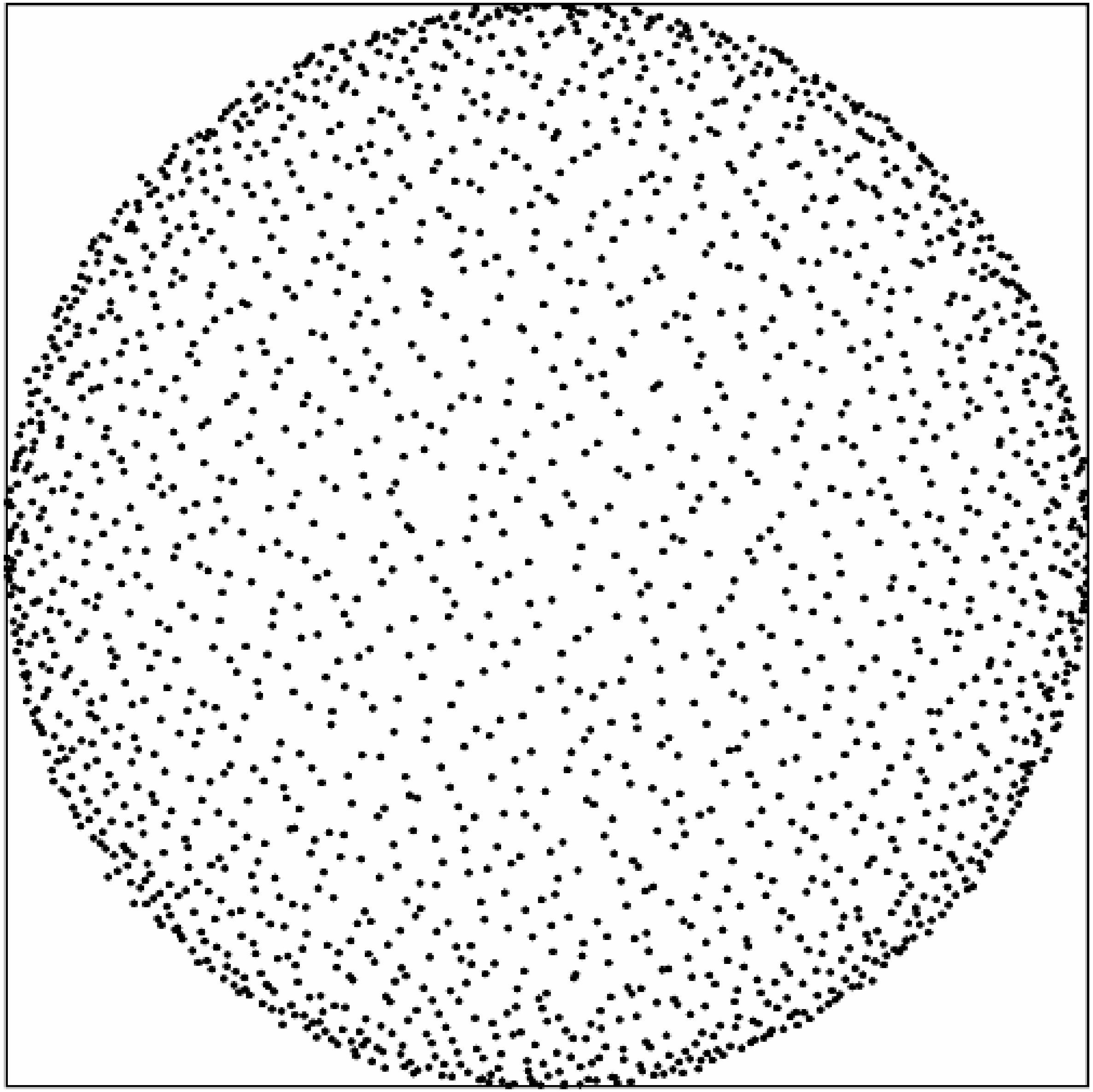
as $n \to \infty$

Theorem (AHRV)

scaled trajectories have subsequential limits which are Hölder($\frac{1}{2}$) with prob 1

as $n \to \infty$

Half-time permutation matrix, n=2000



animation

Conjecture (AHRV)

scaled permutation matrix at time N/2 $\overset{d}{\Longrightarrow}$ Archimedes measure

projection of surface area measure on sphere $S^2 \subset \mathbb{R}^3$ onto $\mathbb{R}^2$

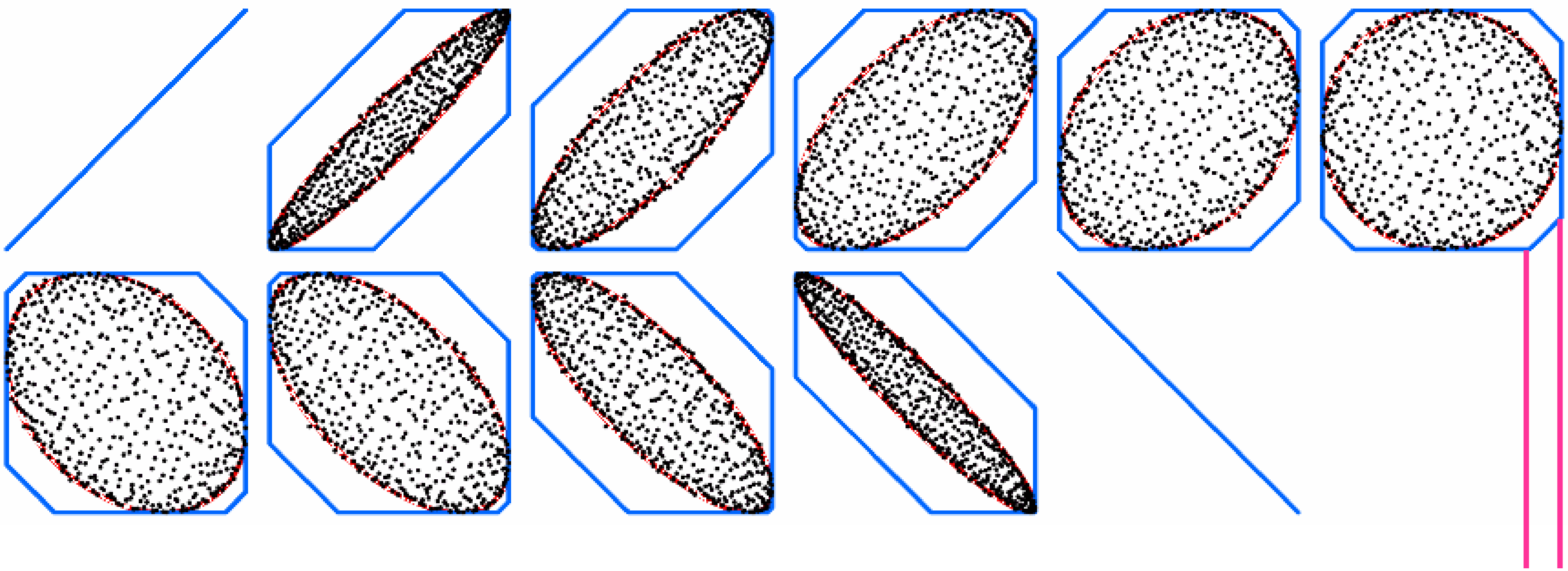
(unique circularly symmetric measure with uniform linear projections;

$$\frac{dx\ dy}{2\pi\sqrt{1-x^2-y^2}}$$ on $x^2+y^2<1$)

scaled permutation matrix at time tN $\overset{d}{\Longrightarrow} \begin{pmatrix} 1 & 0 \\ \cos \pi t & \sin \pi t \end{pmatrix} \circ$ Arch. meas.

Theorem (AHRV)

scaled permutation matrix at time tN
is supported within a certain octagon
with prob $\to 1$ as $n \to \infty$

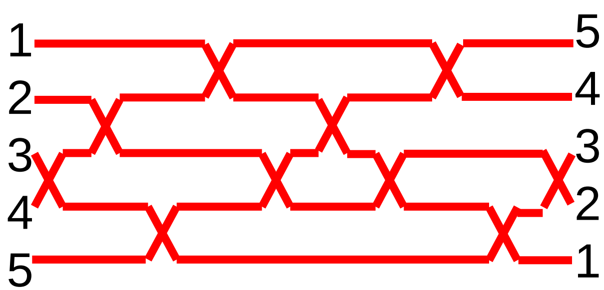


$(1-\frac{1}{2}\sqrt{3}-\varepsilon)n$

Tools in proofs:

1. Bijection (Edelman-Greene 1987)

{sorting networks} $\leftrightarrow$ {standard staircase Young tableaux}

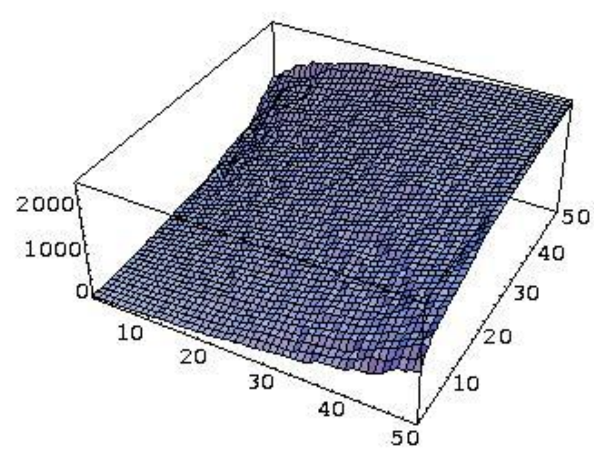


(jeu de taquin algorithm)

1	2	4	8
3	5	6	
7	10		
9			

2. New result for limiting profile of random staircase Young tableau

(from similar result for square tableaux, Pittel-Romik)



Why do we believe the conjectures?

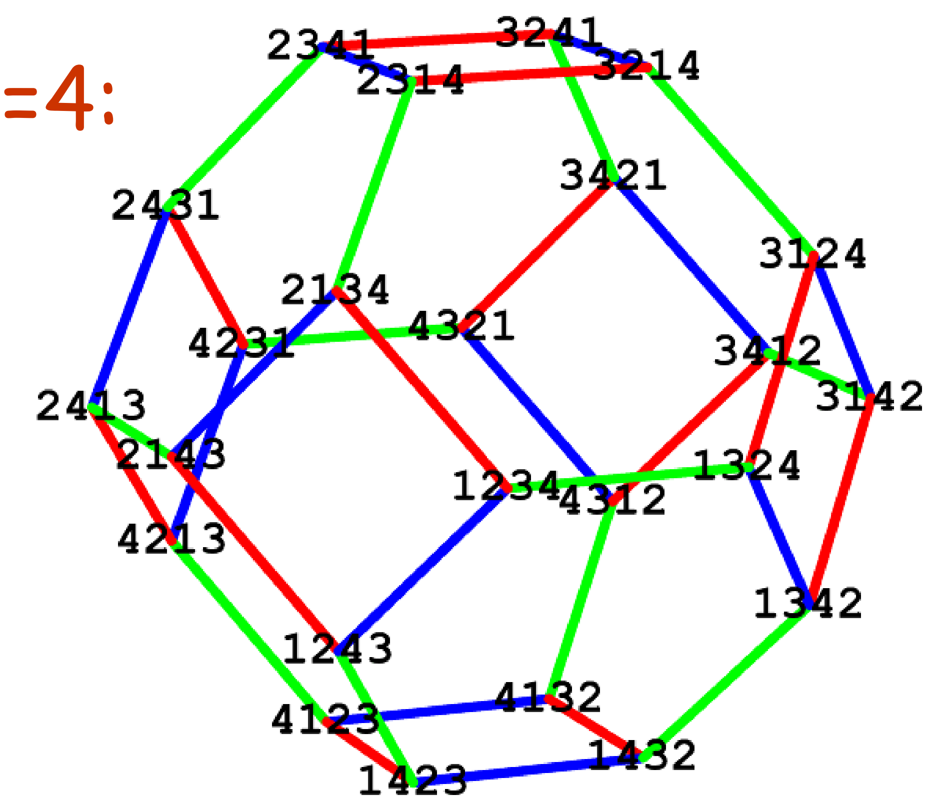
The permutahedron:
embedding of Cayley graph
(S_n, n.n. swaps) in $\mathbb{R}^n$:

$$\sigma \mapsto \sigma^{-1}=(\sigma^{-1}(1),\ldots,\sigma^{-1}(n))\in\mathbb{R}^n$$

n=4:

embeds in
(n-2)-sphere

1…n and n…1
are antipodal



n=5

Conjecture (AHRV)

USN lies close to some great circle on the permutahedron with prob $\to 1$

as $n \to \infty$

e.g. $o(n)$ in $|\ \ |_\infty$

In fact simulations suggest more like $O(\sqrt{n})$!

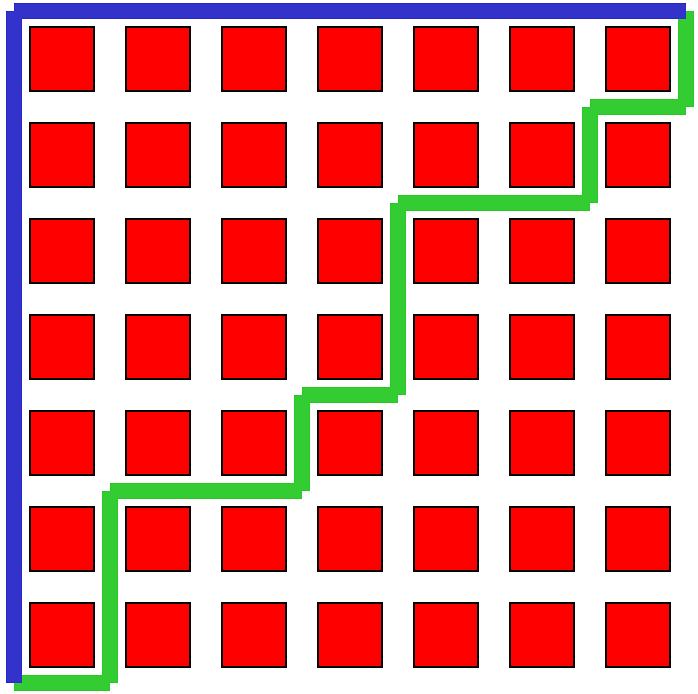
(Again, not true for *every* sorting network, e.g. bubble sort)

Analagous (much easier) fact:

random shortest route

1st St & 1st Ave to nth St & nth Ave

$\approx$ straight line as $n\to\infty$



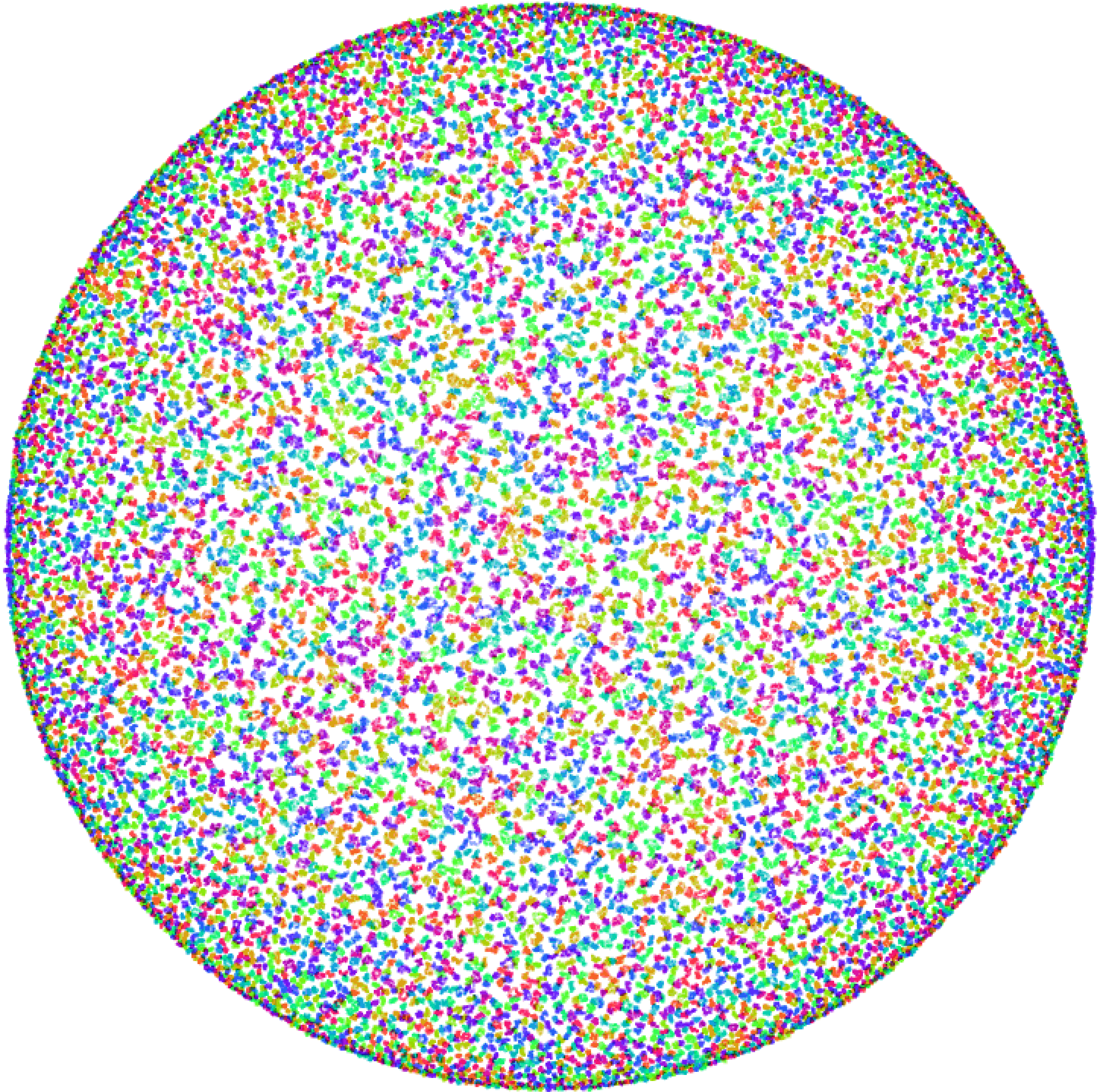
Theorem (AHRV) If a **(non-random)** sorting network lies close to some great circle, then:
(o(n) in $| \ \ |_\infty$)

1. Trajectories $\approx$ Sine curves

2. Half-time permutation $\approx$ Archimedes measure

3. Swap process $\approx$ semicircle x Lebesgue

Simulation



Proof of Theorem:

close to great circle $\Rightarrow$

$\approx$ Sine trajectories (up to a time change)

$\Leftrightarrow$ $\approx$ rotating disc picture

projections uniform $\Rightarrow$ $\approx$ Archimedes

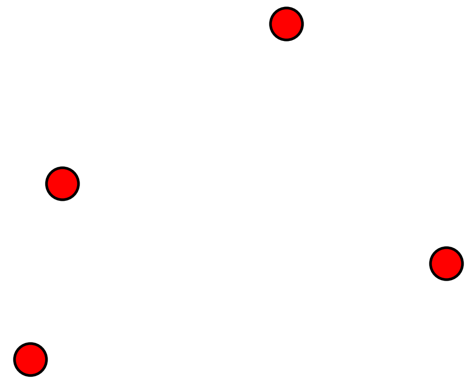
swap rate uniform $\Rightarrow$ rotation uniform

$\Rightarrow$ no time change

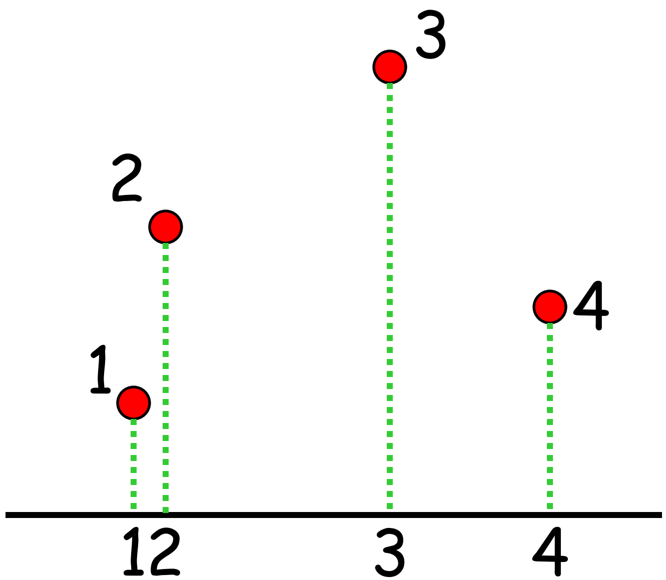
calculation $\Rightarrow$ semicircle law

$\square$

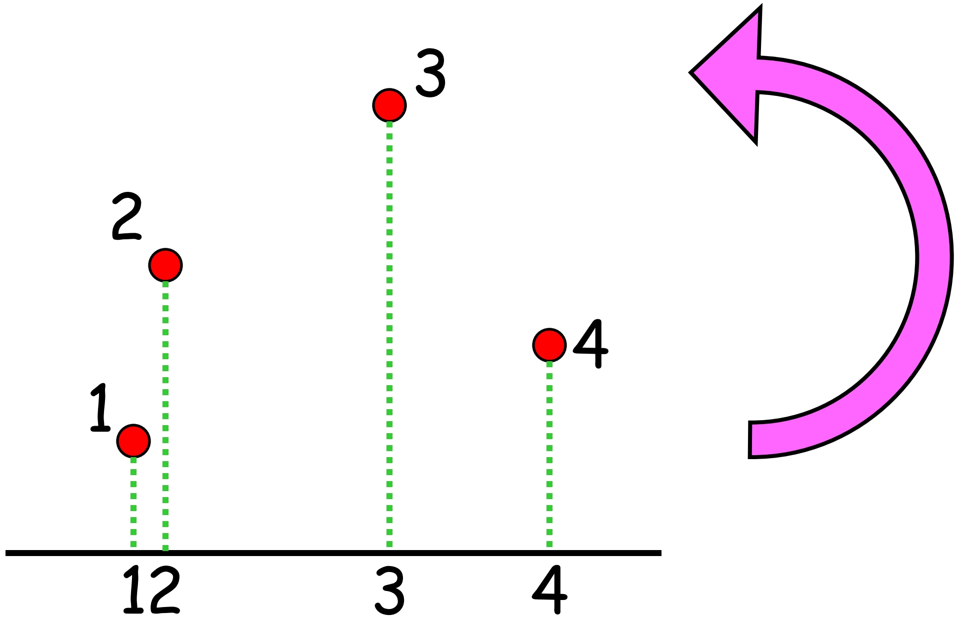
Geometric Sorting Networks



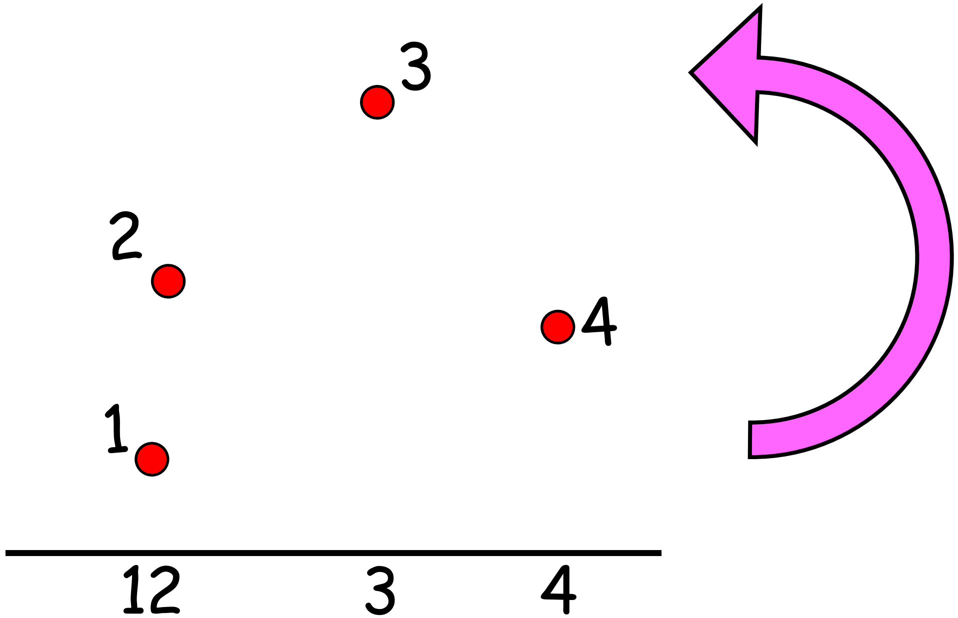
Geometric Sorting Networks



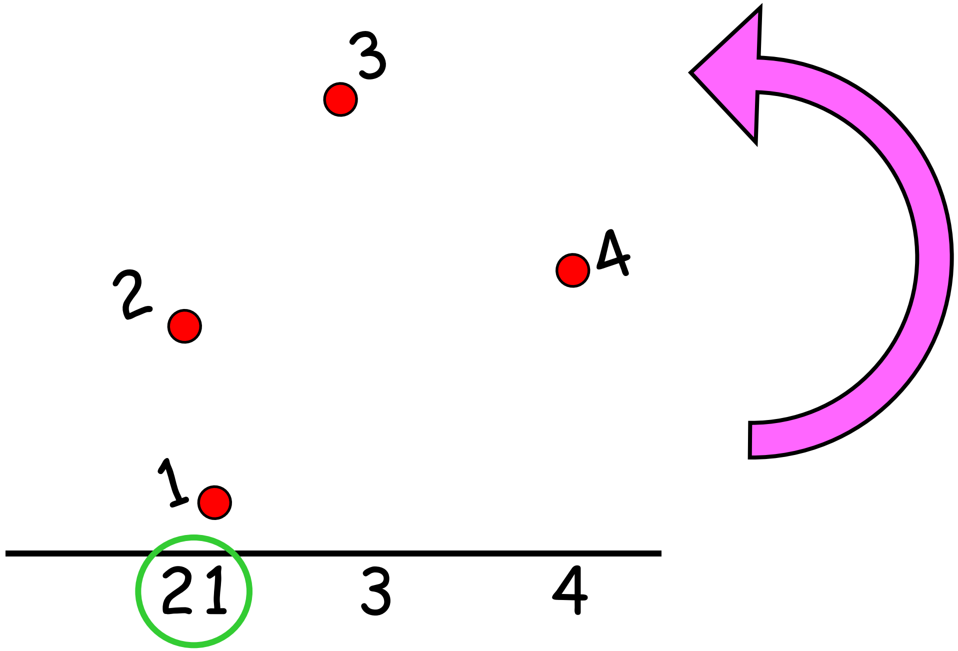
Geometric Sorting Networks



Geometric Sorting Networks



Geometric Sorting Networks



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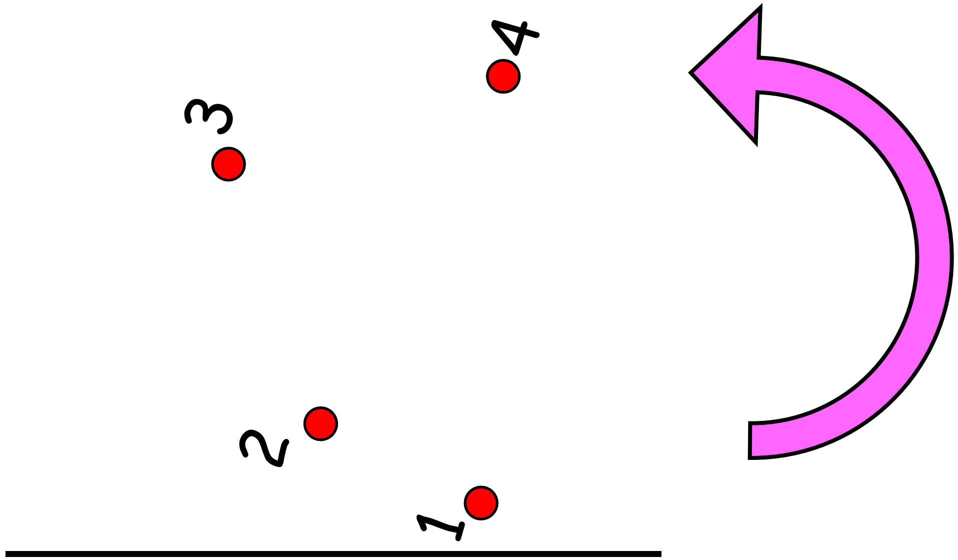
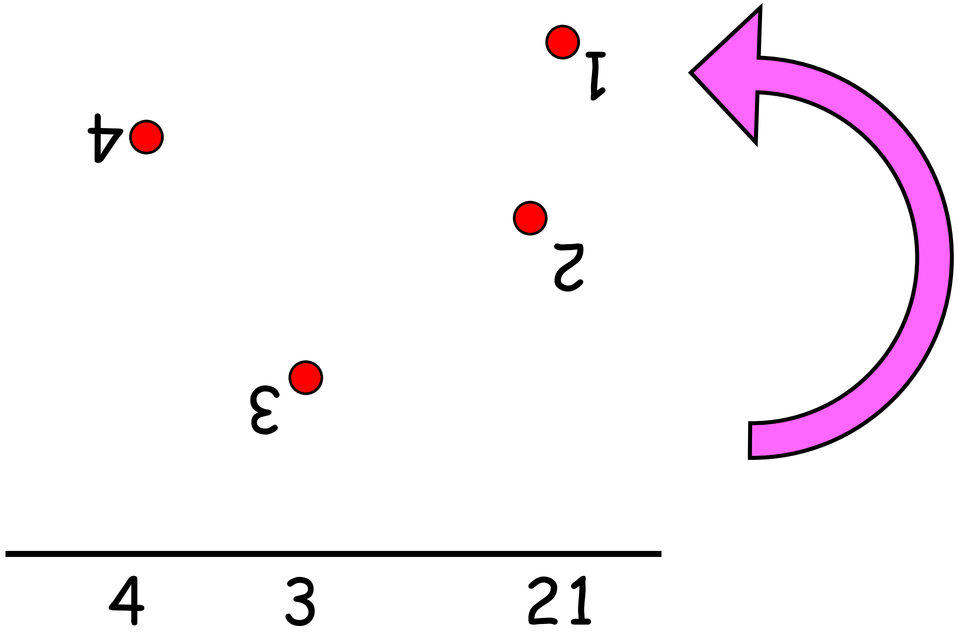
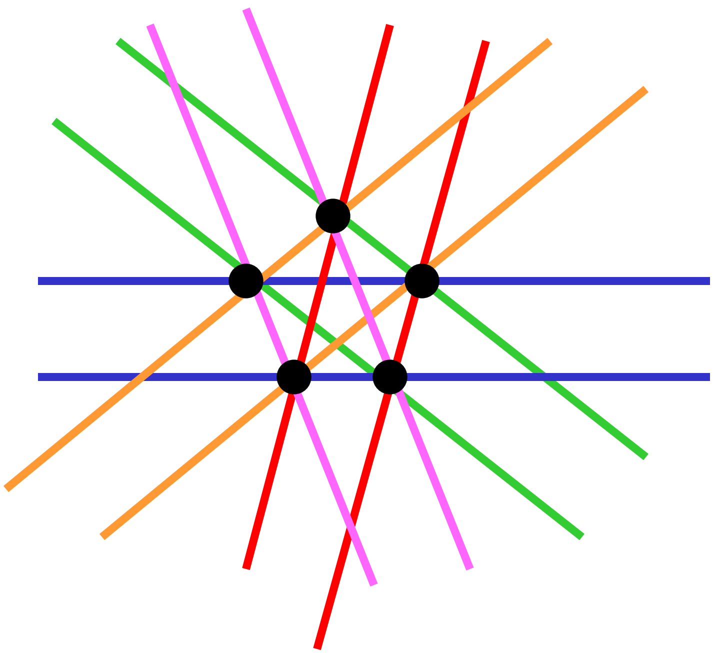


Geometric Sorting Networks



Goodman, Pollack (1980):

- all 4-item sorting networks are geometric
- but not all 5-item ones:



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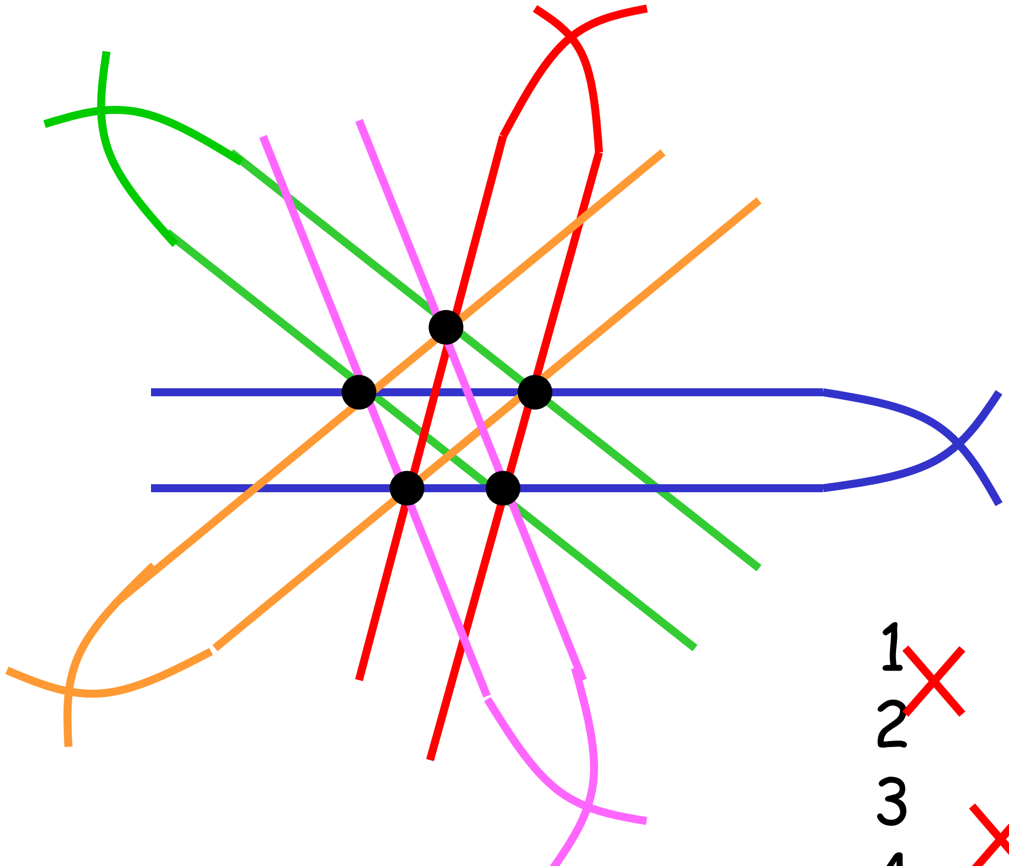
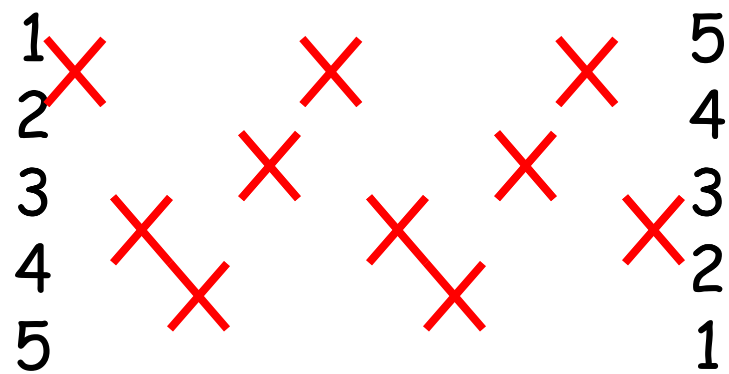




Great circle conjecture says:

USN is "$\approx$ geometric" as $n \to \infty$

but:

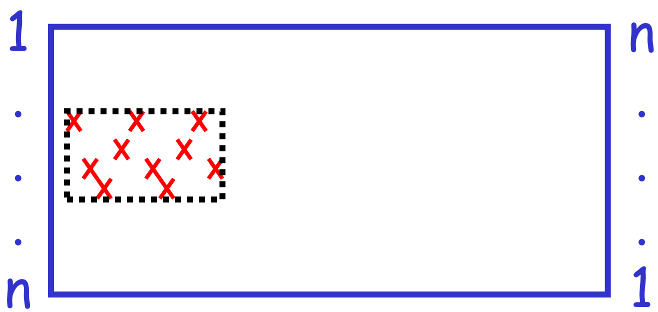
Theorem (Angel, H, Gorin, in prep)

P(USN is geometric) $\to 0$ as $n \to \infty$

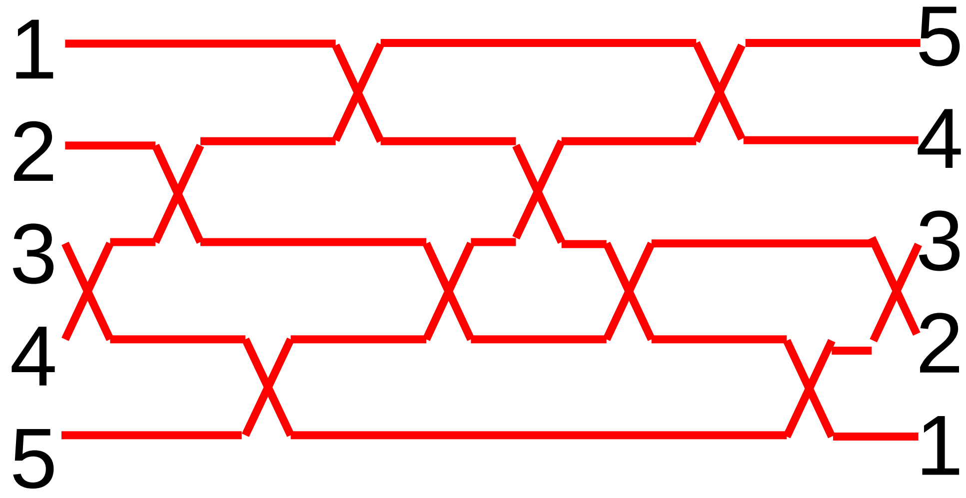
Proof: in fact:

P(USN contains fixed swap pattern) > $1-e^{-cn}$

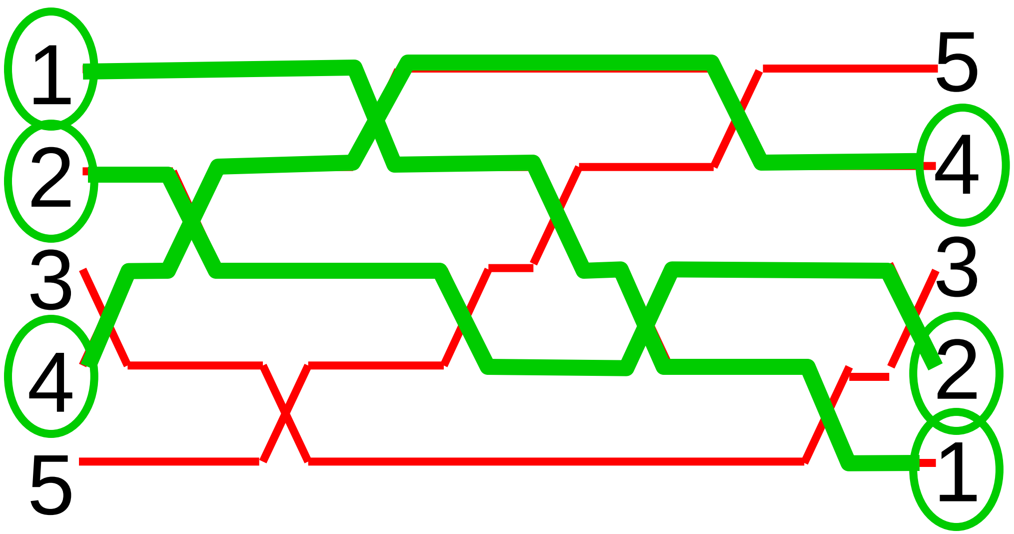
e.g. Goodman-Pollack counterexample



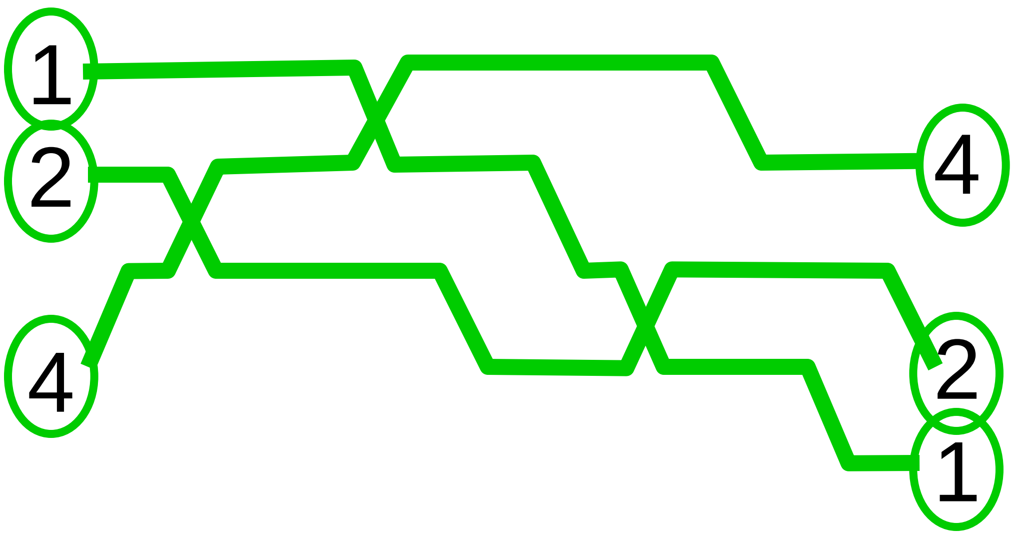
Subnetworks



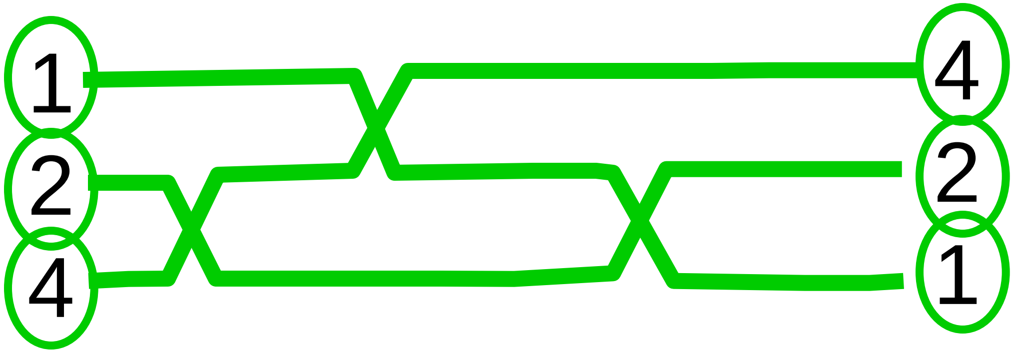
Subnetworks



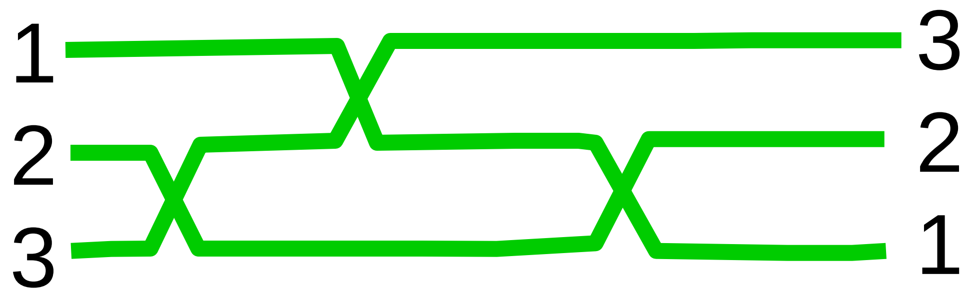
Subnetworks



Subnetworks



Subnetworks



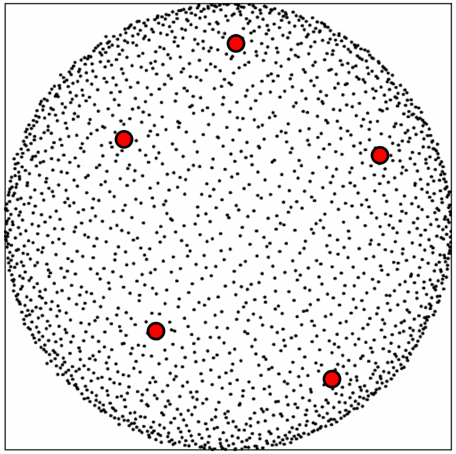
Random Subnetworks

Take an n-item USN. Choose m out of the n items uniformly at random, indep. of USN.

Great circle conjecture $\Rightarrow$

m fixed, $n \to \infty$:

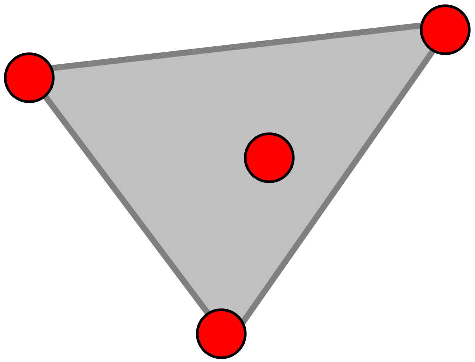
random m-out-of-n network $\xrightarrow{d}$ geom. network of m indep. points from Archimedes distn.



Conjecture (Warrington, 2009)

$$P\left(\begin{array}{c}\text{random}\\ \text{4-out-of-n}\\ \text{network}\end{array} \in \begin{array}{c}\{\text{geom. networks}\\ \text{with 1 point in}\\ \text{hull(other 3)}\}\end{array}\right) = \frac{1}{4}$$

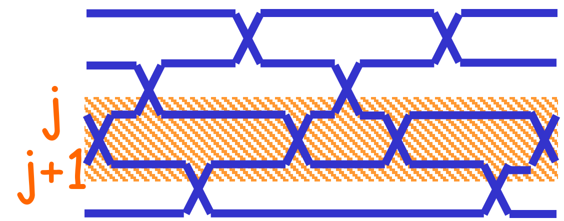
for *all* n !



Theorem (Angel,H 2009)
Warrington's conjecture is true.

Moreover, $\forall$ j < m $\leq$ n,

E(# swaps in location j in random m-out-of-n network)



does not depend on n

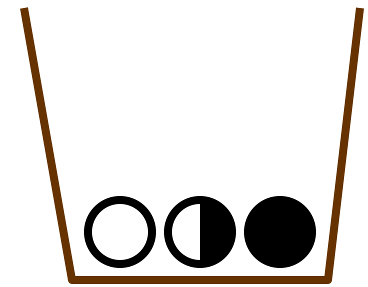
and = $$\frac{(j-\frac{1}{2})\cdots\frac{7}{2}\frac{5}{2}\frac{3}{2}\times(m-j-\frac{1}{2})\cdots\frac{7}{2}\frac{5}{2}\frac{3}{2}}{(j-1)!\times(m-j-1)!}$$

consistent with Archimedes distribution conjecture about n$\rightarrow\infty$ limit

Ingredients of proof

$P(s_1=k)$ = P(k-1 white balls added in first n-2 in **Polya urn**)

1st swap location in USN



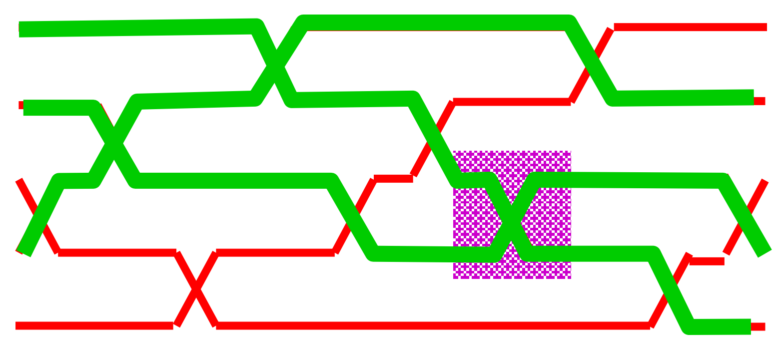
initially $1\frac{1}{2}$W, $1\frac{1}{2}$B

Stationarity of USN

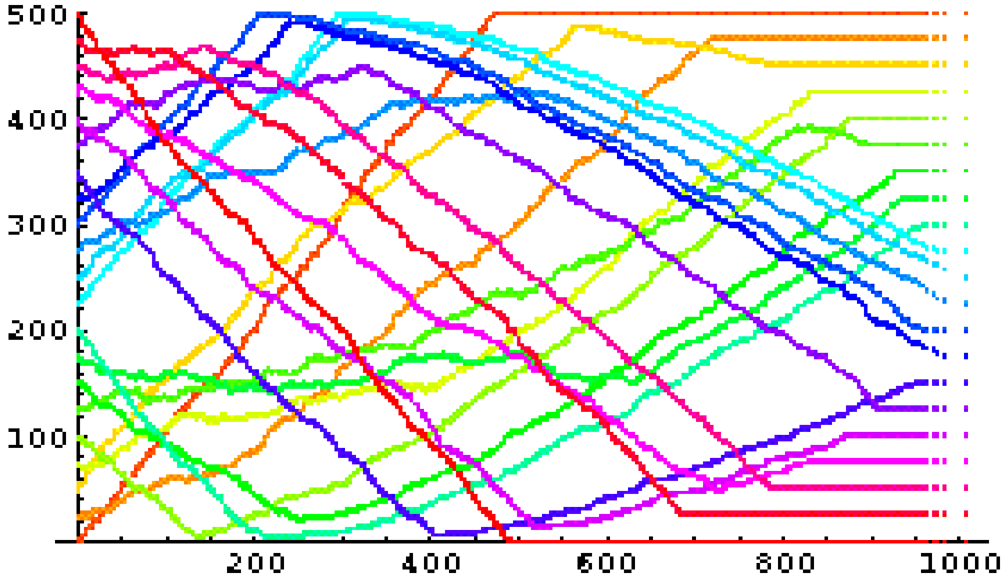
Exchangeability of Polya urn   P(wwwbb) = P(wbwbw)

Compute

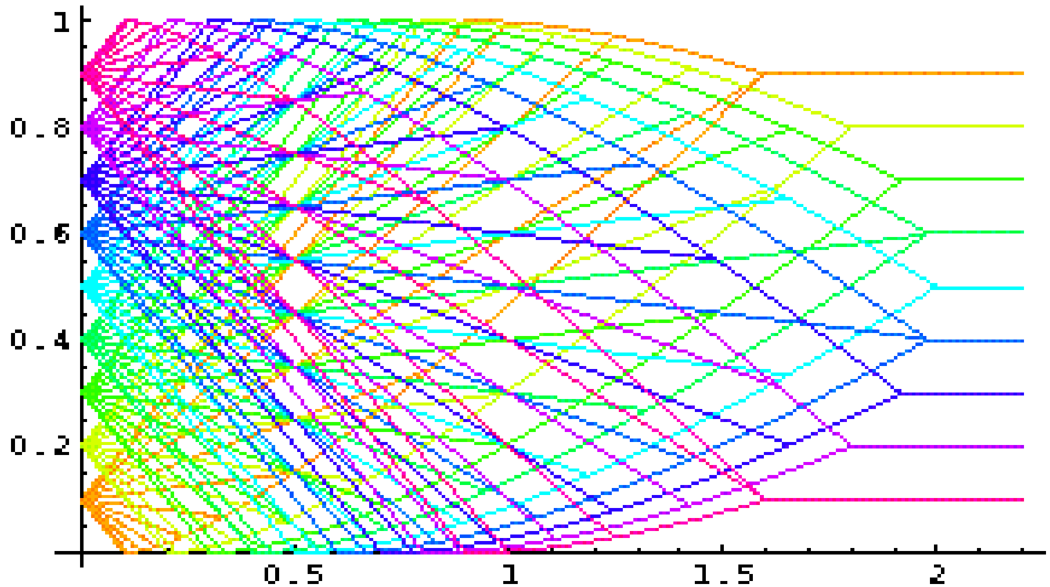
P(given space-time point in USN $\Rightarrow$ swap at location j in subnetwork)

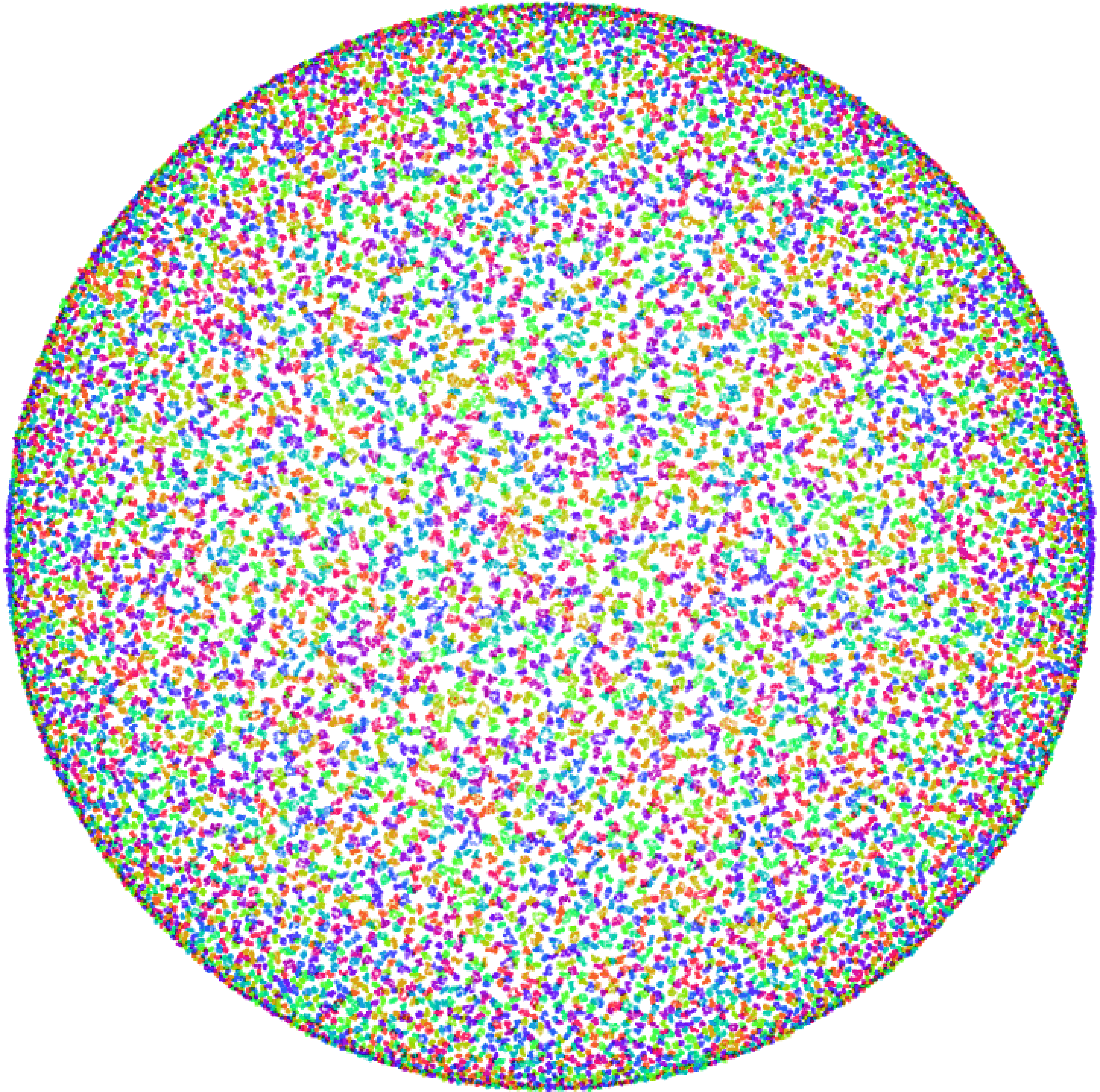


Uniform swap model...

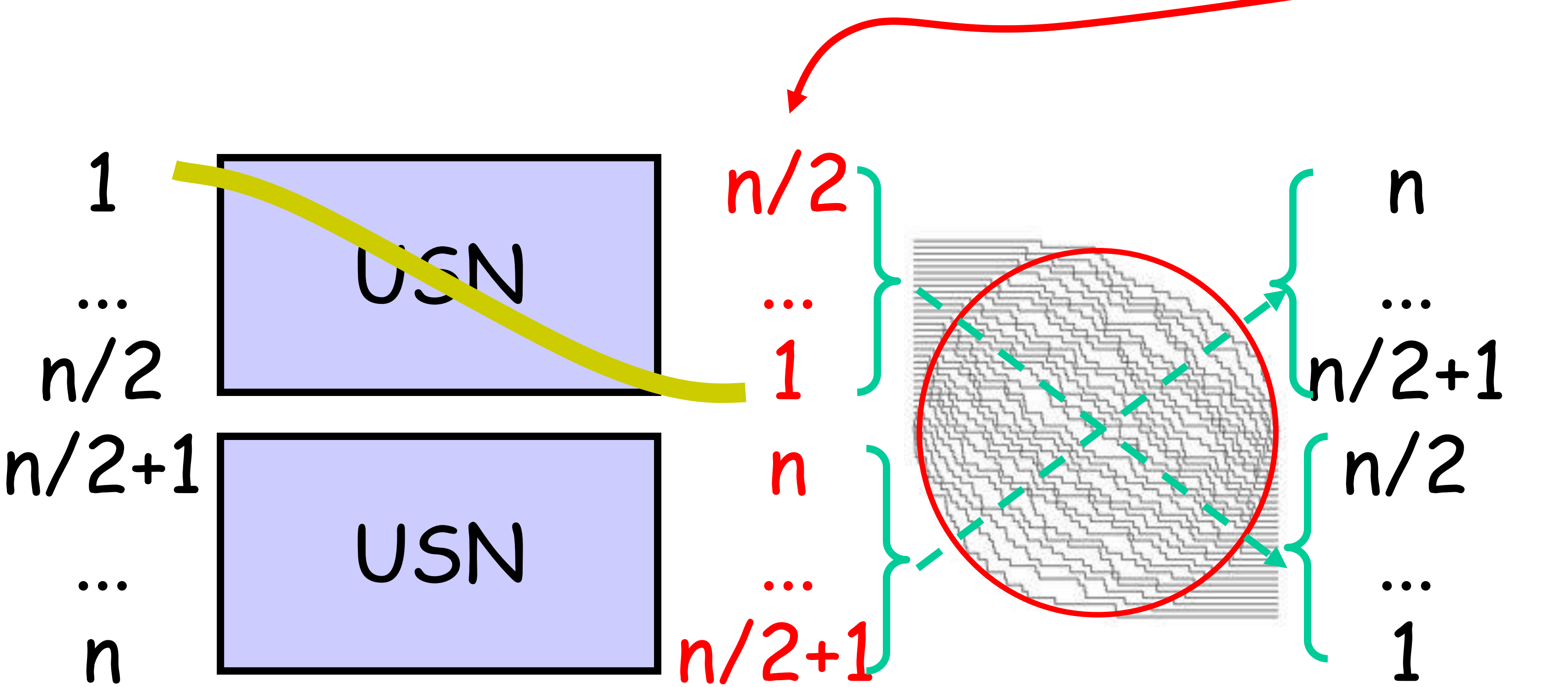


Angel,H,Romik 2008
Amir,Angel,Valko
...



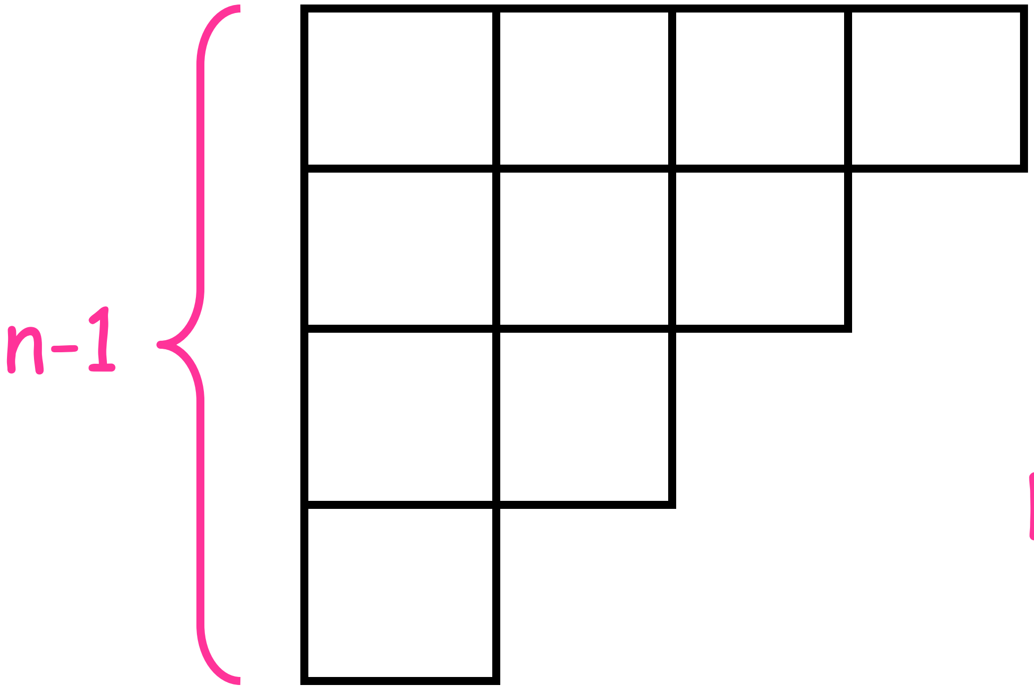


N.B. Not every sorting network lies close to a great circle! E.g. typical network through



(But this permutation is very unlikely).

Staircase Young diagram:



(E.g. n=5)

N cells

Standard staircase Young tableau:

1	2	4	8
3	5	6	
7	10		
9			

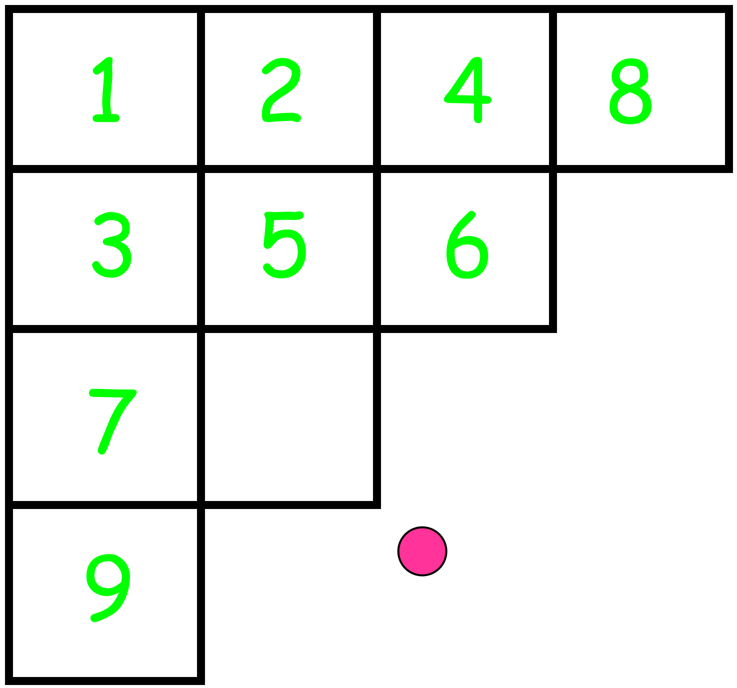
Fill with $1,\cdots,N$ so each row/col increasing

Edelman-Greene algorithm:

1	2	4	8
3	5	6	
7	10		
9			

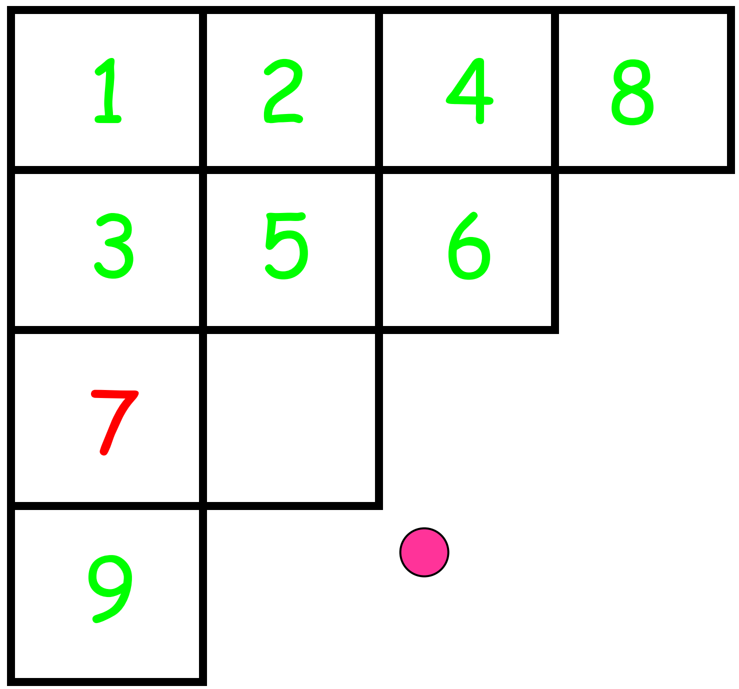
1. Remove largest entry

Edelman-Greene algorithm:



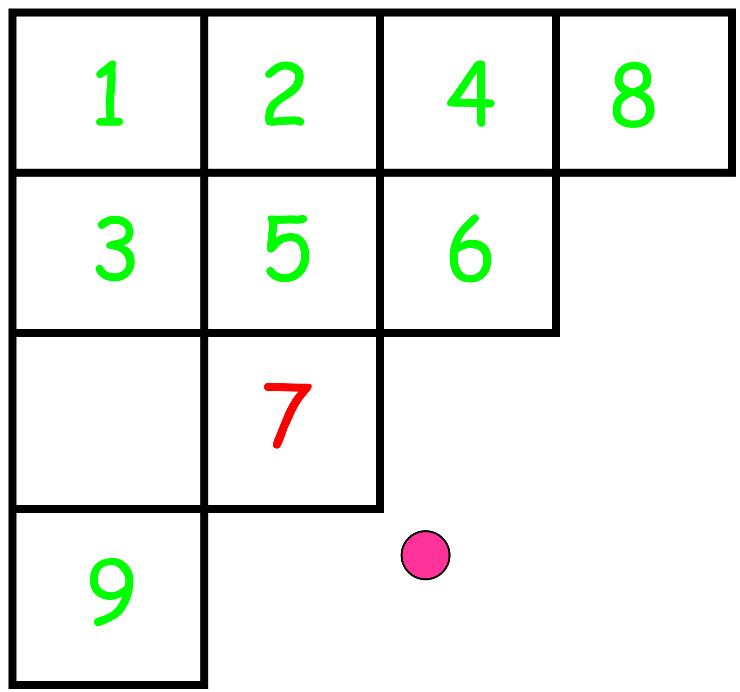
1. Remove largest entry

Edelman-Greene algorithm:



2. Replace with larger of neighbours ↑ ←

Edelman-Greene algorithm:



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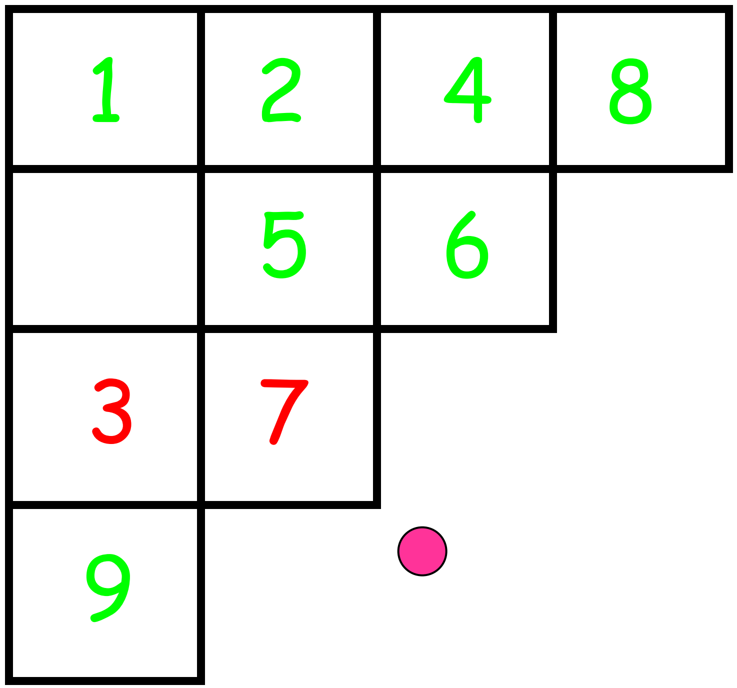
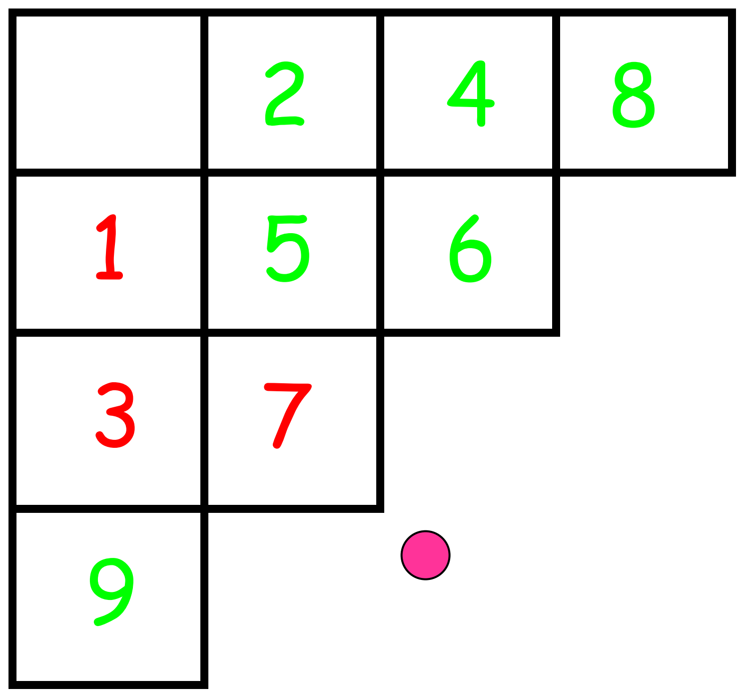
Edelman-Greene algorithm:



2. Replace with larger of neighbours ↑ ←

...repeat

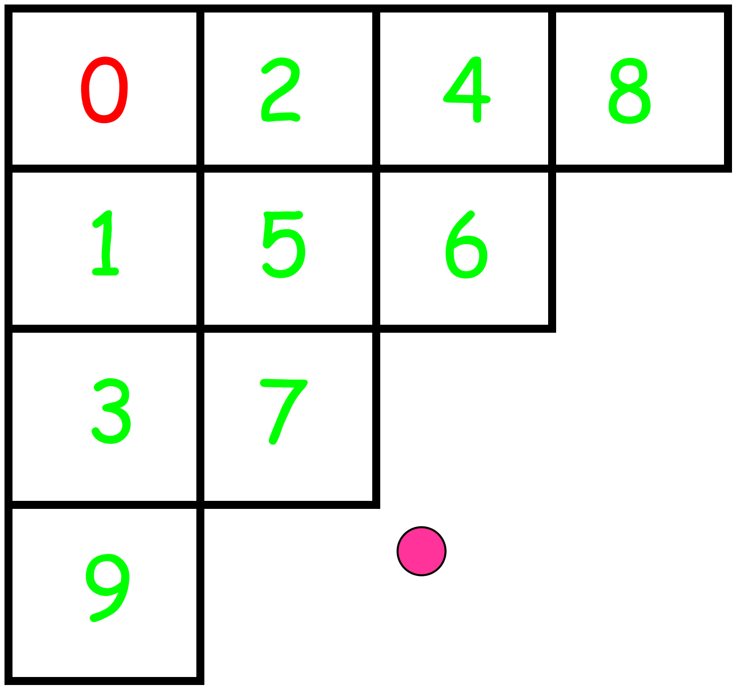
Edelman-Greene algorithm:



2. Replace with larger of neighbours ↑ ←

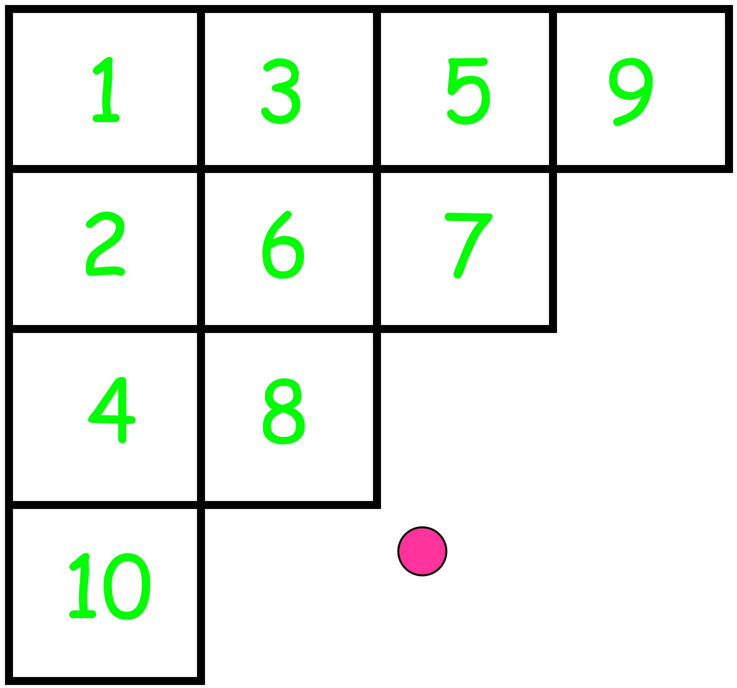
...repeat

Edelman-Greene algorithm:



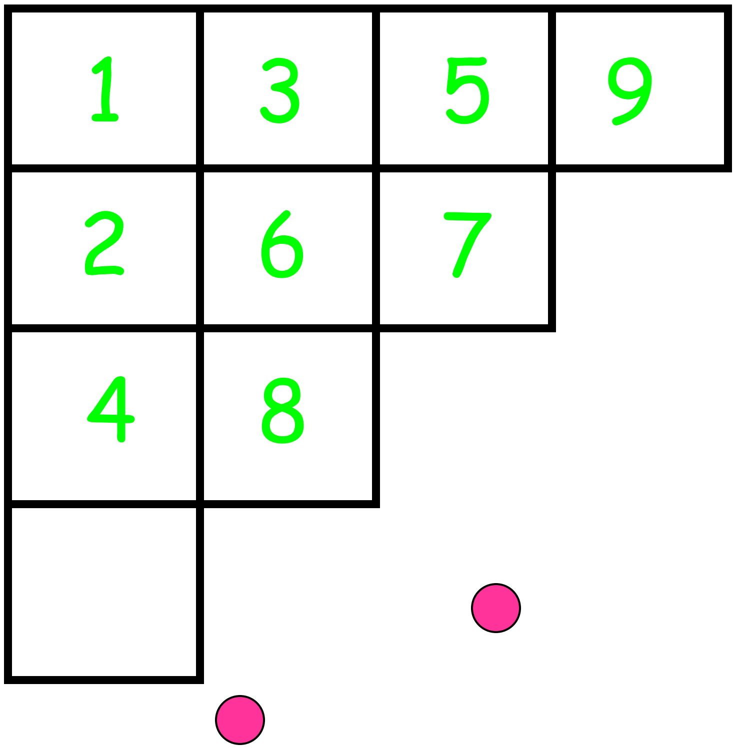
3. Add 0 in top corner

Edelman-Greene algorithm:



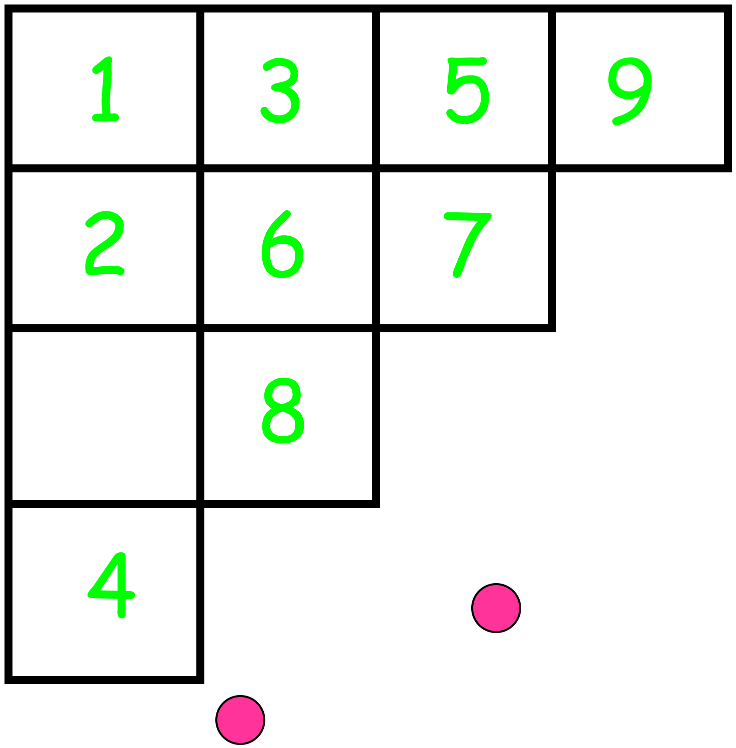
4. Increment

Edelman-Greene algorithm:



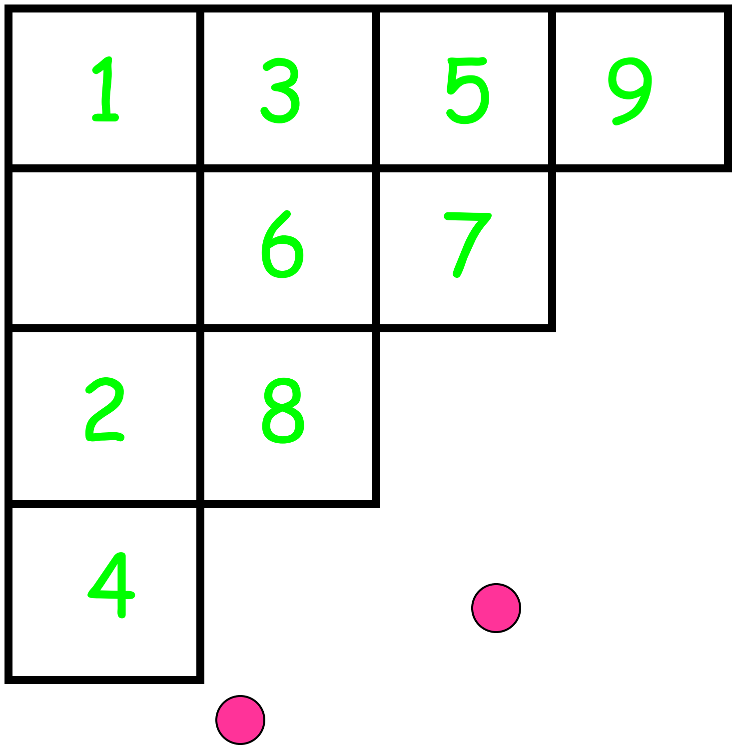
5. Repeat everything...

Edelman-Greene algorithm:



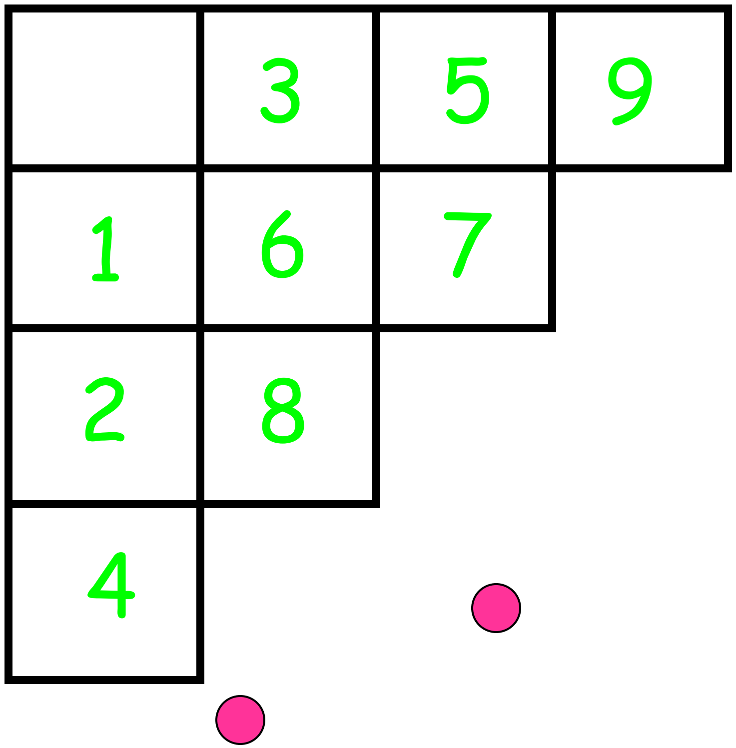
5. Repeat everything...

Edelman-Greene algorithm:



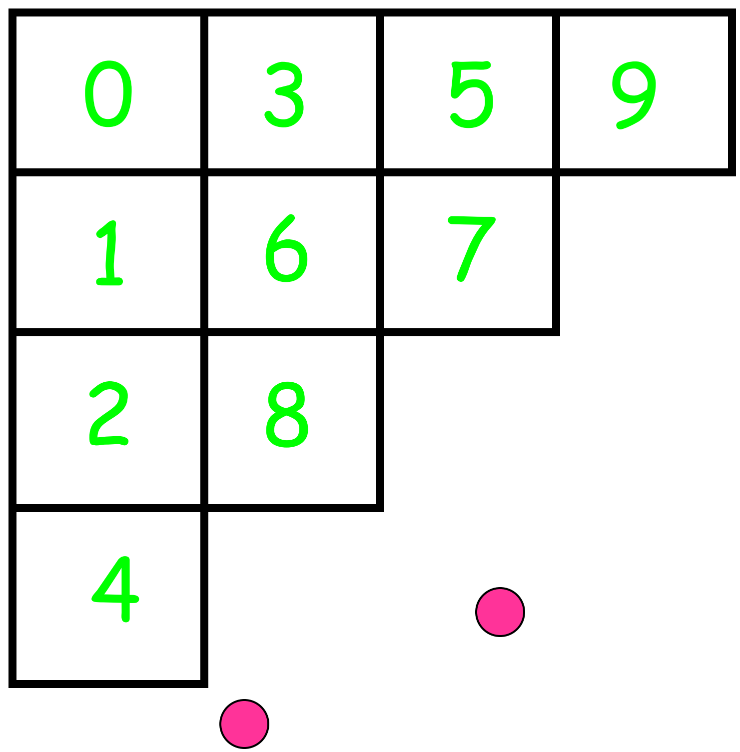
5. Repeat everything...

Edelman-Greene algorithm:



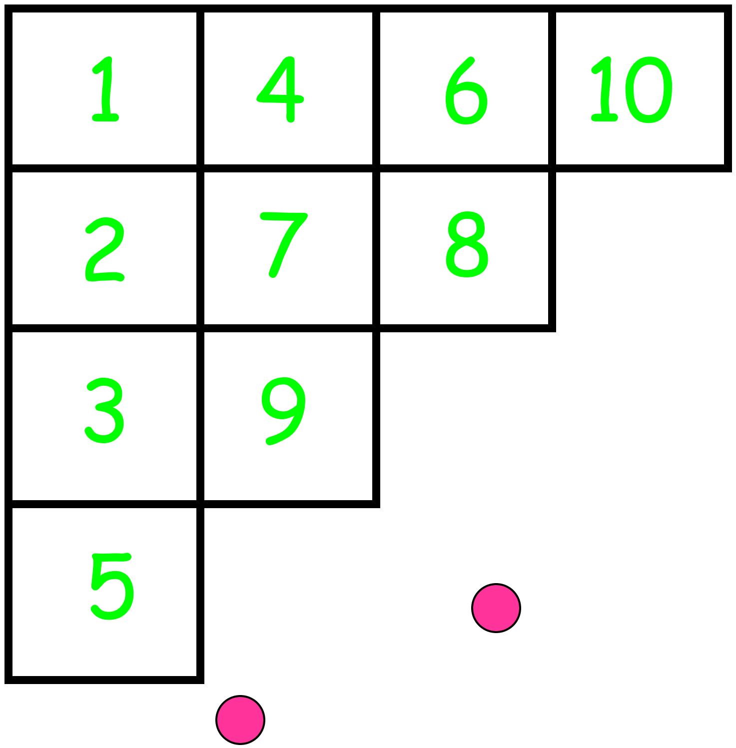
5. Repeat everything...

Edelman-Greene algorithm:



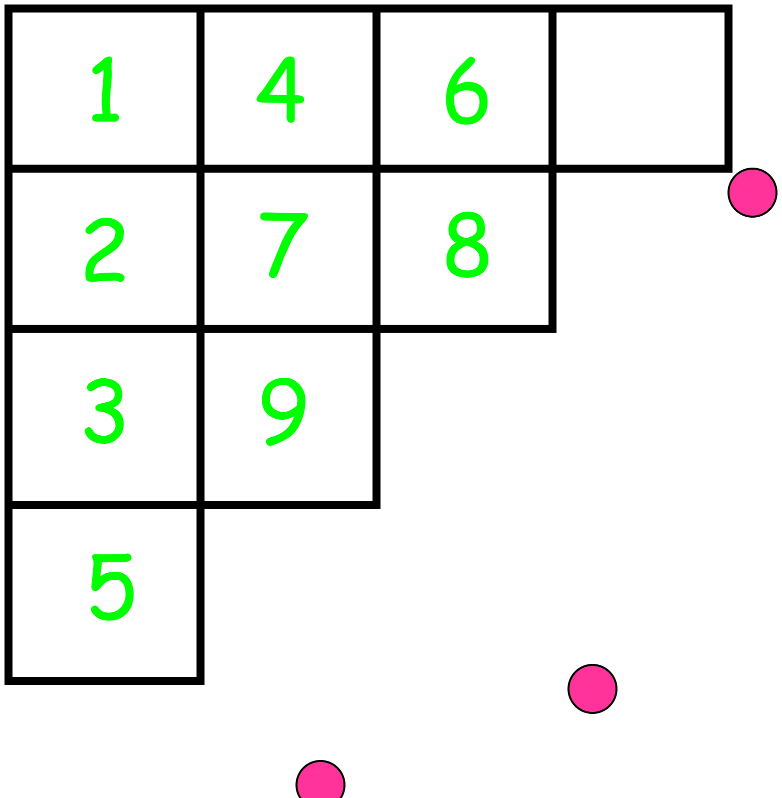
5. Repeat everything...

Edelman-Greene algorithm:



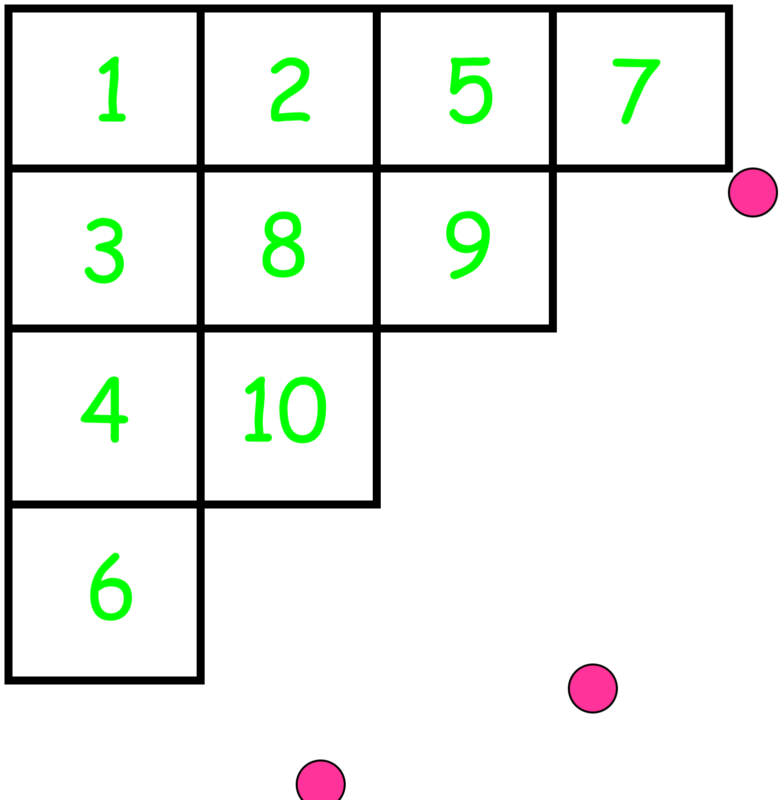
5. Repeat everything...

Edelman-Greene algorithm:



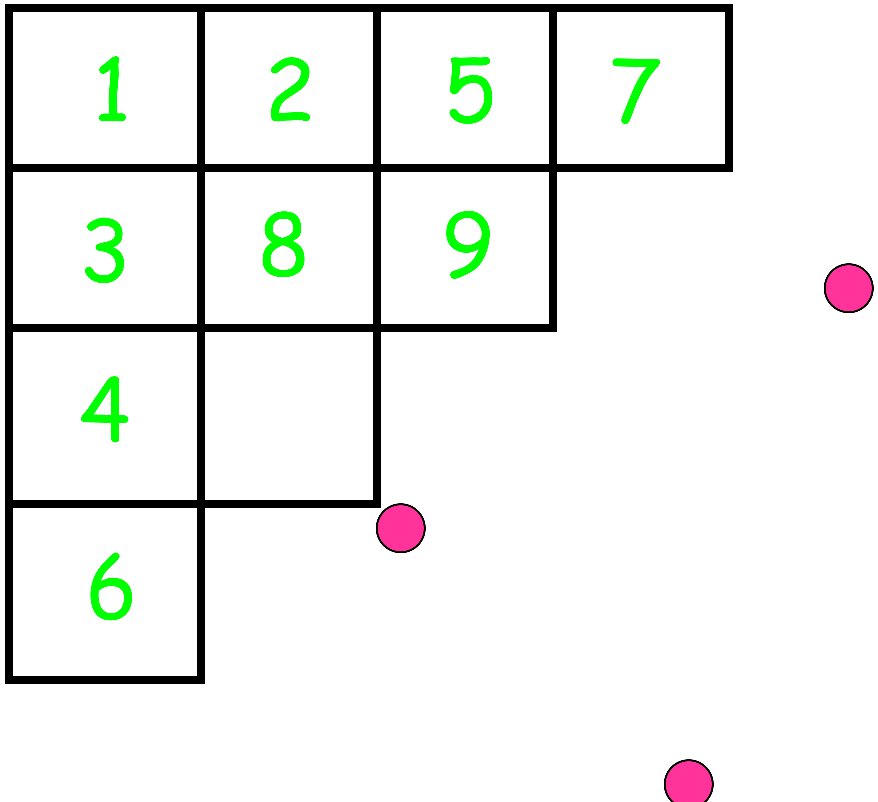
5. Repeat everything...

Edelman-Greene algorithm:



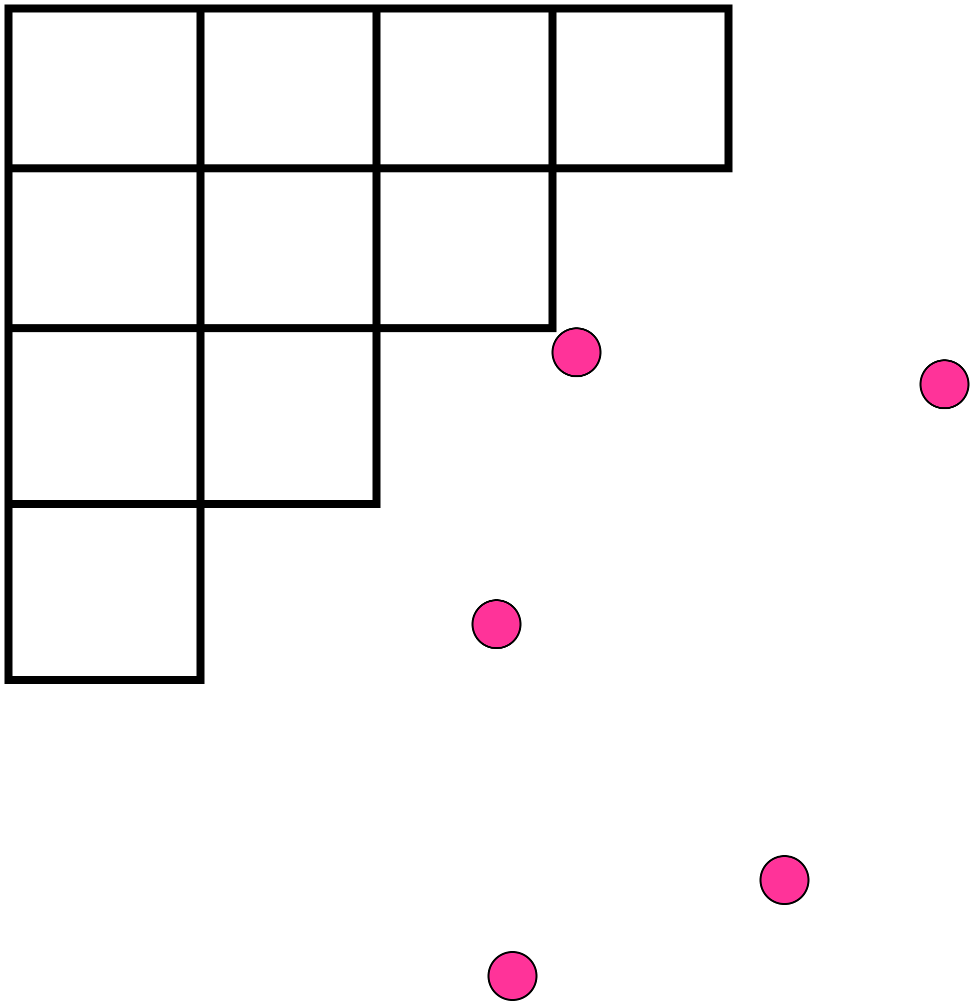
5. Repeat everything...

Edelman-Greene algorithm:

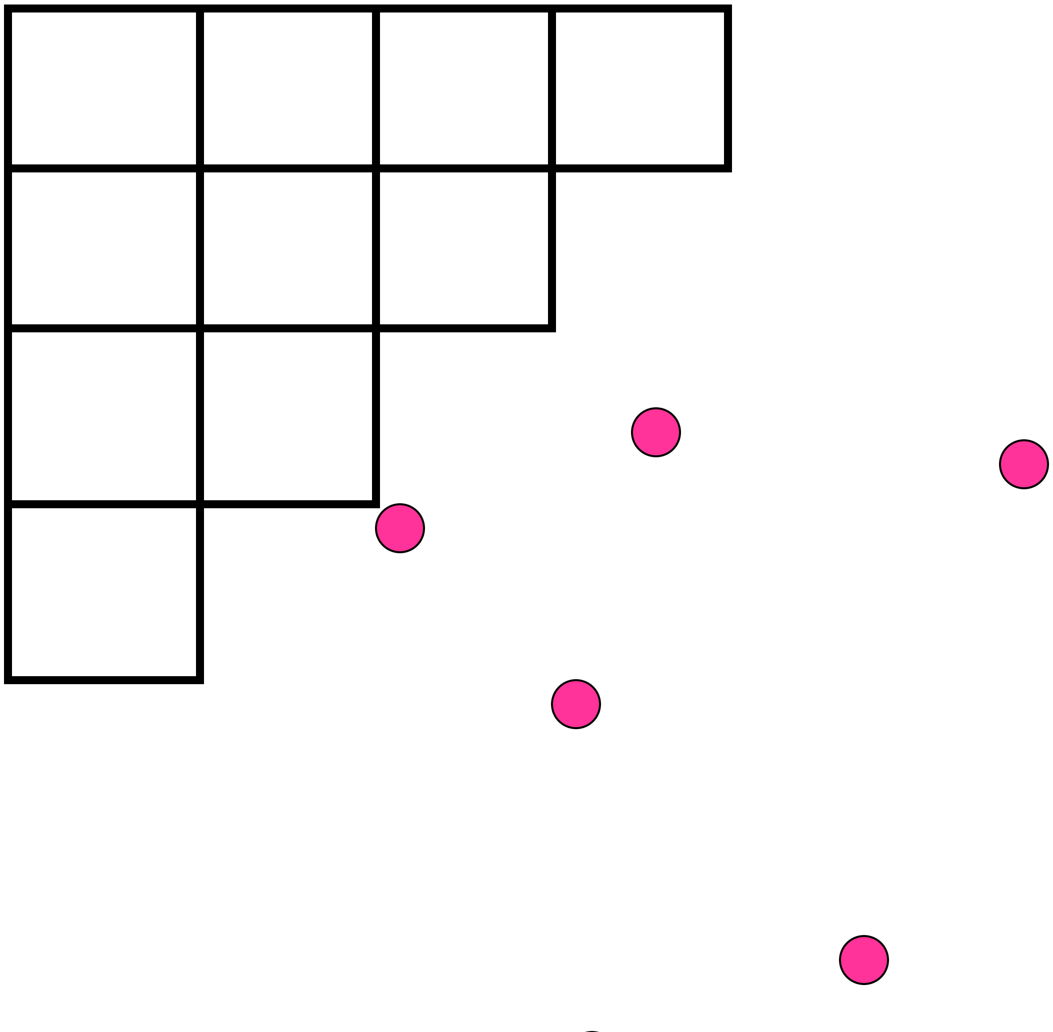


5. Repeat everything...

Edelman-Greene algorithm:



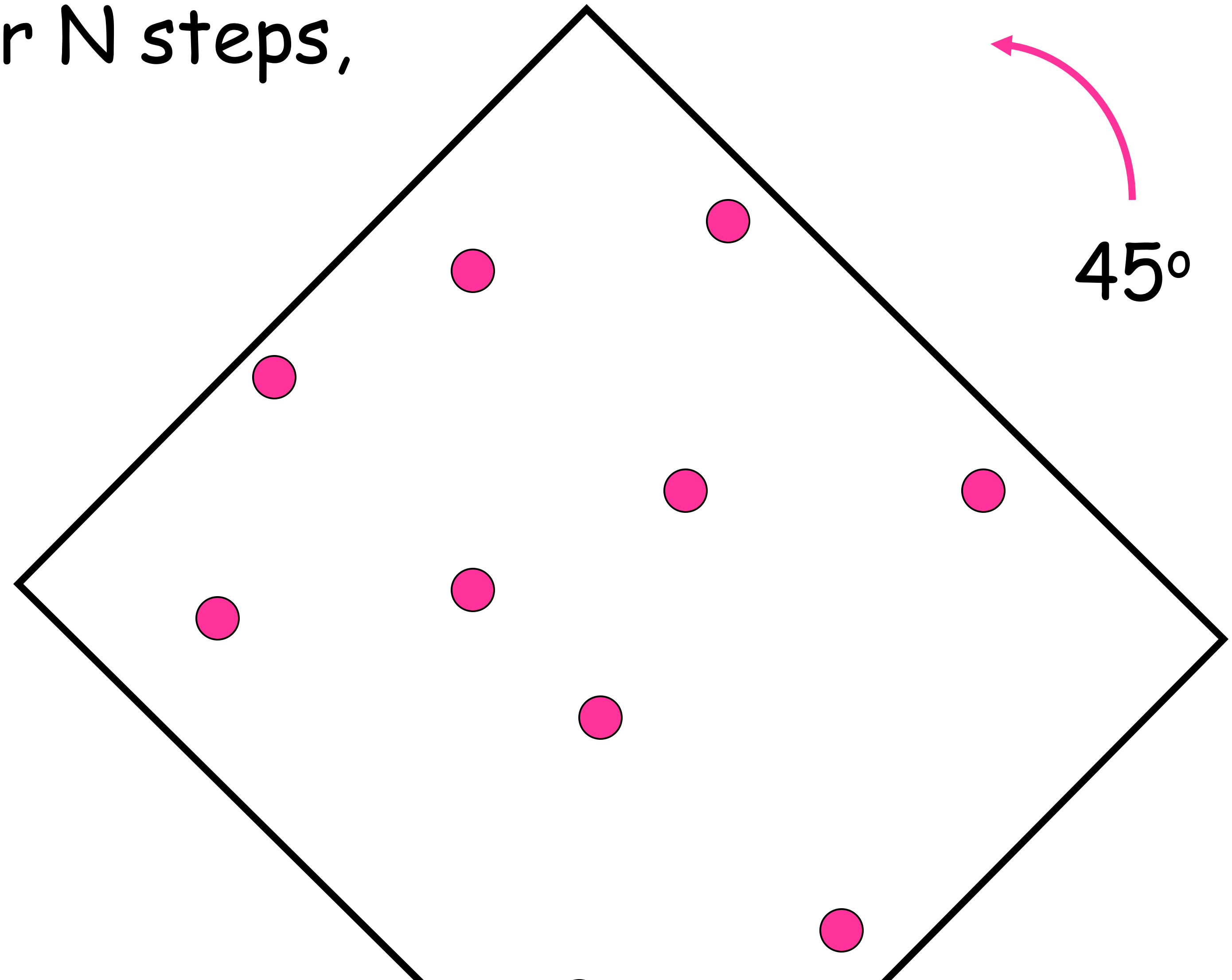
Edelman-Greene algorithm:



etc

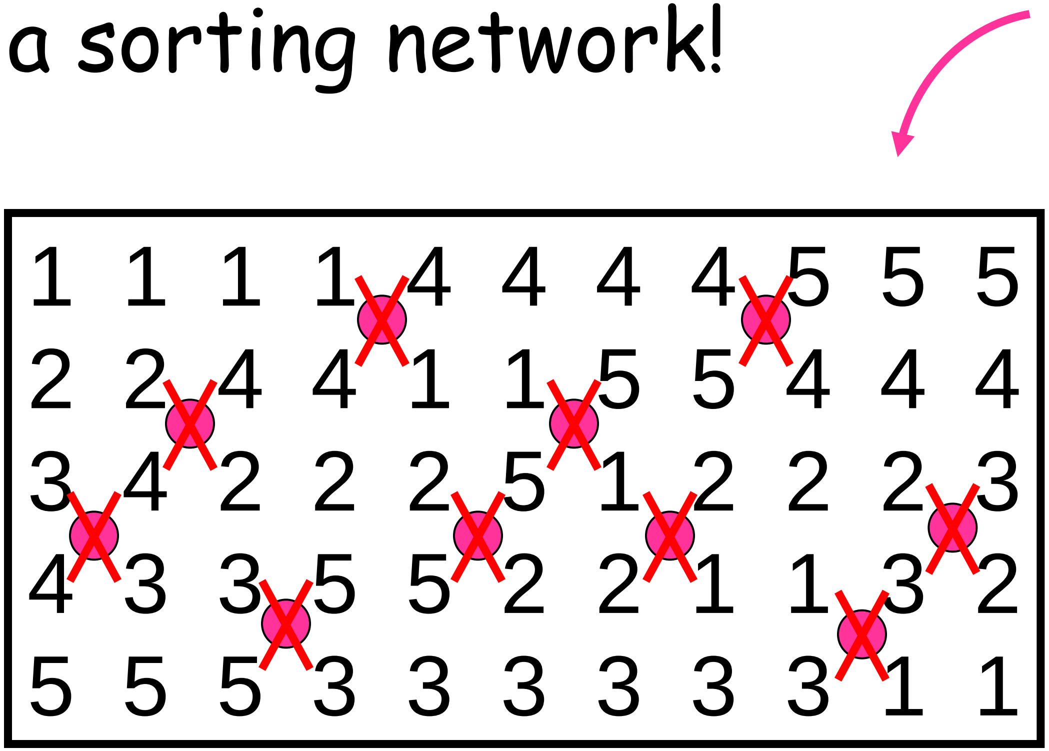
Edelman-Greene Theorem:

After N steps,



Edelman-Greene Theorem:

After N steps,
get swap process
of a sorting network!



Edelman-Greene Theorem:

After N steps,
get swap process
of a sorting network

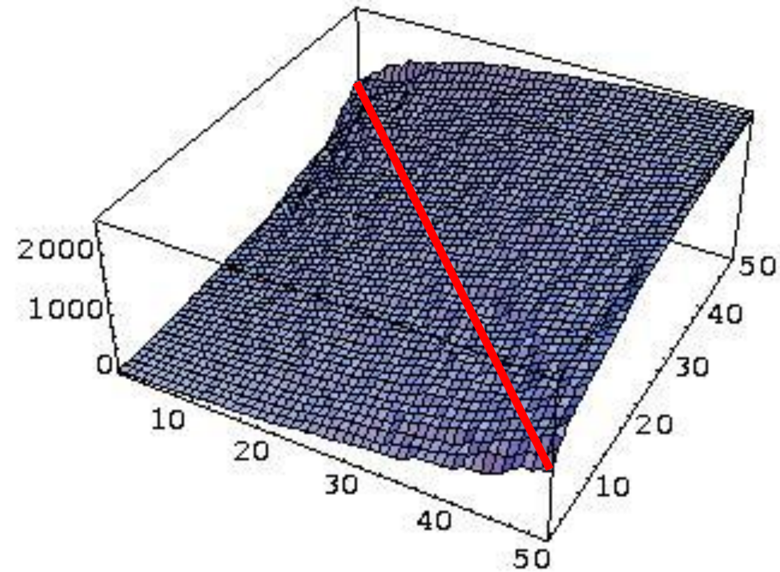
And this is a bijection!

And can explicitly describe inverse!

Theorem (Pittel-Romik): For a uniform random n x n *square* tableau, ∃ limiting shape with contours:

$$h_\alpha(u) = \frac{2}{\pi}[u \tan^{-1}(u/R) + \tan^{-1} R]$$

$$\text{where } R = \frac{\sqrt{\alpha(2-\alpha) - u^2}}{1-\alpha}$$

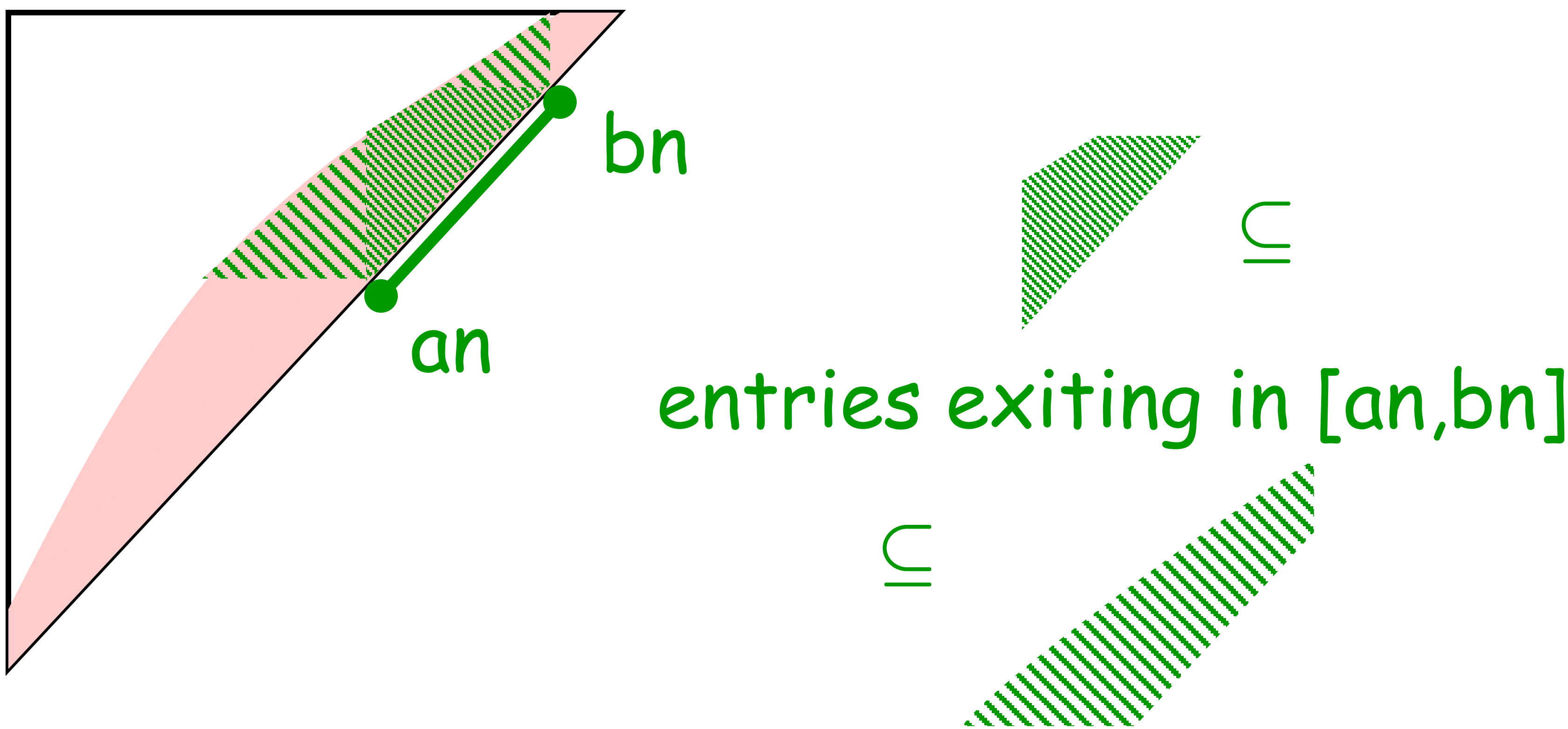


Corollary (AHRV): For uniform random staircase tableau, limiting shape is half of this.

(Proof uses Greene-Nijenhuis-Wilf Hook Walk)

<u>Proof of LLN</u> (swap process $\Rightarrow$ semic. x Leb.)

Swaps in space-time window $[an,bn]\times[0,\varepsilon N]$ come from entries $>(1-\varepsilon)N$ in tableau:

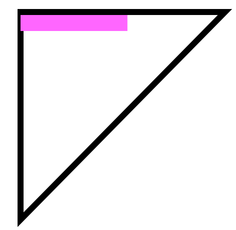


$\approx$ area under contour $\approx$ semicircle

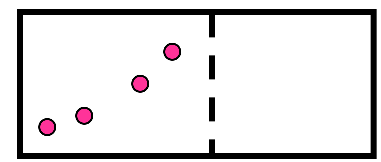
$\square$

Proof of octagon and Holder bounds

Inverse Edelman-Greene bijection
($\approx$ RSK algorithm) $\Rightarrow$

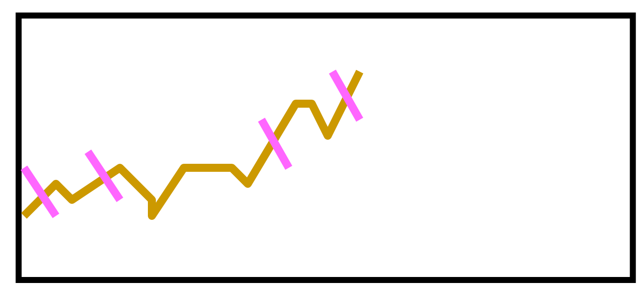


\# entries $<k$ in 1^{st} row



$\geq$ longest $\nearrow$ subseq. of swaps
by time k

$\geq$ furthest any particle moves up
by time k



So can bound this using
limit shape. $\square$

Angel,H,Virag (in prepatation):

Process of first k swaps in positions cn...cn+k $\to$ random limit

as $n \to \infty$

not depending on $c \in (0,1)$