

Notes on Multivariable Calculus I

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About Math 226

Math 226 and 227 are multivariable calculus classes recommended for Honours students in Mathematics at the University of British Columbia. They are aimed at students who have a deeper interest in mathematics, with more emphasis on rigorous proofs and mathematical maturity than in our general-purpose multivariable calculus courses. Accordingly, less time is spent on computational techniques (such as evaluating integrals) and applications (physics, engineering).

Both courses are taught under specific curricular constraints. Most students come to Math 226 directly from single-variable calculus courses emphasizing computations and applications. They are smart, motivated, and they want to see cool mathematics. Many, however, have had little or no exposure to epsilon-delta proofs (let alone the basic concepts of real analysis) and no formal training in rigorous proof writing.

Math 226 covers functions of several variables, their continuity, differentiability, optimization problems, and integration. The intention is to study the material in some depth, include selected advanced topics (Implicit Function Theorem, the precise definition of differentiability), and promote correct mathematical reasoning. Our usual textbook is *Calculus: Several Variables* (also available as the second half of *Calculus: A Complete Course*), by Robert A. Adams and Christopher Essex, published by Pearson. It is an excellent resource, ambitious yet accessible. A fully rigorous treatment (I'm a fan of Folland's *Advanced Calculus*) would be beyond what I can expect to accomplish in this course.

These notes, based on my lecture scripts (see below), reflect my priorities in class. Calculus is rich in big conceptual ideas, often geometric in nature, but also requires considerable attention to detail in bringing those ideas to life. My goal in teaching is to find both a balance between these two aspects and a path for students to go back and forth between the macro and micro pictures as needed. This goal is more aspirational than attained, given that any balance must be based on compromises, but I hope to give my students the means to find their own way around the subject.

Rigorous proofs on a small scale are introduced early and reinforced throughout the course. At the same time, I often break down longer proofs into components, clarifying the main ideas and presenting a flexible toolkit that students can be expected to learn to use. For the main idea, I might start with a simple model case or a calculation that works under stronger assumptions. Then we move in with additional tools such as reductions and approximations. I follow up with practice and homework problems along similar lines, such as “prove this theorem in detail in this specific case” or “assuming we know A, prove B.”

I encourage critical thinking and asking questions such as “how are these concepts different?” or “where did we use this assumption?”. I have included examples to

illustrate what can happen in various counterintuitive situations, and tracked down where the assumptions of the main theorems are used. I also recognize that when a student wants to know why something works, pointing to a formal proof is not always a satisfactory answer. Some of the explanations in these notes evolved over time as I worked to improve my answers to such questions.

Throughout the course, I emphasize working with inequalities and estimates. This is a recurring theme, from evaluating limits to the rigorous concept of differentiability to convergence of improper integrals. I want my students to be able to construct inequality-based proofs, compare orders of magnitude, use approximations with estimates on the error terms. I hope they will find this useful beyond multivariable calculus.

My classes usually include a few extra topics, such as Jordan area and the Riemann integrability criterion, that Math Honours students find attractive. These are also included here, with suggestions for further exploration. On the other hand, some of the conceptually easier parts are abbreviated. Computations are kept reasonably simple to save time. I worked with the remaining material to construct a coherent sequence that I could present in class without rushing.

Most of the problems here are taken from my exams and homework assignments. Some are routine (“compute this quantity using the formula provided”); for more practice of this type, I encourage students to use their textbooks as well as online systems such as WebWork. In homework assignments, I often ask questions that are more challenging, proof-oriented, require some creative thinking, and lend well to collaborations for those so inclined. I ask the students to write their solutions clearly, in full sentences, with correct and complete explanations provided. These are the “proof questions” appearing in most sections of this book. The Worked Problems come with solutions; they often include the examples I used in my lectures. The solutions to the Practice Problems are provided in a separate Solutions Manual volume.

About these notes

I taught Math 226 eight times between 2005 and 2021, and Math 227 seven times between 2006 and 2022. I also taught Math 317, a more conventional calculus course with scope similar to 227, in 2011 and 2014. Some of this repetition was likely due to the Covid pandemic: as classes went online and workloads skyrocketed, the department made the sensible choice to assign most courses to faculty who had taught them previously. Thus, I taught 226 and 227 online in 2020-21, and in person but with a significant online component in 2021-22.

By then, I had a full set of handwritten lecture notes for both courses. These were my actual class scripts, with the material selected and presented accordingly. As I taught the online courses, I rewrote the notes one section at a time, cleaned them up, and posted them on the course website. They were still handwritten at that point,

with hand-drawn pictures, but with further details and explanations added for student use. I also put together a series of typed practice problem sets with solutions, based on my midterm tests, final exams, and homework questions from previous years.

The reception was excellent, with both students and myself relying on these notes to a greater extent than I had expected. I suggested to the department that, with appropriate support, I could make my course materials available to all students and instructors as a free supplementary resource. I asked for a teaching assistant to type up the handwritten part and help compile the files. I also requested a one-course teaching release to compensate for the additional work I was undertaking, and I asked to continue to teach Math 226 and 227 for the duration of the project (initially, I expected 2 years; it took 3, from 2023-24 to 2025-26). The department agreed that there was a need for such a resource, and granted the support I requested.

The TA assigned to this project was Caleb Marshall, a doctoral student at UBC at the time. He worked on it with me from September 2023 to August 2025. Caleb's main responsibility was to type up, in L^AT_EX, my handwritten notes and use computer graphics to make professional-looking versions of my hand-drawn pictures. Transcribing my handwritten notes involved a significant amount of work beyond simple typesetting: some text needed to be added, parts of the material needed to be reorganized, and it took some time (especially at the beginning) to establish a workflow and sort out various technical issues.

The text Caleb produced was lively and engaging, and I have kept many of the expository comments he added. I talked with him frequently about my pedagogical approach, the organization of each section, the level of mathematical rigour, the choice of examples and exercises, and much more. There were many times when a problem needed to be modified or a proof needed clarification. I often asked Caleb for input in such cases so that we could find a solution together, and then let him work out the details. I checked and edited the files before making them available to students.

In the first year (2023-24), we focused on getting the typed content on the page, so that the quality of the illustrations was less of a concern. In the second year, Caleb had more time to work on the graphics, and he trained himself to be an excellent mathematical illustrator. He used Geogebra (a freely available online program) as well as occasional iPad drawings, and became highly competent at both. He also wrote the first drafts of Sections 1.1.5 and 1.1.7 (on proofs and quantifiers), and contributed a few completely new problems with solutions. These include Practice Problem 2 in Section 2.5 (harmonic functions), Practice Problems 2 and 3 in Section 3.1 (linearization), Practice Problem 6 in Section 4.1 (critical points), Practice Problem 4 in Section 4.3 (Lagrange Multipliers), and Practice Problem 7 in Section 5.2 (centroid of a triangle).

We both edited various parts of the text during the 2024-25 academic year, improving the presentation and taking student feedback into account. (The reader might notice two different writing styles.) In Summer 2025, Caleb compiled the Math 226 notes and problem solutions into two volumes, this one and the separate Solutions

Manual. Math 227 notes were compiled later in 2025 by another TA, Mihai Marian. In 2025-2026, the last year of the project, I had more time to reassess the pedagogical approach and review the notes for overall consistency. I reorganized parts of the material, made the final adjustments, and added a new batch of problems and solutions from the 2023-26 exams and homework assignments.

We had minimal resources, with no copy editors or production team. The formatting is often *ad hoc*, based on whatever worked for us for that part of the material. There are no strict rules as to what gets framed in a box or how the examples are numbered. The presentation can be a little bit informal. Nonetheless, my students found these notes useful, and I hope that so will their future readers.

Acknowledgements

This book was written on the Point Grey campus of the University of British Columbia, located on the traditional, ancestral, and unceded territory of the Musqueam people.

I am grateful to the Department of Mathematics at the University of British Columbia, and especially to Daniel Coombs (Head) and Fok-Shuen Leung (Undergraduate Chair), for approving and supporting the project. I owe many thanks to Caleb Marshall for the two years of working on it together. Krysten Casumpang, a staff member in the department, did a great job designing the cover pages.

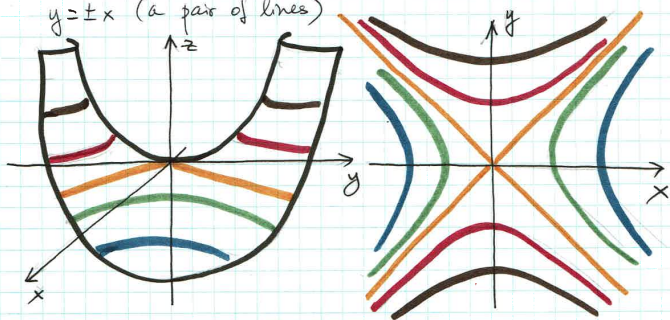
I also would like to acknowledge all Teaching Assistants who helped me run these courses over the years. In Math 226, this includes Matthew Brassil, Nathan Lawrence, Manuel Ruivo de Oliveira, Ethan White, Emmanuel Michta, Summer Sheng, Yuveshen Moorogen, Yuze Cui, Sebastian Gant, Nikou Lei. Unfortunately I have not kept TA records from 2016 and earlier, but nonetheless I would like to thank any of my former TAs who might be reading this.

Last but not least, I'm grateful to all undergraduate students who took Math 226 with me. They have provided valuable and actionable feedback, asked challenging questions, and otherwise pushed me to improve this course. They have also pointed out many typos and errors in earlier versions of these notes. It was a pleasure to work with them.

Izabella Laba

Vancouver, July 2026

2) $f(x,y) = y^2 - x^2$. The graph $z = y^2 - x^2$ is a hyperbolic paraboloid. The level curves $y^2 - x^2 = c$ are hyperbolas in the xy -plane if $c \neq 0$. If $c = 0$, we get $y^2 - x^2 = 0$, so $y = \pm x$ (a pair of lines).

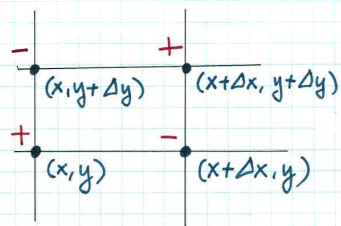


For functions $f(x,y,z)$ of 3 variables, a graph would be a hypersurface $w = f(x,y,z)$ in 4 dimensions. But we can still draw the level surfaces $f(x,y,z) = c$ for fixed c as surfaces in $\mathbb{R}^3$, and then get some idea of what the function "looks like" by varying the constant c . For example, let $f(x,y,z) = z^2 - x^2 - y^2$, then the level surface $z^2 - x^2 - y^2 = 0$ is the cone $z^2 = x^2 + y^2$, and the level surfaces $z^2 - x^2 - y^2 = c$ are hyperboloids (one sheet when $c < 0$, two sheets when $c > 0$).

Idea of proof, in the special case $f_{xy} = f_{yx}$ for a function $f(x,y)$ of 2 variables.

$$\text{Let } Q(x,y,\Delta x,\Delta y) = f(x+\Delta x, y+\Delta y) - f(x, y+\Delta y) - f(x+\Delta x, y) + f(x,y).$$

This adds up values of f at 4 vertices of a rectangle, with alternating $+/-$ signs as indicated.



Also notice that this is symmetric with respect to the order of variables.

Assume that the following double limit exists:

$$\text{(for now)} \quad \lim_{(\Delta x, \Delta y) \rightarrow (0,0)} \frac{Q(x,y,\Delta x,\Delta y)}{\Delta x \Delta y} = L. \quad (*)$$

Then both $f_{xy}(x,y)$ and $f_{yx}(x,y)$ are equal to L .

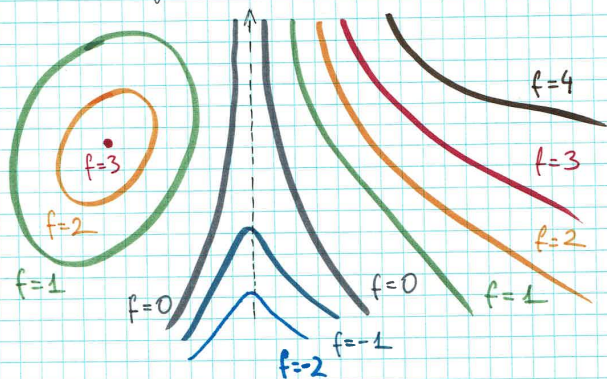
This is because

$$f_y(x,y) = \lim_{\Delta y \rightarrow 0} \frac{1}{\Delta y} (f(x, y+\Delta y) - f(x,y)),$$

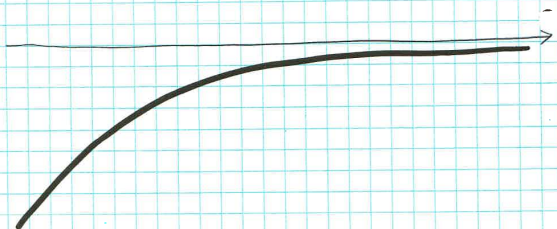
$$f_y(x+\Delta x, y) = \lim_{\Delta y \rightarrow 0} \frac{1}{\Delta y} (f(x+\Delta x, y+\Delta y) - f(x+\Delta x, y))$$

Why is the ^{test} 2nd derivative not useful here?

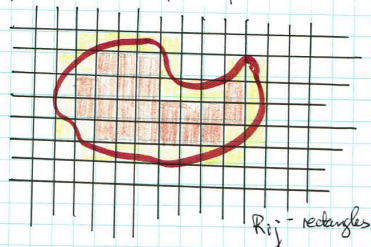
Example: $f(x,y)$ has only one local critical point, with a local but not global maximum at that point.



Cross-section along the dotted line:



Jordan area: for a planar set X , approximate it by grids ^{5/3}



$$S^* = \sum (\text{areas of } R_{ij} \text{ intersecting } X)$$

$$S_* = \sum (\text{areas of } R_{ij} \text{ contained in } X)$$

Then $S_* \leq (\text{area of } X) \leq S^*$,

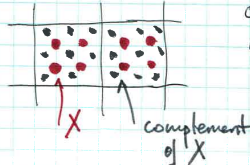
X is Jordan measurable if $\lim S^* = \lim S_*$ as the diameter of the grid $\rightarrow 0$. Jordan area of X (if exists)

X has Jordan area 0 if $\lim S^* = 0$.

Example of a set that does not have Jordan area:

let $X = \{(x,y) \in [0,1]^2 : x \text{ and } y \text{ are both rational}\}$.

Then $S^* = 1$ and $S_* = 0$ for any grid. (Any rectangle contains both points with rational coordinates and points with irrational coordinates.)



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Chapter 1

An Introduction to Euclidean Space

1.1 Sets and Coordinates

1.1.1 Rectangular Coordinates and Euclidean Distance in $\mathbb{R}^3$.

Every point P in $\mathbb{R}^3$ can be written as $P = (x, y, z)$, where x, y and z are real numbers. This corresponds to **rectangular coordinates**, as shown in the following picture.

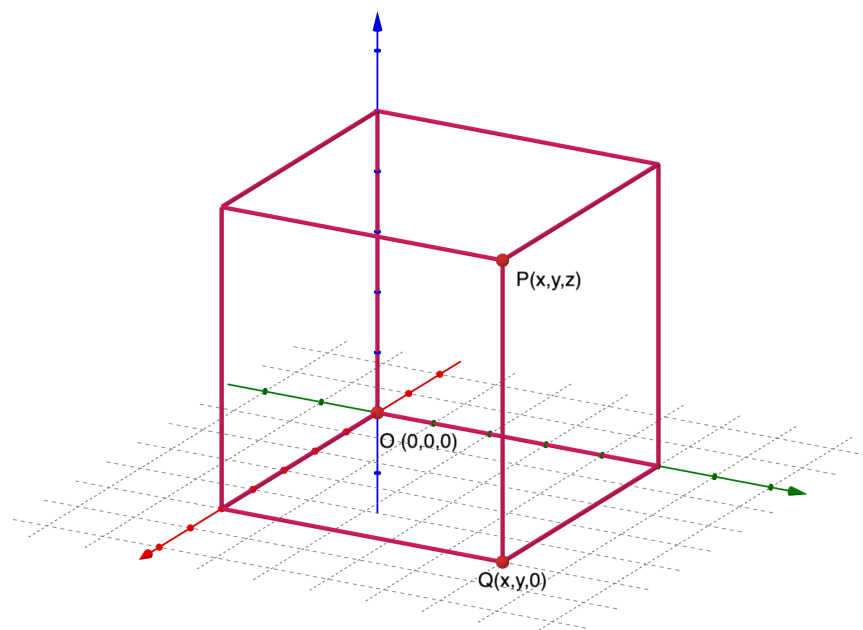


Figure 1.1: Three points O, P, Q in $\mathbb{R}^3$.

We will reserve the notation $O = (0, 0, 0)$ for the **origin**—that is, the point with zero for each coordinate. In this picture, $Q = (x_0, y_0, 0)$ and $P = (x_0, y_0, z_0)$. This is because Q lies in the plane where $z = 0$, and P is collinear with Q on a line parallel to the z -axis.

Notice that O, P and Q form the vertices of a right-triangle. In particular, the segment OP connecting O and P —corresponding to the hypotenuse of this right-triangle—has length

$$|OP| := \sqrt{|OQ|^2 + |PQ|^2} = \sqrt{x_0^2 + y_0^2 + z_0^2}. \quad (1.1.1)$$

In other words, the distance from the origin to P is given by formula (1.1.1). In general, we have the following notion of distance in $\mathbb{R}^3$.

Definition 1.1.1. Suppose that $P = (x_0, y_0, z_0)$ and $R = (x_1, y_1, z_1)$. Then, the (**Euclidean**) **distance** between P and R is denoted $|PR|$ and satisfies,

$$|PR| := \sqrt{(x_0 - x_1)^2 + (y_0 - y_1)^2 + (z_0 - z_1)^2} \quad (1.1.2)$$

Make sure you are comfortable with all of this terminology in your preferred course textbook. We will make ready use of it as we proceed.

1.1.2 Equations and Surfaces in $\mathbb{R}^3$

We will often use equations to represent curves and surfaces in $\mathbb{R}^3$, as follows.

1. A single equation will generally be used to describe a surface. For example, the equation $x^2 + y^2 + z^2 = 9$ describes a sphere in $\mathbb{R}^3$, centred at the origin, with radius $r = 3$. Equations need not involve all variables x, y and/or z . The equation $z = 2$ describes a plane parallel to the xy -plane, passing through the point $(0, 0, 2)$.
2. A system of two equations will generally be used to describe a curve (the intersection of two surfaces). For example, the system of equations

$$\begin{cases} x^2 + y^2 + z^2 = 9 \\ z = 2 \end{cases}$$

represents a circle. All three of the above sets (sphere, plane, intersected circle) are shown in the following figure.

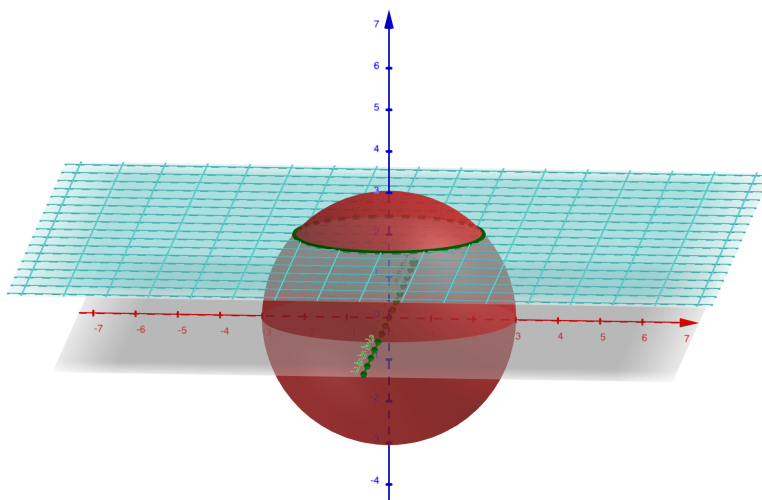


Figure 1.2: The sphere $\{(x, y, z) : x^2 + y^2 + z^2 = 9\}$ and the plane $\{(x, y, z) : z = 2\}$ intersect in a curve (the dark-green circle).

Caution! It is not always true that 1 equation gives a surface and 2 equations give a curve. For example, you can readily check that the equation

$$x^2 + y^2 + z^2 = 0,$$

has only $(0, 0, 0)$ as a solution—which is a point and not a surface. Similarly, the system of equations

$$\begin{cases} x^2 + y^2 + z^2 = 9 \\ z = 4 \end{cases}$$

has no solutions (x_0, y_0, z_0) in $\mathbb{R}^3$! This is because the plane $z = 4$ does not intersect the sphere defined by $x^2 + y^2 + z^2 = 9$. Try plotting this on Geogebra, or another free online 3D calculator. You can find more examples of these ideas in your textbook.

Sometimes, a set in $\mathbb{R}^3$ will be described using an equation or inequality in only one or two variables. This means that any variable not mentioned in the equation can take any value in $\mathbb{R}$. For example, the set

$$\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 4\}$$

is a circle in the xy -plane, centered at $(0, 0)$ and with radius 2. However, the set

$$\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = 4\}$$

is a cylinder of infinite length in the 3-space, parallel to the z -axis. Its cross-section in the xy -plane is the circle above.

1.1.3 Open and Closed Sets

Let us zoom out and consider $\mathbb{R}^n$ —the set of points $(x_1, \dots, x_n)$ in n -dimensional Euclidean space. You can always take $n = 2$ or 3 if it helps you make sense of what follows.

We work with the following definitions:

- If $P \in \mathbb{R}^n$, a **neighbourhood** of P is a ball $B_r(P) := \{Q \in \mathbb{R}^n : |PQ| < r\}$, centred at P for some $r > 0$ (which is usually thought of as very small).

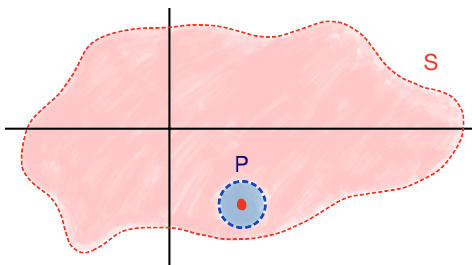


Figure 1.3: A set S is shown in red. The blue disc P is the neighbourhood of some point of S . Notice how P has a **dashed** border—this comes from the $<$ inequality in the definition of a neighbourhood.

- A set $S \subset \mathbb{R}^n$ is **open** if for every point $P \in S$, there is a neighbourhood $B_r(P)$ contained in S . Note: this just has to hold for *some* $r > 0$.
- A set $S \subset \mathbb{R}^n$ is **closed** if its complement $S^c := \mathbb{R}^n \setminus S$ is open.

Let's see some examples.

- Example 1. The ball $\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 < 9\}$ is open. In particular, for each point $C \in B$, we have the neighbourhoods $B_r(C) \subset B$, so long as $0 < r < 3 - |OP|$.

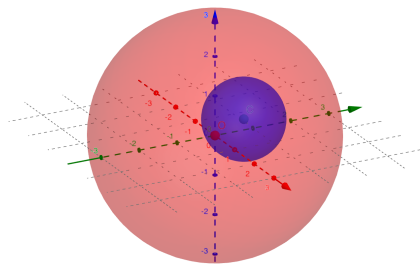


Figure 1.4: Every point $C \in B$ has some neighbourhood $B_r(C)$ contained in B .

- Example 2. The ball $\overline{B} := \{x^2 + y^2 + z^2 \leq 9\}$ in $\mathbb{R}^3$ is **not** open. In fact, for any point $C_1 \in \overline{B}$ satisfying $|OC_1| = 3$, the neighbourhoods $B_r(C_1)$ will necessarily intersect the outside of $\overline{B}$ —that is, the set $\overline{B}^c$ —as well as the inside of $\overline{B}$. However, the ball $\overline{B}$ **is** closed. This is because its complement $\overline{B}^c$ is open. If we take any $C_2 \in \overline{B}^c$, then the distance $|OC_2| > 3$ (this is just the definition of being outside of $\overline{B}$). Since the inequality is strict, we can choose $0 < r < |OC_2| - 3$ such that $B_r(C_2) \subset \overline{B}^c$.

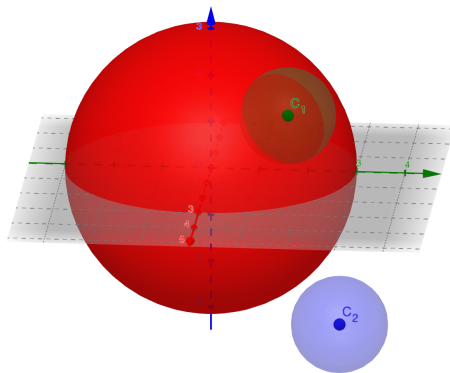


Figure 1.5: The red ball $\overline{B}$ is not open, but it is closed. Compare the neighbourhoods around $C_1 \in \overline{B}$ and $C_2 \in \overline{B}^c$.

- **Example 3.** If S is the rectangle $\{(x, y) \in \mathbb{R}^2 : -2 \leq x \leq 2, -1 < y < 1\}$, then S is **neither** closed **nor** open. Can you figure out why?

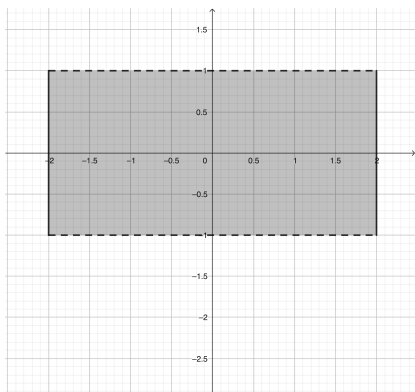


Figure 1.6: Try taking neighbourhoods of points on the solid lines. What happens? For the complement, remember: solid lines become dashed lines and vice-versa.

Having given a few examples of open and closed sets, we can now motivate the definition of boundary. The **boundary**¹ of a set $S \subset \mathbb{R}^n$ is the set of points P such that every neighbourhood of $B_r(P)$ contains both points of S **and also** points of S^c .

For example:

- The open ball B in Example 1 and the closed ball $\overline{B}$ in Example 2 above share the same boundary, the sphere $\{x^2 + y^2 + z^2 = 9\}$.
- The boundary of the rectangle S in Example 3 consists of the four line segments (all solid, no dashes).

We will use $\text{bndry}(S)$ to denote the boundary of S . Boundary points of S may or may not be contained in S . However, if a set is closed, it contains all of its boundary points; similarly, if a set is open, it contains none of its boundary points. See the worked problems below.

Remark 1.1.1. *An alternative description of closed sets in $\mathbb{R}^n$ is the following. The set $S \subset \mathbb{R}^n$ is closed if and only if for every sequence of points $P_1, P_2, \dots$ such that $P_k \in S$ and $\lim_{k \rightarrow \infty} P_k = P$, one has $P \in S$. In words, if a sequence of points in S has a limit then that limit is in S . We will not prove this in this course.*

Again, as a cautious rule-of-thumb: open sets usually are described using strict inequalities like $<$ or $>$; similarly, closed sets are often described using $\leq$ or $\geq$. For example,

$$\{(x, y) \in \mathbb{R}^2 : x + y > 7\} \text{ is an open set.}$$

¹This is the *topological* boundary of a set. It should be distinguished from the “Stokes boundary” of a manifold, which is a different concept that we will study in MATH 227.

whereas

$$\{(x, y) \in \mathbb{R}^2 : -1 \leq y \leq 1\} \text{ is a closed set.}$$

However, expressions involving inequalities can be tricky! So best be careful and figure out what the set actually looks like before deciding whether it is open (or closed, or neither). For example, the set

$$\{(x, y) : x^2 + y^2 \leq 4, x < 10\}$$

is closed, even though it involves one $<$ inequality. This is because the inequality $x^2 + y^2 \leq 4$ actually implies that $x < 10$. So, the set is the closed disc $\{(x, y) : x^2 + y^2 \leq 4\}$ and the $<$ part of the set's definition is redundant. See the picture below.

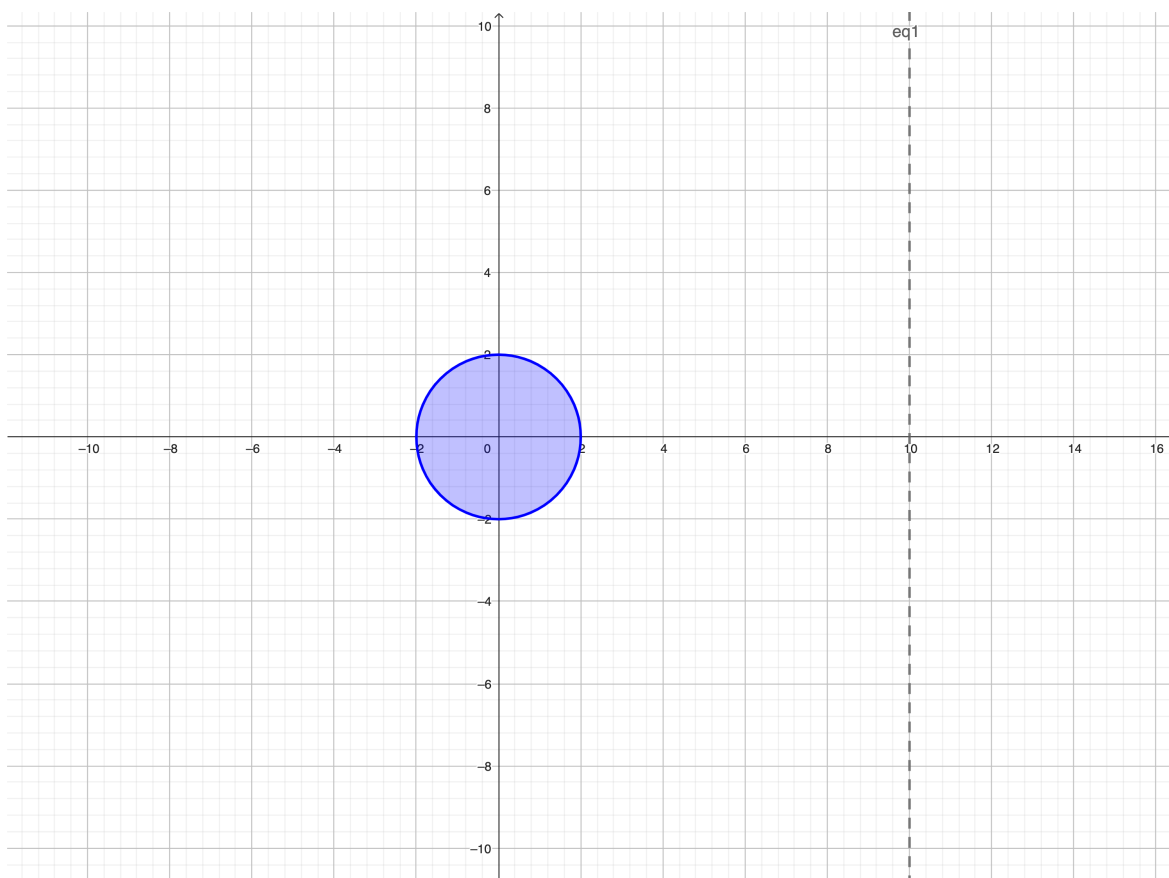


Figure 1.7: The inequality $x < 10$ was redundant. Can you see why in this picture? The vertical line comes from the equation $x = 10$.

Practice working with inequalities in the following way. Verify that,

$$x \leq |x|, \text{ for all } x \in \mathbb{R} \quad \text{and} \quad x^2 \leq x^2 + y^2, \text{ for all } x, y \in \mathbb{R}.$$

Then, use these two inequalities to show that $x^2 + y^2 \leq 4$ implies $x < 10$. Hint: remember that $|x| = \sqrt{x^2}$.

1.1.4 Describing Boundaries: a Final Example

We conclude with a discussion of equations for boundaries of surfaces by considering the following example. Let,

$$S := \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 \leq 9, z \geq 2\}.$$

The boundary of S consists of two parts: part of the sphere above the plane $z = 2$, which is described by the equation

$$\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 9, z \geq 2\}$$

and another disc-like part of the plane $z = 2$, which is defined by the equation

$$\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 \leq 9, z = 2\}.$$

However, notice that if $z = 2$, then $x^2 + y^2 + z^2 = x^2 + y^2 + 4$. So, this last part of the boundary simplifies to

$$\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 \leq 5, z = 2\}.$$

Nifty, because $x^2 + y^2 \leq 5$ is the inequality describing a disc of radius $\sqrt{5}$ in the xy -variables; the $z = 2$ component then lifts this disc to the $z = 2$ plane. You can see these two surfaces in Figure 2.

We will utilize this kind of manipulation of expressions for surfaces and their boundaries very often when we consider derivatives and integrals of multivariate functions. More to come!

1.1.5 Writing Rigorous Proofs

Most of the time, an intuitive understanding of open and closed sets will be sufficient in this course. However, a major goal of this course is for you to become proficient in formulating and writing rigorous mathematical proofs. We will not insist on this level of detail all the time, but you should be able to write out a formal proof if needed.

A “proof” of a mathematical statement is a formal argument justifying why that statement is true, starting from the definitions and theorems that were provided to you and using the principles of logic. This can be different from common-sense explanations. For instance, if you need to explain something informally, using examples can be a great way to do it. We will do this often in this course! But for a formal proof, examples are not sufficient; we have to use general definitions, concepts, and rules of logic.

There are a couple of other things we will practice here that will be important later: using quantifiers (“all”, “some”, “none”), working with inequalities, and working with absolute values. We will review each of these before working on proofs themselves.

It will be important to understand what is a rigorous proof and what is not. When we give a rigorous proof, we will say so explicitly, for example:

Proof. This is how we indicate a rigorous proof. A small square will indicate where the proof ends. The letters QED (from Latin, *quod erat demonstrandum*) are also used in the literature for that purpose. $\square$

When we provide less than that (for example, a sketch of the proof with some details omitted, or a proof that is actually rigorous but only if we make stronger assumptions than those we stated), we will also say that explicitly, with some indication of what would be needed to complete the proof.

We will see many sketches of proofs in this course! All of calculus is based on approximating complicated functions by simpler functions. You have already seen this in single-variable calculus (linear approximation, Taylor polynomials). A typical *outline of a proof* of a theorem in calculus goes like this:

Step 1. X is true for “nice” functions (say, linear functions).

Step 2. A more general function f (say, continuous) can be approximated by a nice function g . Since X is true for g , it is also true for f .

For an *actual proof*, we need to verify that the approximation in Step 2 is good enough so that the property X can indeed be transferred from g to f . This can take the form, for instance, of proving some inequalities for f or estimating the error in the approximation. If you would like to take a peak ahead, an example of this is given in the proof of Theorem 3.1.3 in Section 3.1. The outline of the proof given at the beginning starts with Step 1 (linear functions are continuous) and then states that the same should be true for functions well approximated by linear functions. The actual proof (given below the outline) uses an error estimate.

A major goal of this class is to offer you both fluency in the main ideas in calculus (typically, using various approximations) and the ability to convert such ideas to rigorous mathematics. Too much focus on detail can make it difficult to identify the main ideas. On the other hand, ideas not supported by proofs amount to building castles on quicksand: without an actual proof, you do not really know whether your claims are correct or not. In this class, we will approach this as follows.

- We will provide a representative selection of proofs, illustrating the main methods that are used in arguments of this type. Some of the proofs may be deferred to Worked Problems or Practice Problems, with hints and solutions provided as appropriate.

- In many cases, we will be satisfied with proof outlines rather than detailed proofs. This will save time, avoid long and/or repetitive arguments, and make it easier to focus on the main ideas. However, it is still the goal of this course to give you the tools to be able to work out the details of these proofs if needed. For example, some of the Practice Problems will ask you to prove a specific simple case of a more general theorem.
- Finally, there are also calculus theorems whose proof is beyond the scope of this course. For instance, the theorem on maximizing and minimizing a continuous function on a compact domain (Section 4.2) requires the concept of completeness in mathematical analysis. This is covered in more advanced courses (such as MATH 320 at UBC). In this class, we will just have to state such results without proof.

1.1.6 Inequalities

An important thing about inequalities is that they only work “one way”: in practice, this means that you cannot just manipulate them automatically like equalities. To give a very simple example, if $a < b$ and $b < c$, then $a < c$ by transitivity. But this works in only one direction: if $a < c$, then this does not tell us anything about b . Even if we know that $a < c$ and $a < b$, we still do not know anything about the relationship between c and b .

This has implications for how you need to write proofs and problem solutions involving inequalities. Generally, arguments with inequalities have a more complex structure than a straightforward sequence of equivalences. It is therefore important to include both formulas and sentences indicating your reasoning. To illustrate this, compare the following two examples.

Example 1. $x^2 - 3x + 2 = 0$, $(x - 1)(x - 2) = 0$, $x = 1, 2$.

It is completely clear that someone is simplifying and solving a quadratic equation. No further explanations are needed.

Example 2. $|x^2 - y^2| < 1$, $x^2 < 1/4$ and $y^2 < 1/4$, $\sqrt{x^2 + y^2} < 1/2$.

Can you even tell what is happening here? Perhaps this person is saying this:

If $|x^2 - y^2| < 1$, then $x^2 < 1/4$ and $y^2 < 1/4$, therefore $\sqrt{x^2 + y^2} < 1/2$.

This is obviously incorrect: for instance, if $x = y = 10$, then $|x^2 - y^2| = 0$ but $\sqrt{x^2 + y^2} = 10\sqrt{2} > 1/2$. But it is also possible that they are saying something else:

We need to find a neighbourhood of $(0, 0)$ where $|x^2 - y^2| < 1$. The given inequality holds for example when $x^2 < 1/4$ and $y^2 < 1/4$, and both of these are true when $\sqrt{x^2 + y^2} < 1/2$, so that the last inequality defines a neighbourhood that works.

This would be a correct argument if it had been written this way, which it was not. (We will actually use very similar arguments in the proofs for open and closed sets below, and later in ϵ - δ proofs for limits.) To be clear, it's a very good practice to include sentences in all of your solutions! But in proofs with inequalities, including sentences can really make the difference between a correct solution and something that's either impossible to follow or incorrect as stated.

An important inequality involving the absolute value is the **triangle inequality**:

$$|a + b| \leq |a| + |b|.$$

This looks similar to the triangle inequality in geometry: if P, Q, R are three points in $\mathbb{R}^n$, then

$$|PR| \leq |PQ| + |QR|.$$

Both inequalities are special cases of the general triangle inequality for norms in vector spaces (to be covered in your linear algebra corequisite class). It is useful to think about absolute values as distances on the line, for example $|a - b|$ is the distance from a to b . Then the triangle inequality for the absolute value is a special case of the triangle inequality for points in $\mathbb{R}^n$, if let $n = 1$ and interpret $|a|, |b|, |a + b|$ as the distances on the real line between 0 and a , between 0 and $-b$, and between a and $-b$. It is also useful to remember that if $P = (x, y)$ and $Q = (x', y')$ are two points in the plane, then

$$|x - x'| \leq \sqrt{(x - x')^2 + (y - y')^2} = |PQ|,$$

and similarly for the y -coordinates. A similar statement is also true in higher dimensions. In the rest of this course, we will use these inequalities without proof.

1.1.7 Quantifiers

In formal logic, a *quantifier* is an operation which tells us how many elements x in a given set S possess a given property P . For us, the primary quantifiers we are concerned with are “all” (symbolically: $\forall$) and “exists” (symbolically: $\exists$). “All” tells us that every single element of a given set has a given property; whereas, “exists” tells us that *at least one* (but perhaps more than one) element has a given property. The purpose of introducing quantifiers is to allow us to build formal *truth statements*, which are statements which can either be true or false. The following very simple example allows us to illustrate this purpose.

Consider the following statement: “All positive even numbers are greater than one.” This breaks into three logical components: a set of objects S (“positive even numbers”), a property $\mathcal{P}$ (“greater than one”), and a quantifier $\forall$ (“all”). Together, these three logical components form a statement, which just so happens to be true. Of course, if we change any one of these three components, our statement may no longer be true. For example, the statement, “All even numbers are greater than one”, and the statement “All positive even numbers are greater than three”, are both clearly false.

The quantifiers “all” and “exists” are related to one another through a process called *negation*. Before introducing a more formal definition, let us introduce the concept of negation, and its relation to our quantifiers, through the following example.

Consider the following statement: “All continuous functions are differentiable”. This is a truth statement, with a set of objects S (“continuous functions”), a property $\mathcal{P}(f)$ (“ f is differentiable”), and a quantifier $\forall$ (“all”). Let us label the whole statement as T , and use shorthand to write $T = \forall_{f \in S} \mathcal{P}(f)$. (We may also write the “set of objects” part inline, especially in displayed formulas such as (1.1.3) below. The important thing is to make it clear where the “set of objects” part ends and the property $\mathcal{P}$ begins.) Now, from differential calculus, we know the statement T is false. Indeed, the function $f(x) = |x|$ is a *counterexample* (i.e. a specific element in the set S for which T is false).

This counterexample actually shows that the related statement, “There exists a continuous function which is not differentiable”, is a true statement. Moreover, this related truth statement consists of the same set (“continuous functions”), but importantly has the negative (“is not differentiable”) of our original property (“is differentiable”) and the quantifier has flipped (“all” became “exists”).

This property from the previous example is, in fact, true for general statements. Formally, if S is a given set of elements, $\mathcal{P}$ is a property, $\neg\mathcal{P}$ is its negative², and we take the truth statement $T = \forall_{x \in S} \mathcal{P}$, then T is false if and only if the statement $\neg T = \exists_{x \in S} (\neg\mathcal{P})$ is true. This situation was illustrated by our previous example. Similarly, if T is true, then $\neg T$ is necessarily false. Moreover, if instead we take $T = \exists_{x \in S} \mathcal{P}$ as our truth statement, then T is true if and only if $\neg T = \forall_{x \in S} (\neg\mathcal{P})$ is false (and, again, vice versa). This is the principle of negation, and holds for *any* truth statements which can be formed from a set of objects, a property, and a quantifier “all” or “exists”.

While this discussion of truth or falsehood may feel overly formal, remember: the goal is to establish a general framework for determining whether many different types of logical statements are true or false. A proof is the rigorous and logical process by which we establish a given statement is true. We will spend the next subsection putting together these abstract ideas, and illustrating how quantifiers are utilized in proofs.

1.1.8 A First Foray into Proofs: Open Sets

Now, let’s look at a couple of proofs for open and closed sets.

Question 1. *Prove directly from the definition that the open square $S = \{(x, y) \in \mathbb{R}^2 : -1 < x < 1, -1 < y < 1\}$ is an open set in $\mathbb{R}^2$. (Given a point $P = (x, y) \in S$, find a neighbourhood of P which is contained in S .)*

Let’s first think about what we need to prove. A point $P = (x, y)$ belongs to S if and only if $-1 < x < 1$ and $-1 < y < 1$. In terms of absolute values, $|x| < 1$ and

²Other ways to denote the negation of $\mathcal{P}$ are $!\mathcal{P}$ and $\sim \mathcal{P}$. All are commonly used in math literature.

$|y| < 1$. We need to prove that every such point has a neighbourhood contained in S . In terms of quantifiers,

$$\forall P \in S \exists r > 0 \text{ such that } B_r(P) \subset S. \quad (1.1.3)$$

A good way to prove that something *exists* is to actually point to it. That is, for every $P = (x, y)$ with $|x| < 1$ and $|y| < 1$, we want to find $r > 0$ (possibly depending on P) such that the entire open disc $B_r(P) = \{(x', y') : \sqrt{(x - x')^2 + (y - y')^2} < r\}$ is contained in S .

How do we find such r ? It helps to draw a picture (you should actually do it at this point). From the picture, the closest distance from P to the boundary of the square is the smallest of the numbers $1 - x, x - (-1), 1 - y, y - (-1)$. You could work with that, but it will be easier to write the proof if we notice that this distance is also equal to $\min(1 - |x|, 1 - |y|)$ (can you explain why?). We set r to be equal to this number. Once we know what r should be, we need to use inequalities (usually, the triangle inequality) to prove that $B_r(P)$ is actually contained in S .

(Since this is a formal proof, it would not be enough to check this only for a few selected points in the square, say $P = (0, 1/2)$ and $P = (1/2, 1/2)$. We have to check it for *all* points P in the square. That's why we use the general coordinates (x, y) .)

Here, then, is the proof.

Proof. Let $P = (x, y) \in S$. We need to prove that $B_r(P) \subset S$ for some $r > 0$. Since $-1 < x < 1$ and $-1 < y < 1$, we have $|x| < 1$ and $|y| < 1$. Let $r = \min(1 - |x|, 1 - |y|)$. Then $r > 0$.

We claim that $B_r(P)$ is contained in S . To see this, let $Q = (x', y')$ be a point in $B_r(P)$. Then

$$|x - x'| \leq \sqrt{(x - x')^2 + (y - y')^2} = |PQ| < r.$$

By the triangle inequality, $|x'| \leq |x| + |x' - x| < |x| + r \leq |x| + (1 - |x|) = 1$. Similarly, $|y - y'| < r$, and by the triangle inequality $|y'| < 1$. Therefore $(x', y') \in S$.

This proves that every point $P \in S$ has a neighbourhood contained in S , so that S is open. $\square$

Question 2. Let $S' = \{(x, y, z) \in \mathbb{R}^3 : -1 < x < 1, -1 < y < 1, z = 0\}$. Is S' an open set in $\mathbb{R}^3$? Prove your answer.

Let's first figure out the answer informally. The set S' is a square in the xy -plane in 3 dimensions. For S' to be open, it would have to contain some 3-dimensional neighbourhood of each of its points. But S' is 2-dimensional, so it cannot contain 3-dimensional open balls. We now write this up formally.

Proof. S' is not an open set. For example, $P = (0, 0, 0)$ belongs to S' , but has no neighbourhood contained in S' . To check that last part, let $B_r(P)$ be a neighbourhood

of P in $\mathbb{R}^3$. Then $B_r(P)$ contains the point $Q = (0, 0, r/2)$, but $Q \notin S'$ since the z -coordinate of Q is not 0. $\square$

We mentioned earlier that open and closed sets can be characterized in terms of whether they contain their boundary points. We now prove this.

Question 3. *Prove that a set in $\mathbb{R}^n$ is open if and only if it does not contain any of its boundary points.*

Proof. Let $S \subseteq \mathbb{R}^n$. Recall that a boundary point of S is a point P such that every neighbourhood $B_r(P)$ contains at least one point of S and at least one point not in S .

- We first prove that if S is open then no boundary points of S belong to S . Suppose that S is open, and let P be a boundary point of S . If P were a point in S , then by the definition of an open set, there would be a neighbourhood $B_r(P)$ containing only points of S (and no points that are not in S). That contradicts P being a boundary point.
- We now prove that if no boundary points of S belong to S , then S is open. Suppose that S contains none of its boundary points. We have to check that every $P \in S$ has a neighbourhood $B_r(P) \subseteq S$. Let $P \in S$. For every neighbourhood $B_r(P)$, we have $P \in B_r(P)$, so every such neighbourhood contains at least one point of S (specifically, the point P). If every such neighbourhood also contained points that are not in S , then every neighbourhood would contain both types of points, so P would be a boundary point. But we assumed that S does not contain any of its boundary points, so that's a contradiction. Therefore there must be at least one neighbourhood $B_r(P)$ that does not contain points not in S , so that for that r we have $B_r(P) \subseteq S$.

$\square$

Question 4. *Prove that if S is a closed set in $\mathbb{R}^n$, and P is a boundary point of S , then $P \in S$. You can either work directly with the definitions of a closed set and a boundary point, or else use the statement from Q3.*

Proof. Suppose that S is a closed set in $\mathbb{R}^n$. Let S^c be the complement of S , then S^c is open (by the definition of closed sets). If P is a boundary point of S , then every neighbourhood $B_r(P)$ of P contains both points in S and points in S^c , so that P is also a boundary point of S^c . Since S^c is open, $P \notin S^c$. Therefore $P \in S$. $\square$

Question 5. *Let $S \subset \mathbb{R}^2$ be a nonempty set. Prove directly from the definition that the set*

$$S' = \{P \in \mathbb{R}^2 : \text{there exists } Q \in S \text{ such that } |PQ| < 1\}$$

is an open set. (You need to prove that, given a point $P = (x, y) \in S'$, there is an $r > 0$ such that the neighbourhood $B_r(P)$ is contained in S' . Note that, in the above condition defining S' , the point Q may depend on P .)

Proof. Let $P' = (x, y) \in S'$. Then there exists $Q \in S$ such that $|PQ| < 1$. Let $r = 1 - |PQ|$, so that $r > 0$. It suffices to prove that $B_r(P) \subseteq S'$. To prove this, we need to check that any $P' \in B_r(P)$ belongs to S' . Let $P' \in B_r(P)$, then $|PP'| < r$, so that

$$|P'Q| \leq |P'P| + |PQ| < r + |PQ| = 1,$$

so that the condition for $P' \in S'$ is satisfied (with the same Q). $\square$

Practice Problems

You can use the solutions to Questions 1-4 above as templates for writing proofs of this type. Note the following:

- If you are proving that a set S is open, then for *every* point $P \in S$ you have to find a neighbourhood that works for that point (that is, $B_r(P) \subset S$.) You find that neighbourhood by finding a value of $r > 0$ that works for P . The choice of r is not unique, in fact if some $r > 0$ works for P , then any r' such that $0 < r' < r$ also works.

This strategy applies every time you have to prove that a set S is open, but the way you choose r will depend on the set S (so this part will not be the same for all questions of this type).

- If you are proving that S is not open, then it's enough to find just one point $P \in S$ where the open set condition fails, i.e. no neighbourhood of P is contained in S . Then the reasoning is: identify P , let $r > 0$ be arbitrary, and prove that $B_r(P)$ is not contained in S . To do that last part, for any given $r > 0$ (you should keep it arbitrary, so don't assign it any value), find at least one point Q such that $Q \in B_r(P)$ but $Q \notin S$.

In the first two questions, short answers are sufficient. The remaining problems ask for formal proofs.

1. Decide whether each of the sets below is open, closed, or neither.

- (a) $\{(x, y) \in \mathbb{R}^2 : 0 \leq x < 1, 0 \leq y < 2\}$
- (b) $\{(x, y) \in \mathbb{R}^2 : x + y < 2\}$
- (c) $\{(x, y) \in \mathbb{R}^2 : 0 < \sqrt{x^2 + y^2} \leq 3\}$

- (d) $\{(x, y, z) \in \mathbb{R}^2 : |x| \geq 6\}$
2. The following conditions describe sets in $\mathbb{R}^3$. Are these sets open, closed, or neither? Specify the boundary of each set.
- (a) $x^2 + y^2 + z^2 \geq 16$
 - (b) $x^2 + y^2 \geq 4, z \geq 0$
 - (c) $0 < x < 1, 0 < y < 1, z = 1$
 - (d) $0 < \sqrt{x^2 + y^2} \leq 3$
 - (e) $x \geq 0, x^2 + y^2 + z^2 \geq 4$
 - (f) $x > 0, 0 \leq y \leq 4$
3. Prove directly from the definition that the set $S = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 < 4\}$ is an open set in $\mathbb{R}^3$. (Given a point $P = (x, y, z) \in S$, find a neighbourhood of P which is contained in S .)
4. Let $S' = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 < 4, z \geq 0\}$. Is S' an open set in $\mathbb{R}^3$? Is it closed? Prove your answers (again, using points and neighbourhoods).
5. Prove directly from the definition that the set $S = \{(x, y) \in \mathbb{R}^2 : x + y > 0\}$ is an open set in $\mathbb{R}^2$. (Given a point $P = (x, y) \in S$, find an $r > 0$ such that the neighbourhood $B_r(P)$ is contained in S .)
6. Let $S' = \{(x, y) \in \mathbb{R}^2 : x + y > 0, x - y \leq 0\}$. Is S' an open set in $\mathbb{R}^2$? Prove your answers (again, using points and neighbourhoods).
7. Let $S = \{(x, y) \in \mathbb{R}^2 : |x| \leq 1, |y| < 1\}$. Prove directly from the definition (again, using points and neighbourhoods) that S is **not** a closed set in $\mathbb{R}^2$.
8. Prove from the definition that the set $S := \{(x, y) : x^2 + 4y^2 < 1\}$ is open in $\mathbb{R}^2$. As above, this means: for each point $P \in S$, find an $r > 0$ such that $B_r(P) \subset S$.

Hint: The calculations in this question might be a bit longer, depending on how you choose your $r > 0$. It may help if you do **not** try to make r as large as possible. Rather, it suffices to find *some value* $r > 0$ such that $B_r(P) \subset S$. A discussion of this idea is given in the Solution Manual.

(We will figure out later how to use calculus methods to find the distance from a point to a curve. This could be used to find the optimal value of r . However, we do not need that for the purpose of this exercise.)

9. Let $S \subset \mathbb{R}^2$ be an open set. Prove directly from the definition that the set

$$S' = \{(x, y) \in \mathbb{R}^2 : (2x, 2y) \in S\}$$

is also an open set. (We know that S is open. This means that, given a point $P = (x, y) \in S$, there is an $r > 0$ such that the neighbourhood $B_r(P)$ is contained in S . You have to prove that the same condition is satisfied for all $P' \in S'$, possibly with a different value of r .)

1.2 Vectors, Dot Products, and Projections

1.2.1 Vectors

A vector $\mathbf{v}$ in $\mathbb{R}^2$ or $\mathbb{R}^3$ has two aspects: magnitude (length) and direction. Geometrically, we can represent a vector by an ordered pair of points, say $\mathbf{v} = \vec{AB}$. For example, if $A := (a_1, a_2, a_3)$ and $B := (b_1, b_2, b_3)$ are points in $\mathbb{R}^3$, then the vector $\vec{AB} = \langle b_1 - a_1, b_2 - a_2, b_3 - a_3 \rangle$ looks like an arrow rooted at A with tip B . Any vector $\mathbf{v}$ has many such representations. Taking points C and D in $\mathbb{R}^3$ which satisfy $d_i - c_i = b_i - a_i$ for $i = 1, 2, 3$ produces the same vector $\mathbf{v} = \vec{CD} = \vec{AB}$.

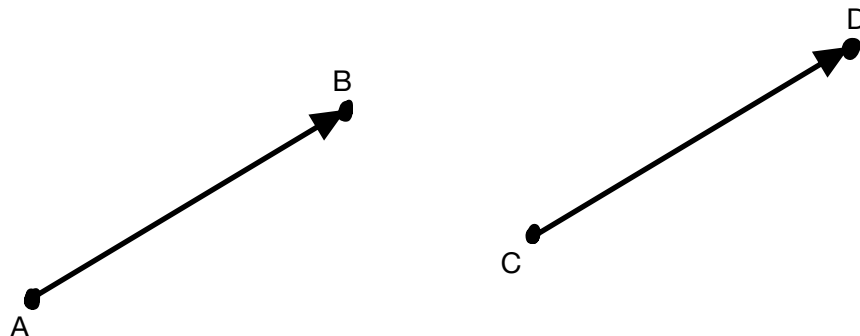
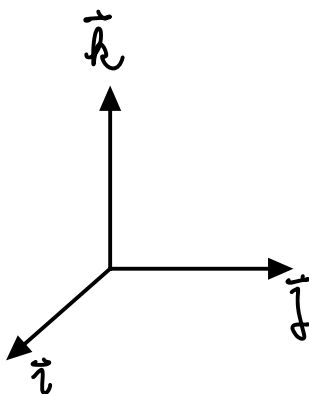


Figure 1.8: The same vector in $\mathbb{R}^2$ can be represented by either one of the ordered pairs of points (A, B) and (C, D) .

As an example, we can write vector $\mathbf{v} := \langle 1, 1, 1 \rangle$ as $\vec{AB}$ and $\vec{CD}$ where A, B, C, D all satisfy,

$$(1, 1, 1) = \underbrace{(3, 3, 3)}_A - \underbrace{(2, 2, 2)}_B = \underbrace{(\sqrt{3} + 1, \pi + 1, 72.1 + 1)}_C - \underbrace{(\sqrt{3}, \pi, 72.1)}_D.$$

Not all representations are equally aesthetically pleasing or easy to use; so, it's a good reminder to choose our representations carefully.

Figure 1.9: The standard basis vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ in $\mathbb{R}^3$.

For a vector $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$, the real numbers v_1, v_2, v_3 are the **components** of $\mathbf{v}$. The vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ are the **standard basis vectors** in $\mathbb{R}^3$. They are given by the equations,

$$\mathbf{i} := \langle 1, 0, 0 \rangle \quad \mathbf{j} := \langle 0, 1, 0 \rangle \quad \mathbf{k} := \langle 0, 0, 1 \rangle.$$

In 3-dimensional space, they form the corner of a cube with side-length 1.

In $\mathbb{R}^2$, we will only use the vectors $\mathbf{i} := \langle 1, 0 \rangle$ and $\mathbf{j} := \langle 0, 1 \rangle$. In $\mathbb{R}^n$, for $n > 3$, one usually uses the notation $\mathbf{e}_1, \dots, \mathbf{e}_n$, where $\mathbf{e}_k$ is the vector with zeroes in all entries $j \neq k$ and 1 in the k -th entry.

1.2.2 Notation for vectors

Different sources (textbooks, WebWork, instructors) will have different preferences for their notation for vectors. For example, if $\mathbf{v}$ is a vector in $\mathbb{R}^3$, we can write

$$\mathbf{v} = \vec{v} = \langle v_1, v_2, v_3 \rangle = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}.$$

In particular, the boldface notation $\mathbf{v}$ means the same thing as the over-arrow notation $\vec{v}$. You can use whichever you like. In these notes, we (mostly) retain the notation $\mathbf{v}$, but we will use the notation $\vec{AB}$ to denote the vector with tail at A and tip at B . In handwritten materials where boldface is difficult to render, it is preferable to use $\vec{v}$. Similarly, we will mostly use angular brackets as in $\langle v_1, v_2, v_3 \rangle$ for vectors and round brackets as in (x_1, x_2, x_3) for points, but the round bracket notation is also often used for vectors.

We also have the following notations:

- The zero vector in $\mathbb{R}^3$ is $\mathbf{0} := \langle 0, 0, 0 \rangle$; in $\mathbb{R}^2$, the definition is similar.
- The **length** of a vector $\mathbf{v} := \langle v_1, v_2, v_3 \rangle$ is $|\mathbf{v}| = \sqrt{v_1^2 + v_2^2 + v_3^2}$. This length is, in fact, a **norm** on $\mathbb{R}^3$, a concept you have likely encountered in your linear algebra

course. However, there are other norms on $\mathbb{R}^3$ which are **not** the length. The length of a vector is always nonnegative, and $|\mathbf{v}| = 0$ if and only if $\mathbf{v} = \mathbf{0}$.

- A **unit vector** is a vector of length 1. If $\mathbf{v}$ is any vector with $|\mathbf{v}| \neq 0$, then $\mathbf{u} := \mathbf{v}/|\mathbf{v}|$ is a unit vector in the direction of $\mathbf{v}$.

1.2.3 The dot product

One of the most important definitions for vectors in $\mathbb{R}^3$ is the following.

Definition 1.2.1. If $\mathbf{u} := \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} := \langle v_1, v_2, v_3 \rangle$ then the **dot product** $\mathbf{u} \cdot \mathbf{v}$ is the real number

$$\mathbf{u} \cdot \mathbf{v} := u_1v_1 + u_2v_2 + u_3v_3.$$

The dot product is also called the **scalar product** or **inner product** in other resources. (Strictly speaking, “inner product” and “scalar product” are more general concepts that can be defined on vector spaces other than $\mathbb{R}^n$. You will see that in linear algebra.) The dot product in $\mathbb{R}^2$ has a similar definition, just without the third coordinate. The proposition below lists a few basic properties of the dot product (you should try and prove them yourself!).

Proposition 1.2.2. Let $\mathbf{u}, \mathbf{v}$ and $\mathbf{w}$ be vectors in $\mathbb{R}^3$ and let c be some real scalar. Then,

- (i) $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$ (commutativity)
- (ii) $\mathbf{u} \cdot \mathbf{u} = |\mathbf{u}|^2$ (relation to the length norm $|\cdot|$)
- (iii) $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$ (distributive law)
- (iv) $(c\mathbf{u}) \cdot \mathbf{v} = c(\mathbf{u} \cdot \mathbf{v}) = \mathbf{u} \cdot (c\mathbf{v})$ (scalar multiplication)

There is also an important geometric interpretation of the dot product.

Proposition 1.2.3. Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in $\mathbb{R}^3$. Choose the angle θ between $\mathbf{u}$ and $\mathbf{v}$ satisfying $0 \leq \theta \leq \pi$. Then,

$$\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}| \cos \theta.$$

This is based on the Cosine Law in geometry. Consider the triangle spanned by the vectors $\mathbf{u}$ and $\mathbf{v}$, with the angle θ between them. The vector $\mathbf{v} - \mathbf{u}$ completes the triangle. The Cosine Law is a relation between the sides of the triangle:

$$|\mathbf{v} - \mathbf{u}|^2 = |\mathbf{u}|^2 + |\mathbf{v}|^2 - 2|\mathbf{u}||\mathbf{v}| \cos \theta.$$

By (ii) above, we have $|\mathbf{v} - \mathbf{u}|^2 = (\mathbf{v} - \mathbf{u}) \cdot (\mathbf{v} - \mathbf{u})$. Expand this and simplify using the properties above... what do you get? Can you finish the proof?

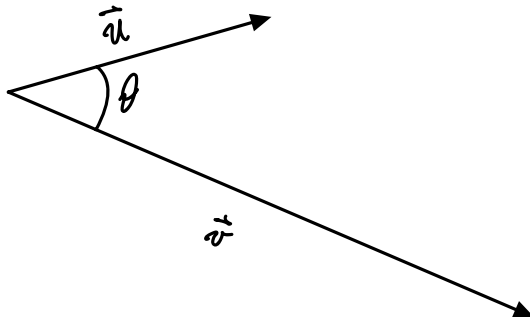


Figure 1.10: The angle between two vectors $\mathbf{u}$ and $\mathbf{v}$

Now, you might be scratching your head and asking: why did we choose the angle interior to $\mathbf{u}$ and $\mathbf{v}$ in the previous figure? Well, it turns out, we could have picked either! This is because

$$\cos(\theta_{\text{interior}}) = \cos(-\theta_{\text{interior}}) = \cos(2\pi - \theta_{\text{interior}}) = \cos(\theta_{\text{exterior}}).$$

To show this, we used that the cosine function is even and has period 2π radians.

1.2.4 Angles between vectors

The dot product can be used to find a formula for the angle between vectors in terms of their components. If $\mathbf{u}$ and $\mathbf{v}$ make angle θ with one another, one has

$$\cos \theta := \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}||\mathbf{v}|}$$

Example 1.2.4. Find the angle between the vectors $\mathbf{i} + \mathbf{k}$ and $2\mathbf{i} - \mathbf{j}$.

Solution: Let $\mathbf{u} := \mathbf{i} + \mathbf{k}$ and $\mathbf{v} := 2\mathbf{i} - \mathbf{j}$. Then $\mathbf{u} \cdot \mathbf{v} = 2 + 0 + 0 = 2$. Moreover, $|\mathbf{u}| = \sqrt{1+1} = \sqrt{2}$ and $|\mathbf{v}| = \sqrt{4+1} = \sqrt{5}$. So, $\cos(\theta) = \frac{2}{\sqrt{10}} \approx .63$. So, we have:

$$\theta = \arccos(0.63) \approx 0.866 \text{ radians.}$$

Note: you need a calculator to determine $\arccos(0.63)$ to three decimal places.

Remark 1.2.1. An important special case is the situation when $\theta = \frac{\pi}{2}$ (which is another way of saying that $\mathbf{u}$ and $\mathbf{v}$ are perpendicular). Then $\cos(\theta) = 0$, so that $\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}| \cos \theta = 0$.

Conversely, if $\mathbf{u} \cdot \mathbf{v} = 0$, then either $|\mathbf{u}| = 0$, or $|\mathbf{v}| = 0$, or $\cos(\theta) = 0$. Assume first that $\mathbf{u} \cdot \mathbf{v} = 0$ but $\mathbf{u}$ and $\mathbf{v}$ are nonzero. Then $\cos \theta = 0$, so that $\theta = \frac{\pi}{2}$ (in the definition of the dot product, we used the internal angle $0 \leq \theta \leq \pi$). Hence, $\mathbf{u}$ and $\mathbf{v}$ are perpendicular. To keep things consistent, we will say that the zero vector $\mathbf{0}$ is perpendicular to all vectors by convention.

Let's summarize this as

$$\mathbf{u} \perp \mathbf{v} \iff \mathbf{u} \cdot \mathbf{v} = 0.$$

Example 1.2.5. Let $\mathbf{u} := \langle 2, -1, 3 \rangle$, $\mathbf{v} := \langle 0, 4, 2 \rangle$ and $\mathbf{w} := \langle 1, -1, -1 \rangle$. Then, $\mathbf{u} \cdot \mathbf{v} = -4 + 6 \neq 0$ and $\mathbf{u} \cdot \mathbf{w} = 2 + 1 - 3 = 0$. So, $\mathbf{u}$ is perpendicular to $\mathbf{w}$, but is not perpendicular to $\mathbf{v}$.

1.2.5 Scalar and Vector Projections

Let $\mathbf{w}$ and $\mathbf{v}$ be two vectors with $\mathbf{v} \neq \mathbf{0}$. We want to find two vectors $\mathbf{a}$ and $\mathbf{b}$ such that:

- $\mathbf{w} = \mathbf{a} + \mathbf{b}$,
- $\mathbf{a}$ is parallel to $\mathbf{v}$ (but may point in the opposite direction),
- $\mathbf{b}$ is perpendicular to $\mathbf{v}$.

Then $\mathbf{a}$ is the **vector projection** of $\mathbf{w}$ on $\mathbf{v}$ (notation: $\mathbf{a} = \mathbf{w}_\mathbf{v}$).

We first note (see the picture below) that $\mathbf{a}$ and $\mathbf{b}$ depend only on the *direction* of $\mathbf{v}$. Therefore, in the calculation for $\mathbf{a}$ and $\mathbf{b}$, we will use instead the unit vector $\mathbf{u} = \mathbf{v}/|\mathbf{v}|$ in the same direction. Since $\mathbf{a}$ is parallel to $\mathbf{v}$, therefore also to $\mathbf{u}$, we have $\mathbf{a} = s\mathbf{u}$ for some real number s that we will call the **scalar projection** of $\mathbf{w}$ on $\mathbf{v}$. We find s from the condition that $\mathbf{b} = \mathbf{w} - \mathbf{a}$ should be perpendicular to $\mathbf{u}$:

$$\begin{aligned} 0 &= (\mathbf{w} - \mathbf{a}) \cdot \mathbf{u} = (\mathbf{w} - s\mathbf{u}) \cdot \mathbf{u} \\ &= \mathbf{w} \cdot \mathbf{u} - s\mathbf{u} \cdot \mathbf{u} = \mathbf{w} \cdot \mathbf{u} - s, \end{aligned}$$

where at the end we used that $\mathbf{u} \cdot \mathbf{u} = |\mathbf{u}|^2 = 1$. Therefore $s = \mathbf{w} \cdot \mathbf{u}$. This gets us the following formulas for the vector and scalar projections:

Suppose $\mathbf{w}$ and $\mathbf{v}$ are vectors in $\mathbb{R}^3$, with $\mathbf{v} \neq \mathbf{0}$. Then, the vector projection and scalar projection of $\mathbf{w}$ on $\mathbf{v}$ are given by the formulae,

$$\begin{aligned} \text{(Scalar projection)} \quad s &= \frac{\mathbf{w} \cdot \mathbf{v}}{|\mathbf{v}|} = |\mathbf{w}| \cos \theta, \\ \text{(Vector projection)} \quad \mathbf{a} = \mathbf{w}_\mathbf{v} &= s \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{\mathbf{w} \cdot \mathbf{v}}{|\mathbf{v}|^2} \mathbf{v} \end{aligned}$$

where θ is the angle between $\mathbf{w}$ and $\mathbf{v}$.

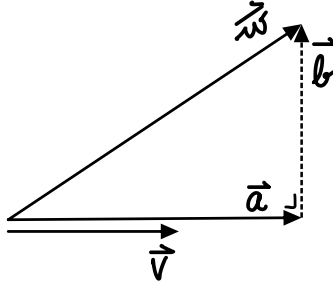


Figure 1.11: Projections of $\mathbf{w}$ onto $\mathbf{v}$, with $\mathbf{a}$ as the vector projection of $\mathbf{w}$ onto $\mathbf{v}$. Since $\mathbf{w}$ points in the same direction as $\mathbf{v}$, we have $+\|\mathbf{w}\|$ as the scalar projection. The vectors $\mathbf{v}$ and $\mathbf{w}$ are *always collinear*! In this diagram, $\mathbf{v}$ is drawn slightly off from $\mathbf{a}$ to make the picture less messy.

It is important to remember that the vector projection is a vector and the scalar projection is a scalar (just a number!). The fraction

$$\frac{\mathbf{w} \cdot \mathbf{v}}{|\mathbf{v}|^2}$$

in the vector projection formula is also just a number, so that $\mathbf{a}$ (equal to that number times $\mathbf{v}$) is a vector parallel to $\mathbf{v}$. The length of $\mathbf{a}$ is $|\mathbf{a}| = |s\mathbf{u}| = |s|$, since $\mathbf{u}$ has length 1. Notice that the number s is positive if the angle θ is acute and negative if θ is obtuse (see Figure 1.12 below). Thus $s = \pm|\mathbf{a}|$, and we choose $+$ or $-$ based on whether $\mathbf{a}$ points in the same (or opposite) direction as $\mathbf{v}$.

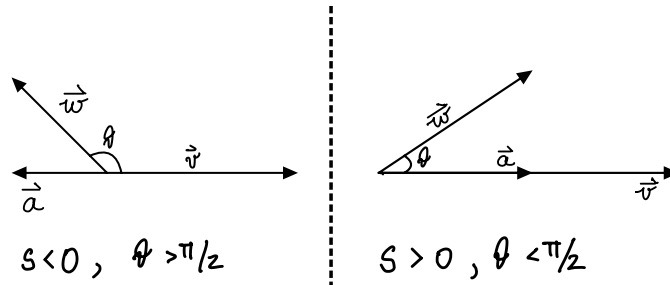


Figure 1.12: If $\theta < \frac{\pi}{2}$, then $\cos \theta > 0$; in this situation, we see that $\mathbf{a} = \mathbf{w}_{\mathbf{v}}$ points in the same direction as $\mathbf{v}$. When $\theta > \frac{\pi}{2}$, we have $\cos \theta < 0$; in this case, we see that $\mathbf{a}$ points in the opposite direction from $\mathbf{v}$.

Worked Problems

Here are some examples of how we use the dot product and vector projection.

1. (Simple) If $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \mathbf{i}$, then $\mathbf{u} \cdot \mathbf{v} = u_1$.
2. (Less Simple) Write the vector $2\mathbf{i} - \mathbf{k}$ as a sum $\mathbf{u} + \mathbf{v}$, where $\mathbf{u}$ is parallel to $\mathbf{i} + 3\mathbf{j}$ and $\mathbf{v}$ is perpendicular to $\mathbf{u}$.

We have that $\mathbf{u}$ is the vector projection of $2\mathbf{i} - \mathbf{k}$ onto $\mathbf{i} + 3\mathbf{j}$. Therefore,

$$\mathbf{u} = \frac{(2\mathbf{i} - \mathbf{k}) \cdot (\mathbf{i} + 3\mathbf{j})}{|\mathbf{i} + 3\mathbf{j}|^2}(\mathbf{i} + 3\mathbf{j}) = \frac{1}{5}(\mathbf{i} + 3\mathbf{j}).$$

To find $\mathbf{v}$, recall the relationship of the perpendicular vector that of the projection of $(2\mathbf{i} - \mathbf{k})$ on $(\mathbf{i} + 3\mathbf{j})$. Namely, we must have:

$$\mathbf{v} = (2\mathbf{i} - \mathbf{k}) - \frac{1}{5}(\mathbf{i} + 3\mathbf{j}) = \frac{9}{5}\mathbf{i} - \frac{3}{5}\mathbf{j} - \mathbf{k}.$$

3. (Even Less Simple) Find all vectors $\mathbf{v}$ that satisfy all of the following conditions:
 - (a) $|\mathbf{v}| = 2$,
 - (b) $\mathbf{v}$ is perpendicular to $\mathbf{i} - 2\mathbf{j} + \mathbf{k}$,
 - (c) $\mathbf{v}$ makes a 45 degrees angle with $\mathbf{i} - \mathbf{k}$.

Let $\mathbf{v} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ be any vector in $\mathbb{R}^3$. We want to find the equations that x, y, z must solve so that $\mathbf{v}$ satisfies the three conditions stated. For convenience let $\mathbf{u} = \mathbf{i} - 2\mathbf{j} + \mathbf{k}$ and $\mathbf{w} = \mathbf{i} - \mathbf{k}$. Then condition (b) says that $\mathbf{v} \cdot \mathbf{u} = 0$, and so from the definition of dot product we have

$$x - 2y + z = 0. \tag{1.2.1}$$

To work with condition (c) we use the relation between the dot product and the angle between two vectors, which gives

$$\mathbf{v} \cdot \mathbf{w} = |\mathbf{v}||\mathbf{w}| \cos 45^\circ.$$

We may use the definition of the dot product on the left hand side, the fact that $|\mathbf{v}| = 2$, compute that $|\mathbf{w}| = \sqrt{2}$ and $\cos 45^\circ = 1/\sqrt{2}$, to obtain

$$x - z = 2.$$

This allows us to solve for z in terms of x only, namely $z = x - 2$, and so we can now eliminate z from the rest of the problem. For example, plugging back into (1.2.1) and rearranging gives

$$y = x - 1.$$

Finally, note that condition (a) may be written as $|\mathbf{v}|^2 = 4$, that is,

$$x^2 + y^2 + z^2 = 4.$$

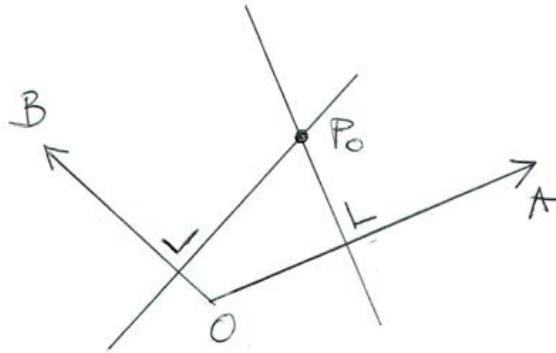
Using what we found above to eliminate both y and z gives $x^2 + (x-1)^2 + (x-2)^2 = 4$, which simplifies to $3x^2 - 6x + 1 = 0$. This is a quadratic equation in x , so we may use the quadratic formula to get that

$$x = 1 \pm \frac{\sqrt{6}}{3}.$$

For each of these values of x we can find the corresponding values of y and z from above, and hence we are left with two vectors $\mathbf{v}_1 = (1 + \frac{\sqrt{6}}{3})\mathbf{i} + \frac{\sqrt{6}}{3}\mathbf{j} + (-1 + \frac{\sqrt{6}}{3})\mathbf{k}$ and $\mathbf{v}_2 = (1 - \frac{\sqrt{6}}{3})\mathbf{i} - \frac{\sqrt{6}}{3}\mathbf{j} - (1 + \frac{\sqrt{6}}{3})\mathbf{k}$ as the only possible solution candidates. One should now check by direct computation that these do indeed satisfy conditions (a)-(c).

4. Let $\mathbf{u}$ and $\mathbf{v}$ be two non-zero vectors in $\mathbb{R}^2$ that are not parallel to each other. Let a, b be two real numbers. Prove that there is exactly one vector $\mathbf{r}$ in $\mathbb{R}^2$ such that its scalar projection on $\mathbf{u}$ is a and its scalar projection on $\mathbf{v}$ is b .

(Optional: what would be the corresponding statement in $\mathbb{R}^3$?)



Proof 1: geometric. Let's represent $\mathbf{u}$ and $\mathbf{v}$ as $\vec{OA}$ and $\vec{OB}$ in the plane. Then all vectors whose scalar projections on $\mathbf{u}$ is a can be represented as $\vec{OP}$, where P lies on the line $\frac{\mathbf{u}}{|\mathbf{u}|} \cdot \vec{OP} = a$. This line is perpendicular to $\mathbf{u}$, or equivalently, to the line through O and A . Similarly, all vectors whose scalar projections on $\mathbf{v}$ is b can be represented as $\vec{OQ}$, where Q lies on the line $\frac{\mathbf{v}}{|\mathbf{v}|} \cdot \vec{OQ} = b$. That line is perpendicular to $\mathbf{v}$. Since $\mathbf{u}$ and $\mathbf{v}$ are not parallel, the two lines perpendicular to them are not parallel, either. Therefore they intersect at exactly one point P_0 , so that the vector $\mathbf{r} = \vec{OP}_0$ is the only one that satisfies both conditions.

Proof 2: using linear equations. Let $\mathbf{r} = x\mathbf{i} + y\mathbf{j}$, $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j}$, $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j}$. Then the two projection conditions say that

$$\mathbf{r} \cdot \mathbf{u} = a|\mathbf{u}|, \quad \mathbf{r} \cdot \mathbf{v} = b|\mathbf{v}|,$$

or in terms of components,

$$\begin{aligned} u_1x + u_2y &= a|\mathbf{u}|, \\ v_1x + v_2y &= b|\mathbf{v}|. \end{aligned}$$

If $\mathbf{u}$ and $\mathbf{v}$ are not parallel, then the coefficient matrix

$$\begin{pmatrix} u_1 & u_2 \\ v_1 & v_2 \end{pmatrix}$$

is non-degenerate (i.e. has non-zero determinant), so the linear system of equations has a unique solution for (x, y) . Alternatively, you can just solve the system of equations and find (x, y) from it.

Answer to the optional question: if $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ are three non-coplanar vectors in $\mathbb{R}^3$, and if a, b, c are real numbers, then there is exactly one vector $\mathbf{r}$ in $\mathbb{R}^3$ such that its respective scalar projections on $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ are a, b, c .

5. Let $Q = (2, 0, -1)$. Describe the set of points $P = (x, y, z)$ such that the vector $\vec{PQ}$ makes equal angles with the vectors $\mathbf{u} = \mathbf{i} + \mathbf{j}$ and $\mathbf{v} = 3\mathbf{j} - 4\mathbf{k}$. (Give an equation describing that set.)

Let $\mathbf{w} = \vec{PQ} = (2 - x)\mathbf{i} - y\mathbf{j} + (-1 - z)\mathbf{k}$. For the two angles to be equal, we need their cosines to be equal:

$$\frac{\mathbf{w} \cdot \mathbf{u}}{|\mathbf{u}||\mathbf{w}|} = \frac{\mathbf{w} \cdot \mathbf{v}}{|\mathbf{v}||\mathbf{w}|}, \quad \text{so that} \quad \frac{\mathbf{w} \cdot \mathbf{u}}{|\mathbf{u}|} = \frac{\mathbf{w} \cdot \mathbf{v}}{|\mathbf{v}|}.$$

Plugging in, we get $|\mathbf{u}| = \sqrt{1 + 1} = \sqrt{2}$, $|\mathbf{v}| = \sqrt{9 + 16} = \sqrt{25} = 5$, so that we need to have

$$\frac{(2 - x) - y}{\sqrt{2}} = \frac{-3y - 4(-1 - z)}{5},$$

$$5(2 - x - y) = \sqrt{2}(-3y + 4z + 4).$$

After simplification, we get $5x + (5 - 3\sqrt{2})y + 4\sqrt{2}z = 10 - 4\sqrt{2}$. (We will see soon that this is the equation of a plane.)

Practice Problems

Try the following problems to test your understanding of the material. Solutions can be found in the Solutions Manual.

1. What is the distance from the point $(1, 2, 3)$ in $\mathbb{R}^3$ to the nearest point on the x -axis?
2. Express the vector $\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ as a sum of vectors $\mathbf{u}$ and $\mathbf{v}$, where $\mathbf{u}$ is parallel to the vector $\mathbf{i} - \mathbf{j}$ and $\mathbf{v}$ is perpendicular to $\mathbf{u}$.
3. Let $\mathbf{u} = 5\mathbf{i} + 7\mathbf{k}$. Find vectors $\mathbf{v}$ and $\mathbf{w}$ such that: $\mathbf{u} = \mathbf{v} + \mathbf{w}$, $\mathbf{v}$ is parallel to the vector $\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$, and $\mathbf{w}$ is perpendicular to the vector $\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$.
4. Find all values of t for which the vector $\mathbf{v} = (t - 2)\mathbf{i} + 2\mathbf{j} + (2t + 1)\mathbf{k}$ is perpendicular to the vector $\mathbf{w} = 2t\mathbf{i} + (3 - 4t)\mathbf{j} + 2\mathbf{k}$.
5. Let $Q = (1, 0, -1)$. The set of points $P = (x, y, z)$ such that the vector $\vec{PQ}$ makes equal angles with the vectors $\mathbf{u} = 2\mathbf{i} - \mathbf{j}$ and $\mathbf{v} = \mathbf{i} + \mathbf{j} + 3\mathbf{k}$ can be described by an equation in x, y, z . Find that equation.
6. Let $\mathbf{u}$ and $\mathbf{v}$ be two non-zero vectors in $\mathbb{R}^2$ that are not parallel and not perpendicular to each other. Consider the following sequence of vectors $\mathbf{w}_1, \mathbf{w}_2, \dots$ in $\mathbb{R}^2$:
 - $\mathbf{w}_1$ is the vector projection of $\mathbf{u}$ in the direction of $\mathbf{v}$,
 - $\mathbf{w}_2$ is the vector projection of $\mathbf{w}_1$ in the direction of $\mathbf{u}$,
 - $\mathbf{w}_3$ is the vector projection of $\mathbf{w}_2$ in the direction of $\mathbf{v}$,
 - $\mathbf{w}_4$ is the vector projection of $\mathbf{w}_3$ in the direction of $\mathbf{u}$,
 - continue in this manner, alternating projections in the directions of $\mathbf{u}$ and $\mathbf{v}$.

Find the general formula for the length of $\mathbf{w}_n$ for $n = 1, 2, \dots$, in terms of $|\mathbf{u}|$, $|\mathbf{v}|$, and the angle θ between $\mathbf{u}$ and $\mathbf{v}$. What happens as $n \rightarrow \infty$? (Optional: you may want to draw a picture.)

1.3 The Cross Product

1.3.1 Definitions and Basic Properties

We introduce a foundational definition for the study of vector geometry in $\mathbb{R}^3$.

Definition 1.3.1. *Given two vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^3$, the **cross product** $\mathbf{u} \times \mathbf{v}$ is the unique vector in $\mathbb{R}^3$ such that:*

- (1) $\mathbf{u} \times \mathbf{v}$ is perpendicular to both $\mathbf{u}$ and $\mathbf{v}$;
- (2) $|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}||\mathbf{v}|\sin\theta$, where θ is the angle between $\mathbf{u}$ and $\mathbf{v}$.
- (3) The triple $(\mathbf{u}, \mathbf{v}, \mathbf{u} \times \mathbf{v})$ forms a **right-handed triad** – essentially, it's oriented the same way as the $(\mathbf{i}, \mathbf{j}, \mathbf{k})$ triple, going counterclockwise from $\mathbf{u}$ to $\mathbf{v}$ to $\mathbf{u} \times \mathbf{v}$. Your textbook may have other (different but equivalent) descriptions.

We first check that this determines $\mathbf{u} \times \mathbf{v}$ uniquely. Assume first that $\mathbf{u}$ and $\mathbf{v}$ are not parallel. Then (1) says that $\mathbf{u} \times \mathbf{v}$ is perpendicular to the plane spanned by $\mathbf{u}$ and $\mathbf{v}$, so this determines the direction of $\mathbf{u} \times \mathbf{v}$ up to the choice of sign. That choice is then determined by (3). Then (2) gives the magnitude of $\mathbf{u} \times \mathbf{v}$, and we are done.

Now, if $\mathbf{u}$ and $\mathbf{v}$ are parallel, there are many vectors that are perpendicular to both of them. However, in that case the angle between $\mathbf{u}$ and $\mathbf{v}$ is either 0 or π , so that $\sin\theta = 0$ and $|\mathbf{u} \times \mathbf{v}| = 0$. Therefore $\mathbf{u} \times \mathbf{v}$ is the zero vector in this case.

The following theorem tells us how to calculate the cross product.

Theorem 1.3.2. *If $\mathbf{u} := u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$ and $\mathbf{v} := v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$, the cross product is given by*

$$\begin{aligned}\mathbf{u} \times \mathbf{v} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \mathbf{k} \\ &= (u_2v_3 - u_3v_2)\mathbf{i} - (u_1v_3 - u_3v_1)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k}\end{aligned}$$

The formal 3×3 determinant (with vectors in the first row!) is evaluated as indicated above. We first expand the 3×3 determinant in the first row (as if the vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$ were just numbers) – you will see this in your linear algebra class. Then, for each of the remaining 2×2 determinants we use the fact that $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$ for any $a, b, c, d \in \mathbb{R}$. The easiest way to remember the formula for the cross product is to memorize the 3×3 determinant formula.

Sketch of Proof of Theorem 1.3.2. Let $\mathbf{w}$ be the vector on the left-hand side of the equality in Theorem 1.3.2. We need to verify the conditions (1), (2) and (3) from Definition 1.3.1. We work in order.

- To check (1), we need to check that the dot products $\mathbf{u} \cdot \mathbf{w}$ and $\mathbf{v} \cdot \mathbf{w}$ are zero. Write $\mathbf{u} \cdot \mathbf{w} = u_1w_1 + u_2w_2 + u_3w_3$, and then substitute the components of $\mathbf{w}$. If you like using linear algebra, you may notice that

$$\mathbf{u} \times \mathbf{w} = \begin{vmatrix} u_1 & u_2 & u_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

and the last determinant is zero because two of the rows are identical. Or you could actually calculate the dot product in terms of the components of $\mathbf{u}$ and $\mathbf{v}$ and simplify to get 0.

- To check (2), we first rewrite the scalar $|\mathbf{u} \times \mathbf{v}|$ in a form that will be easier to use. Write,

$$\begin{aligned} |\mathbf{u} \times \mathbf{v}|^2 &= |\mathbf{u}|^2 |\mathbf{v}|^2 \sin^2 \theta \\ &= |\mathbf{u}|^2 |\mathbf{v}|^2 (1 - \cos^2 \theta) \\ &= |\mathbf{u}|^2 |\mathbf{v}|^2 \left(1 - \left(\frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}| |\mathbf{v}|} \right)^2 \right) \\ &= |\mathbf{u}|^2 |\mathbf{v}|^2 - (\mathbf{u} \cdot \mathbf{v})^2 \end{aligned}$$

Now, you need a calculation to evaluate both this quantity and the quantity

$$|\mathbf{w}|^2 = \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}^2 + \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix}^2 + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}^2$$

and check that they are equal. We omit the calculation because it is a bit long, but it should be completely straightforward from here.

- To prove (3), we have to verify that the triad $\mathbf{u}, \mathbf{v}, \mathbf{w}$ is right-handed. To do this, we will set up our coordinate system so that the calculations are easy. It turns out that what works best is to make the x -axis parallel to $\mathbf{u}$, then rotate the coordinate system so that $\mathbf{v}$ lies in the xy -plane and has a positive y -component. (If the y -component of v is zero for all such rotations, then $\mathbf{u}$ and $\mathbf{v}$ are parallel and the cross product is zero.) This means that $\mathbf{u} = u_1 \mathbf{i}$ and $\mathbf{v} = v_1 \mathbf{i} + v_2 \mathbf{j}$ with $u_1 > 0$ and $v_2 > 0$. But then $w_3 = u_1 v_2$ is positive, so the right-handed property holds.

□

Remark 1.3.1. *The previous is **not a complete proof!** However, it should give you a good start. What is missing is, first, you need to complete the missing calculations.*

Second, and this is a little bit more delicate, you need to verify that (3) does not depend on the placement of $\mathbf{u}$ and $\mathbf{v}$ in the coordinate system. An important idea here is that the cross product is invariant under rigid motions: if R is a rotation or a translation or a composition of both, then $(R\mathbf{u}) \times (R\mathbf{v}) = R(\mathbf{u} \times \mathbf{v})$. This follows from the geometric definition of $\mathbf{u} \times \mathbf{v}$. To complete the proof, one should check that the coordinate formula in Theorem 1.3.2 is also invariant under rigid motions. This can be done using the invariance of determinants under orthogonal transformations in linear algebra.

We list a few basic properties of the cross product.

Proposition 1.3.3. *For vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ in $\mathbb{R}^3$, the cross product satisfies the following,*

$$(1) \quad \mathbf{u} \times \mathbf{u} = \mathbf{0}$$

$$(2) \quad \mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u})$$

$$(3) \quad (c\mathbf{u}) \times \mathbf{v} = c(\mathbf{u} \times \mathbf{v})$$

$$(4) \quad (\mathbf{u} + \mathbf{v}) \times \mathbf{w} = \mathbf{u} \times \mathbf{w} + \mathbf{v} \times \mathbf{w}$$

1.3.2 Examples

We calculate a few cross products.

Example 1.3.4. *Recall that $\mathbf{i} = \langle 1, 0, 0 \rangle$, $\mathbf{j} = \langle 0, 1, 0 \rangle$ and $\mathbf{k} = \langle 0, 0, 1 \rangle$. Let us practice with the following calculations.*

(1)

$$\mathbf{i} \times \mathbf{j} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = 0\mathbf{i} + 0\mathbf{j} + 1\mathbf{k} = \mathbf{k}$$

(2) *Now perform the necessary calculations to show that $\mathbf{j} \times \mathbf{i} = -\mathbf{k}$. In general, one always has $\mathbf{u} \times \mathbf{v} = -\mathbf{v} \times \mathbf{u}$.*

(3) *Let $\mathbf{u} = 3\mathbf{i} + \mathbf{k}$ and $\mathbf{v} = \mathbf{i} - \mathbf{j} + 2\mathbf{k}$. Then,*

$$\begin{aligned} \mathbf{u} \times \mathbf{v} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 0 & 1 \\ 1 & -1 & 2 \end{vmatrix} = (0 + 1)\mathbf{i} - (6 - 1)\mathbf{j} + (-3 - 0)\mathbf{k} \\ &= \mathbf{i} - 5\mathbf{j} - 3\mathbf{k} \end{aligned}$$

Warning! It is entirely possible that the cross product $(\mathbf{u} \times \mathbf{v}) \times \mathbf{w}$ might **not** be equal to the cross product $\mathbf{u} \times (\mathbf{v} \times \mathbf{w})$. In other words, the cross product operation $\times$ is **not** associative! For example,

$$(\mathbf{i} \times \mathbf{j}) \times \mathbf{j} = \mathbf{k} \times \mathbf{j} = -\mathbf{i},$$

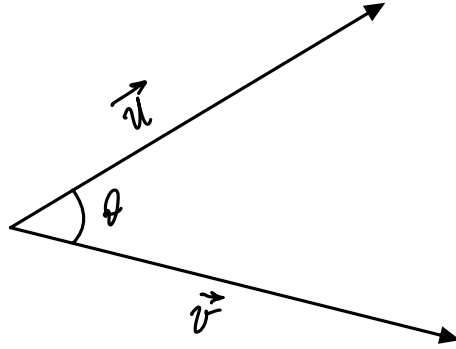
and yet

$$\mathbf{i} \times (\mathbf{j} \times \mathbf{j}) = \mathbf{i} \times \mathbf{0} = \mathbf{0}.$$

1.3.3 Applications

The cross product is used to calculate areas and volumes of certain shapes in $\mathbb{R}^3$. We provide some applications below.

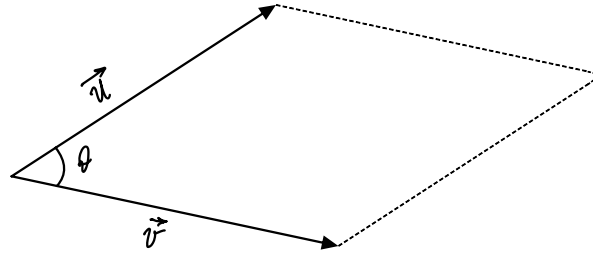
Application 1.3.5 (Areas of a Triangle and Parallelogram). *Suppose that $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^3$ make angle θ with one another; see the figure below.*



The vectors $\mathbf{u}$ and $\mathbf{v}$ form two sides of a triangle. Calculating this area using trigonometry gives,

$$AREA_{TRI} = \frac{1}{2}|\mathbf{u}||\mathbf{v}|\sin\theta = \frac{1}{2}|\mathbf{u} \times \mathbf{v}|.$$

Similarly, we can calculate the area of the parallelogram; see the figure below.



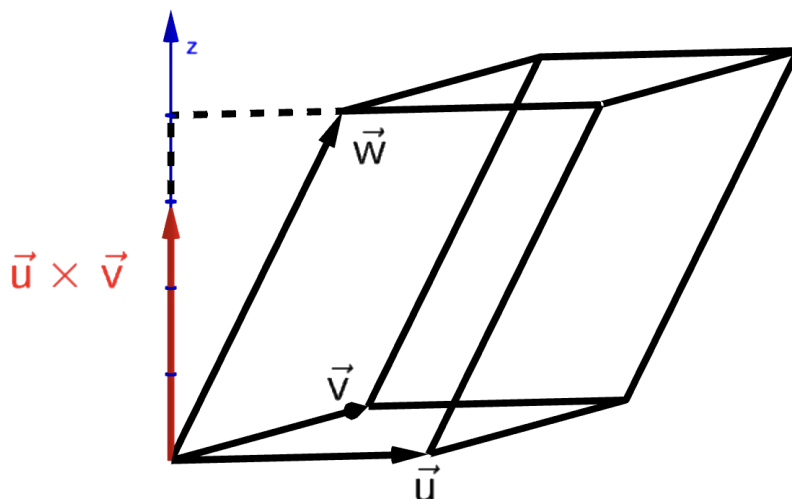
This area is

$$AREA_{PAR} = 2(AREA_{TRI}) = |\mathbf{u} \times \mathbf{v}|$$

The cross product also gives a vector perpendicular to a plane. This will be used extensively in what follows.

Application 1.3.6 (Vector perpendicular to a plane). *Let $\mathbf{u}$ and $\mathbf{v}$ be (non-parallel) vectors in $\mathbb{R}^3$. Consider any 2-dimensional plane parallel to both $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^3$. Then $\mathbf{w} = \mathbf{u} \times \mathbf{v}$ is perpendicular to the plane.*

Application 1.3.7 (Volume of a Parallelepiped). *Suppose $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are three non-parallel vectors in $\mathbb{R}^3$. We may view these vectors as one corner of a box with parallelogram faces. This shape is called a **parallelepiped**. See the figure below.*



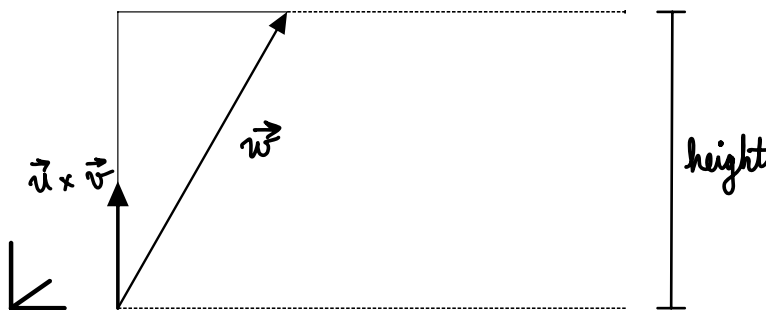
In order to calculate the area of this shape (the parallelepiped), we need to calculate the **area of the base** and the **height**. We know that,

$$AREA_{BASE} = |\mathbf{u} \times \mathbf{v}|$$

and we can calculate the height as follows,

$$HEIGHT = |(\text{scalar projection of } \mathbf{w} \text{ on } \mathbf{u} \times \mathbf{v})| = \left| \frac{\mathbf{w} \cdot (\mathbf{u} \times \mathbf{v})}{|\mathbf{u} \times \mathbf{v}|} \right|,$$

where the outer $|\cdot|$ denote the absolute value. Note that, by **height**, we mean the height of $\mathbf{w}$ above the plane spanned by $\mathbf{u}$ and $\mathbf{v}$. See the figure below.



Now putting together everything so far, we see that the volume of the parallelepiped is given by

$$VOL_{PAR} = (AREA_{BASE})(HEIGHT) = |\mathbf{u} \times \mathbf{v}| \cdot \left| \frac{\mathbf{w} \cdot (\mathbf{u} \times \mathbf{v})}{|\mathbf{u} \times \mathbf{v}|} \right| = |\mathbf{w} \cdot (\mathbf{u} \times \mathbf{v})|$$

You can check yourself that the order of $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ does not matter in calculating the area of the parallelepiped.

The expression $\mathbf{w} \cdot (\mathbf{u} \times \mathbf{v})$ is sometimes called the **scalar triple product** of $\mathbf{w}$, $\mathbf{u}$ and $\mathbf{v}$. To evaluate it, we should use the component representations of $\mathbf{u} \times \mathbf{v}$. That is,

$$\begin{aligned} \mathbf{w} \cdot (\mathbf{u} \times \mathbf{v}) &= w_1 \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} - w_2 \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} + w_3 \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \\ &= \begin{vmatrix} w_1 & w_2 & w_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} \end{aligned}$$

This is not the last time you will see this 3×3 determinant appear in this course. More to come.

Worked Problems

We have some worked problems related to the cross product and its relation to areas and volumes.

1. Find the area of the triangle in $\mathbb{R}^3$ with vertices $(0, 0, 0)$, $(3, 0, 1)$, $(1, -1, 2)$.

Let $P = (3, 0, 1)$ and $Q = (1, -1, 2)$. Then $\vec{OP} = 3\mathbf{i} + \mathbf{k}$ and $\vec{OQ} = \mathbf{i} - \mathbf{j} + 2\mathbf{k}$. These are the same vectors that appeared in Example 1.3.4(3). So, the area of the triangle OPQ is given by

$$\frac{1}{2}|\vec{OP} \times \vec{OQ}| = \frac{1}{2}|\mathbf{i} - 5\mathbf{j} - 3\mathbf{k}| = \frac{1}{2}\sqrt{1 + 25 + 9} = \frac{\sqrt{35}}{2}.$$

2. Find the volume of the parallelepiped spanned by the vectors $\mathbf{u} = \langle 1, 0, -1 \rangle$, $\mathbf{v} = \langle 1, 3, 3 \rangle$ and $\mathbf{w} = \langle 0, 1, -2 \rangle$.

The volume of the parallelepiped is given by $|\mathbf{w} \cdot (\mathbf{u} \times \mathbf{v})|$. The scalar product is given by

$$\begin{aligned}\mathbf{w} \cdot (\mathbf{u} \times \mathbf{v}) &= \begin{vmatrix} 0 & 1 & -2 \\ 1 & 0 & -1 \\ 1 & 3 & 3 \end{vmatrix} = -1 \begin{vmatrix} 1 & -1 \\ 1 & 3 \end{vmatrix} - 2 \begin{vmatrix} 1 & 0 \\ 1 & 3 \end{vmatrix} \\ &= -1(3 + 1) - 2(3 - 0) = -10.\end{aligned}$$

Recall: volumes cannot be negative, so the volume is $|-10| = 10$.

Practice Problems

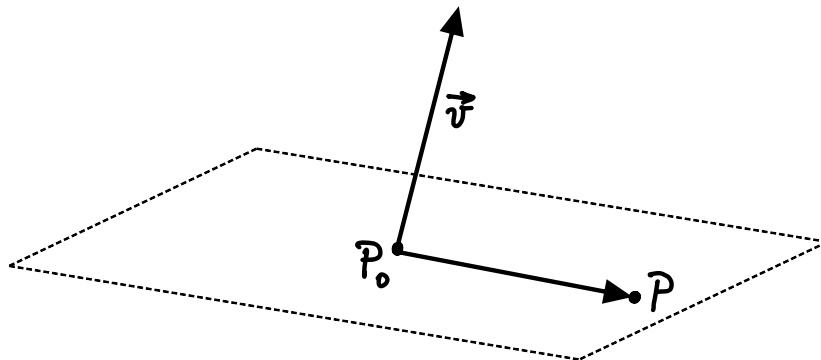
1. Give an example of three non-zero vectors $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ such that $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = 4$.
2. Give an example of three non-zero vectors $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$, such that $\mathbf{v} \times \mathbf{w} \neq \mathbf{0}$ but $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = 0$.
3. Find the area of the triangle in $\mathbb{R}^3$ with vertices $(0, 1, 1)$, $(2, 2, 1)$, $(1, 0, 3)$.
4. Find the volume of the parallelepiped in $\mathbb{R}^3$ spanned by the vectors $\mathbf{i} - \mathbf{j} + 4\mathbf{k}$, $3\mathbf{j} - \mathbf{k}$, $2\mathbf{i} + \mathbf{j}$.

1.4 Lines and Planes

1.4.1 Planes in $\mathbb{R}^3$.

A plane in $\mathbb{R}^3$ is determined by a point on the plane and a vector perpendicular to it. We will use this fact to determine the **standard form** of an equation of a plane in $\mathbb{R}^3$.

Suppose that a plane passes through a point $P_0 = (x_0, y_0, z_0)$ and is simultaneously perpendicular to a vector $\mathbf{v} = A\mathbf{i} + B\mathbf{j} + C\mathbf{k}$ (we assume that $\mathbf{v} \neq \mathbf{0}$). Then for every point $P = (x, y, z)$ in the plane, the vector $\vec{P_0P}$ connecting these points must be perpendicular to $\mathbf{v}$. See the figure below.



However, this means that

$$0 = \mathbf{v} \cdot \vec{P_0P} = \langle A, B, C \rangle \cdot \langle x - x_0, y - y_0, z - z_0 \rangle.$$

Calculating the dot product gives us the following important equation for a plane.

Proposition 1.4.1. *Suppose that a plane passes through $P_0 = (x_0, y_0, z_0)$ and is perpendicular to the vector $\mathbf{v} = A\mathbf{i} + B\mathbf{j} + C\mathbf{k}$. Then, any point $P = (x, y, z)$ in the plane satisfies the equation*

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0,$$

or equivalently,

$$Ax + By + Cz = D,$$

where $D = Ax_0 + By_0 + Cz_0$. We call $\mathbf{v}$ the **normal vector** to the plane.

The procedure above can be reversed in the sense that, given a plane equation $Ax + By + Cz = D$, we can read the normal vector $\mathbf{v} = A\mathbf{i} + B\mathbf{j} + C\mathbf{k}$ from this equation. However, the normal vector is not unique and any other vector in the same direction would also work. For example, one vector perpendicular to the plane $x + 2y + 3z = 10$ is $\mathbf{v} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$, and another is $2\mathbf{v} = 2\mathbf{i} + 4\mathbf{j} + 6\mathbf{k}$.

1.4.2 Finding the Equation of a Plane in $\mathbb{R}^3$

Let us consider some examples.

1. Find the equation of the plane passing through $P_0 = (2, 1, 3)$ which is perpendicular to $\mathbf{v} = \mathbf{i} - 7\mathbf{k}$.

We are given a point P_0 in the plane and the normal vector $\mathbf{v}$. Using Proposition 1.4.1, the plane is given by the equation

$$1(x - 2) + 0(y - 1) - 7(z - 3) = 0.$$

This simplifies to $x - 2 - 7z + 21 = 0$, or $x - 7z = -19$.

Note: in general, in questions asking for the equation of a plane, you do not need to perform the simplification at the end. We did the calculation here to show you how to move from one form to another. For example, if you need to do some further work with that plane, the simplified form can be easier to use.

2. Find the equation of the plane that passes through the points $(0, 1, -1)$, $(3, 0, 2)$, $(1, 1, 0)$.

Here we are not given the normal vector $\mathbf{v}$, so we need to be clever and construct it ourselves. Let us label the points $P = (0, 1, -1)$, $Q = (3, 0, 2)$, and $R = (1, 1, 0)$. Notice that the vectors $\vec{PQ}$ and $\vec{PR}$ are **both** contained in the plane (and, in fact, **span the plane**). Thus, to find a vector perpendicular to the plane, we construct the cross product of $\vec{PQ}$ and $\vec{PR}$,

$$\begin{aligned}\mathbf{v} = \vec{PQ} \times \vec{PR} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -1 & 3 \\ 1 & 0 & 1 \end{vmatrix} = (-1 + 0)\mathbf{i} - (3 - 3)\mathbf{j} + (0 + 1)\mathbf{k} \\ &= -\mathbf{i} + \mathbf{k}.\end{aligned}$$

We now use this $\mathbf{v}$ and the provided point P and apply the plane equation to obtain,

$$0 = -(x - 0) + (z + 1).$$

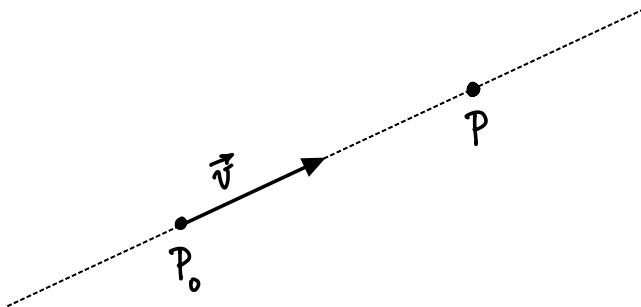
Simplifying, this gives the equation $x - z = 1$.

Remark 1.4.1. We could have used other combinations of the given points to construct our plane equation. For example, we could have used the cross product $\vec{PQ} \times \vec{QR}$ instead of the cross product $\mathbf{v} = \vec{PQ} \times \vec{PR}$. In this situation, we would get either the same equation, or else an **equivalent** one (such as $-2x + 2z = -2$ instead of $x - z = 1$). Check this for yourself!

1.4.3 Lines in $\mathbb{R}^3$

A line in $\mathbb{R}^3$ is determined by a point and a vector parallel to the line. This is the foundation for all line equations in this subsection.

If a line passes through P_0 and is parallel to $\mathbf{v}$, then for any point P on the line, we necessarily have $\vec{P_0P} = t\mathbf{v}$ for some $t \in \mathbb{R}$. This is the **vector parametric** equation of a line.



We may rewrite this equation in components. If $P_0 = (x_0, y_0, z_0)$, $P = (x, y, z)$, and $\mathbf{v} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$, then

$$\langle x - x_0, y - y_0, z - z_0 \rangle = t\langle a, b, c \rangle.$$

By comparing the components of the above vector equation, we arrive at the following system of **scalar parametric equations**,

$$\begin{cases} x = x_0 + at \\ y = y_0 + bt \\ z = z_0 + ct \end{cases}$$

Again, this procedure has a partial converse: if the equation of a line is given in the above form, then this tells us that the vector $a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ is parallel to the line.

Here's yet another way to view lines. Let $\mathbf{r}_0 = \vec{OP}_0 = \langle x_0, y_0, z_0 \rangle$ and $\mathbf{r} = \vec{OP} = \langle x, y, z \rangle$. Then $\vec{P_0P} = \mathbf{r} - \mathbf{r}_0$. Therefore, another way of rewriting the scalar parametric equations is

$$\mathbf{r} - \mathbf{r}_0 = t\mathbf{v} \iff \mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$$

These are the main forms of line equations that we will use, but a line can also be expressed in other ways. For example, a line in $\mathbb{R}^3$ can be written using only two linear equations (as opposed to the three scalar equations listed above). If a, b, c are all non-zero, we may eliminate the t variable by writing

$$t = \frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}.$$

If (say) $a = 0$, then we have instead

$$x = x_0; \quad \frac{y - y_0}{b} = \frac{z - z_0}{c}.$$

Worked Problems

1. Find the scalar parametric equations of the line which passes through the point $(4, 5, -2)$ and is perpendicular to the plane through the three points $P = (1, 0, 1)$, $Q = (3, 2, 0)$, $R = (-1, 1, 2)$.

The line should be parallel to the normal vector to the plane in question. For the normal vector, we will use the cross product of two of the vectors spanned by P, Q, R :

$$\mathbf{n} = \vec{PQ} \times \vec{PR} = \langle 2, 2, -1 \rangle \times \langle -2, 1, 1 \rangle = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 2 & -1 \\ -2 & 1 & 1 \end{vmatrix} = 3\mathbf{i} + 6\mathbf{k}.$$

Hence the vector parametric equation is $\mathbf{r} = (4 + 3t)\mathbf{i} + 5\mathbf{j} + (6t - 2)\mathbf{k}$, and the scalar parametric equations are

$$x = 4 + 3t, \quad y = 5, \quad z = 6t - 2.$$

2. The planes $x + y + z = 6$ and $2x - y - 3z = 0$ intersect in a line. Find the vector parametric equation of this line

We need to find a point P_0 on the line and a vector $\mathbf{v}$ parallel to the line in order to have enough information to write our equation.

To find a point, let us set $z = 0$ in both of our plane equations. This produces the system of two variable linear equations,

$$\begin{cases} x + y = 6 \\ 2x - y = 0 \end{cases}$$

which has the solution $(x_0, y_0) = (2, 4)$ (you should verify this!). So, the point $P_0 = (2, 4, 0)$ lies on our line.

To find a vector, notice that the two planes are perpendicular to the vectors $\mathbf{u} = \mathbf{i} + \mathbf{j} + \mathbf{k}$ and $\mathbf{v} = 2\mathbf{i} - \mathbf{j} - 3\mathbf{k}$, respectively. Any vector parallel to their intersection must simultaneously be perpendicular to $\mathbf{u}$ and $\mathbf{v}$. One way to find such a vector is to use the cross product:

$$\begin{aligned} \mathbf{w} = \mathbf{u} \times \mathbf{v} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 1 \\ 2 & -1 & -3 \end{vmatrix} = (-3 + 1)\mathbf{i} - (-3 - 2)\mathbf{j} + (-1 - 2)\mathbf{k} \\ &= -2\mathbf{i} + 5\mathbf{j} - 3\mathbf{k}. \end{aligned}$$

The vector equation is $\mathbf{r} = \vec{OP_0} + t\mathbf{w}$. Substituting our work, this gives

$$\mathbf{r} = \langle 2, 4, 0 \rangle + t\langle -2, 5, -3 \rangle$$

Remark 1.4.2. *This is not the only way to solve this problem. For example, the point $(0, 9, -3)$ also lies on the intersection line. You would find this point by setting $x = 0$ and solving the corresponding system of equations for y and z . The cross product $\mathbf{v} \times \mathbf{u}$ is also perpendicular to the intersection line. Using these would give us the equation,*

$$\mathbf{r} = \langle 0, 9, -3 \rangle + t\langle 2, -5, 3 \rangle.$$

This equation looks different. However, it still describes the same line!

Other methods could also be used, for example we could use linear algebra to solve the system of two linear equations and get a family of solutions for x, y, z depending on a parameter t . The advantage of the method used above is that a version of it can be applied to intersections of surfaces more general than planes. We will see it later in this course.

3. The points P, Q, R, S in $\mathbb{R}^3$ have coordinates $P = (2, 0, 0)$, $Q = (1, 1, 0)$, $R = (0, 0, 1)$, $S = (1, 2, 3)$.

(a) Find the area of the triangle $\triangle PQR$.

We have $\vec{PQ} = \langle -1, 1, 0 \rangle$ and $\vec{PR} = \langle -2, 0, 1 \rangle$, so that

$$\vec{PQ} \times \vec{PR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 1 & 0 \\ -2 & 0 & 1 \end{vmatrix} = \mathbf{i} + \mathbf{j} + 2\mathbf{k}.$$

The area of the triangle is

$$\frac{1}{2}|\vec{PQ} \times \vec{PR}| = \frac{1}{2}|\mathbf{i} + \mathbf{j} + 2\mathbf{k}| = \frac{1}{2}\sqrt{6}.$$

(b) Find the equation (in the form $Ax + By + Cz = D$) of the plane through the points P, Q, R .

The plane passes through P and is perpendicular to the vector $\vec{PQ} \times \vec{PR}$ from (a), so the equation is

$$1(x - 2) + 1(y - 0) + 2(z - 0) = 0,$$

which simplifies to $x + y + 2z = 2$.

(c) Find the parametric equation of the line which passes through S and is perpendicular to the plane from (b).

The line passes through S and is parallel to the vector from (a), so the equation is

$$\mathbf{r} = \langle 1, 2, 3 \rangle + t\langle 1, 1, 2 \rangle.$$

4. Let $P = (2, 5, 1)$. Find the point Q on the line $x = t + 1, y = 3t, z = t + 3$ which is closest to P .

Let $Q = (t + 1, 3t, t + 3)$ be a point on the line, for some t that we need to find. For Q to be the closest point to P , the vector $\vec{PQ} = \langle t + 1 - 2, 3t - 5, t + 3 - 1 \rangle = \langle t - 1, 3t - 5, t + 2 \rangle$ should be perpendicular to the line, hence to its direction vector $\mathbf{u} = \langle 1, 3, 1 \rangle$. We set up the equation for that:

$$\vec{PQ} \cdot \mathbf{u} = t - 1 + (3t - 5) \cdot 3 + (t + 2) = 0,$$

which simplifies to

$$t - 1 + 9t - 15 + t + 2 = 0, \quad 11t = 14, \quad t = \frac{14}{11}.$$

Therefore Q has coordinates

$$x = \frac{14}{11} + 1 = \frac{25}{11}, \quad y = 3 \cdot \frac{14}{11} = \frac{42}{11}, \quad z = \frac{14}{11} + 3 = \frac{47}{11}.$$

Practice Problems

- Give an example of a line parallel to the plane $x + y + z = 0$ but not contained in that plane. Give the equation of the line in the vector parametric form $\mathbf{r} = \mathbf{a} + t\mathbf{v}$.
- The points P, Q, R in $\mathbb{R}^3$ have coordinates $P = (1, 1, -1)$, $Q = (2, -1, 0)$, $R = (2, 3, 4)$.
 - Find the area of the triangle $\triangle PQR$.
 - Find the angle at P in the triangle $\triangle PQR$. (An answer in the form $\cos^{-1}(\cdot)$ or $\sin^{-1}(\cdot)$ is sufficient.)
 - Find the equation (in the form $Ax + By + Cz = D$) of the plane through the points P, Q, R .
- Prove that the line $x = t + 1, y = 3t, z = t + 3$ is parallel to the plane $4x - y - z = 2$.
- Find the equation (in the form $Ax + By + Cz = D$) of the plane that contains the line $x = t + 1, y = 3t, z = t + 3$ and is perpendicular to the plane $4x - y - z = 2$.
- Find the equation of the plane that contains the line $x = t, y = 2t, z = 0$, and the point $(3, -1, -1)$.

6. Write the vector $\mathbf{w} = \mathbf{i} - \mathbf{j} + 4\mathbf{k}$ as a sum of two vectors $\mathbf{w} = \mathbf{u} + \mathbf{v}$ such that $\mathbf{u}$ is parallel to the plane $x + 2y - 2z = 0$ and the other is perpendicular to it.
7. Find the equations (in whichever form you prefer) of the line of intersection of the planes $3x - y + z = 2$ and $x + 2y - z = -4$.
8. Find vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ in $\mathbb{R}^2$ such that the set of points whose position vector $\mathbf{r}$ satisfies the inequalities

$$\mathbf{r} \cdot \mathbf{a} \leq 1, \mathbf{r} \cdot \mathbf{b} \leq 1, \mathbf{r} \cdot \mathbf{c} \leq 1 \quad (1.4.1)$$

is the triangle with vertices $(1, 2), (2, -2), (-3, 0)$.

9. Let $\mathbf{v}$ and $\mathbf{w}$ be two nonzero vectors in $\mathbb{R}^3$ that are perpendicular to each other. Prove that the set

$$\{P = (x, y, z) \in \mathbb{R}^3 : \vec{OP} \times \mathbf{v} = \mathbf{w}\}$$

is a line.

1.5 Quadric Surfaces

Quadric surfaces are surfaces which are described by 2nd order equations in the variables x, y, z . An equation is of the **second order** if it is polynomial in each of its variables and has total degree two. (For example, $x^2 + y + xz - 3z^2 = 6$ is an equation of the second order, but $x^2y^2 - z = 0$ is not.)

Quadric surfaces provide interesting geometric examples. A major reason for us to be interested in quadric surfaces is that we will use them to approximate surfaces with more complicated equations.

Each of the following subsections introduces one type of quadric surfaces in $\mathbb{R}^3$. They are: spheres/ellipsoids, cylinders, paraboloids, hyperboloids and cones. We give a short description of each, with accompanying computer-rendered images. We are not attempting a complete classification of quadric surfaces (although that can be done—classifying surfaces given by various types of algebraic equations is a major area of study in algebraic geometry). Instead, our focus will be on recognizing—and working with—certain basic types of surfaces that will be useful in the sequel.

1.5.1 Spheres and Ellipsoids

These surfaces are defined by the equations:

$$\begin{array}{ll} \text{(Sphere)} & x^2 + y^2 + z^2 = r^2 \\ \text{(Ellipsoid)} & \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \end{array}$$

where a, b, c, r are all real, non-zero numbers. Note that the equations above do not change if we replace r by $-r$, a by $-a$, and so on; for convenience, we will always assume that $a, b, c, r > 0$. A sphere is actually a special kind of ellipsoid, with $a = b = c = r$ for some $r > 0$. For an ellipsoid centred at some point (x_0, y_0, z_0) instead of the origin $(0, 0, 0)$, we have the equation

$$\frac{(x - x_0)^2}{a^2} + \frac{(y - y_0)^2}{b^2} + \frac{(z - z_0)^2}{c^2} = 1.$$

The other surfaces described below can be translated and rescaled in a similar way.

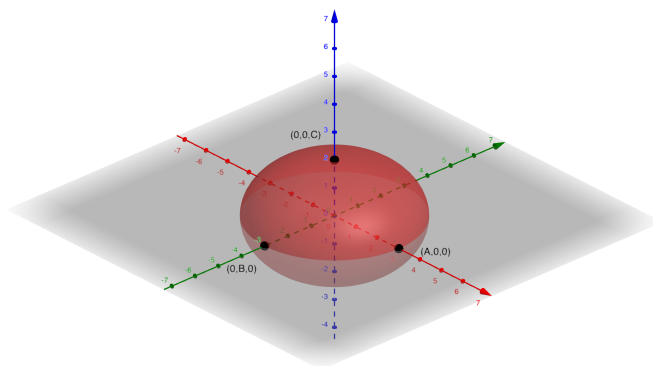


Figure 1.13: An ellipsoid centred at the origin.

1.5.2 Cylinders

The “usual” circular cylinder about the z -axis has the equation

$$\text{(Circular Cylinder)} \quad x^2 + y^2 = r^2$$

Notice that the z variable does not make an appearance in the equation defining the circular cylinder! This means that the surface is *invariant under translations* parallel to the z -axis: if we move the surface by a vector parallel to the z -axis, we get the same surface. Additionally, the circular cylinder is also symmetric about the z -axis.

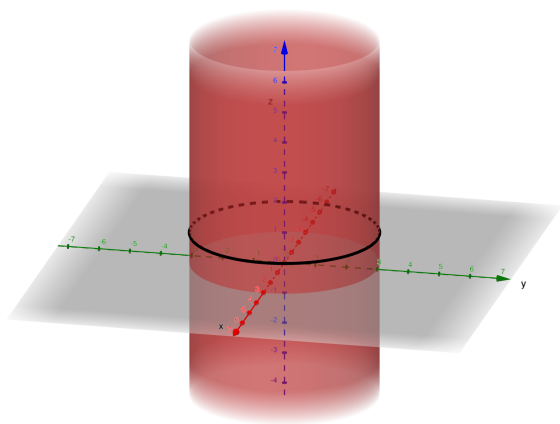


Figure 1.14: A circular cylinder about the z -axis with $r = 3$.

The dark curve in the previous figure is given by the system of equations

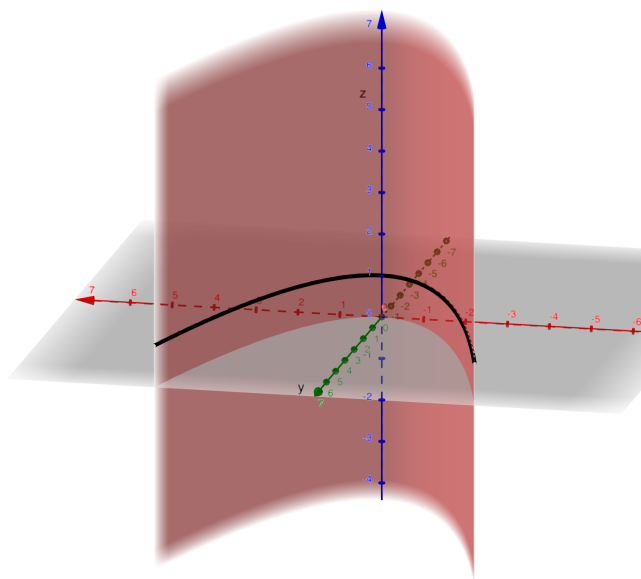
$$\begin{cases} x^2 + y^2 = 9 \\ z = 1 \end{cases}$$

These are the equations for a circle of radius 3 and the equation for the plane $z = 1$. In words, the intersection of the circular cylinder with a plane parallel to the xy -axis is the circle $x^2 + y^2 = r^2$. Hence the name—*circular* cylinder.

We can also consider elliptic and parabolic cylinders. These are given by the equations

$$\begin{array}{ll} \text{(Elliptic Cylinder)} & \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \\ \text{(Parabolic Cylinder)} & \frac{x^2}{a^2} - y = 0 \end{array}$$

Notice that, just like the circular cylinder, both of these equations omit the z variable. This means that these surfaces are also invariant under translations parallel to the z -axis. Moreover, the intersection of these surfaces with any plane parallel to the xy -axis is an ellipse (resp. parabola).

Figure 1.15: A parabolic cylinder about the z -axis.

1.5.3 Paraboloids

In these surfaces, one variable occurs only in the first power, so that the equation is of order one in that variable. Here are some examples of paraboloids where the z -variable occurs only in the first power:

(Elliptic Paraboloid)

$$z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$$

(Hyperbolic Paraboloid)

$$z = \frac{y^2}{b^2} - \frac{x^2}{a^2}$$

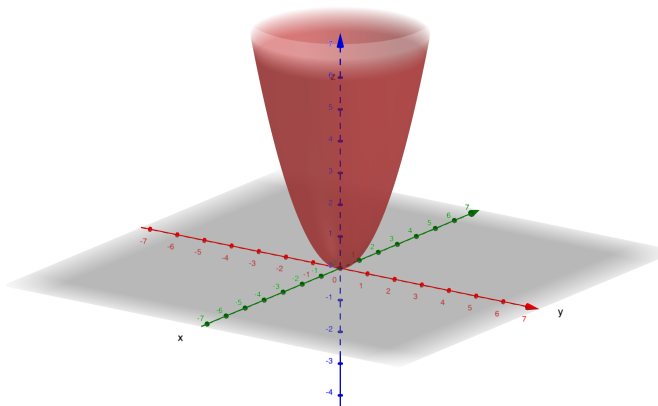


Figure 1.16: An elliptic paraboloid oriented about the z axis.

Similar to the case of spheres and ellipsoids, we give the case $a = b$ a special name: this equation describes a **circular** paraboloid. Recall that a circle is a special case of an ellipse.

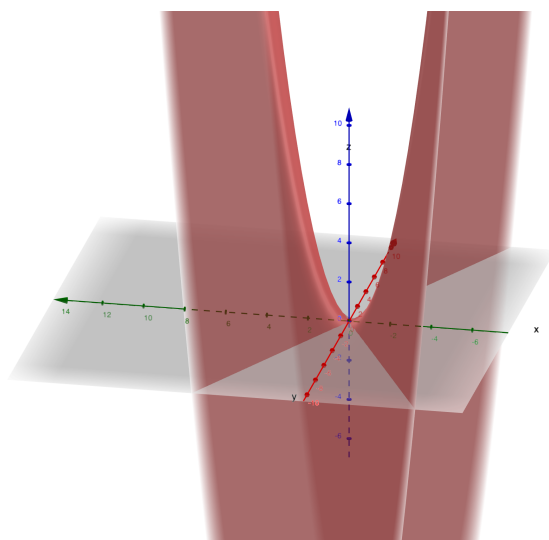


Figure 1.17: A hyperbolic paraboloid.

Figure 5 shows the hyperbolic paraboloid intersecting the xy plane in a pair of straight lines. This can be confirmed using the equation $z = \frac{y^2}{b^2} - \frac{x^2}{a^2}$ above: setting $z = 0$, we get $\frac{y^2}{b^2} - \frac{x^2}{a^2} = 0$, meaning that either $\frac{y}{b} = \frac{x}{a}$ or $\frac{y}{b} = -\frac{x}{a}$. These are the two lines you see in the xy -plane.

What's less clear from the picture is that a hyperbolic paraboloid actually contains many more straight lines! Let us show this algebraically. First, rewrite the equation

$$z = \frac{y^2}{b^2} - \frac{x^2}{a^2} \text{ as}$$

$$z = \left(\frac{y}{b} + \frac{x}{a}\right) \left(\frac{y}{b} - \frac{x}{a}\right).$$

Now, fix a real number $c \neq 0$, and consider the line L at the intersection of the two planes

$$\begin{cases} \frac{y}{b} + \frac{x}{a} = \frac{z}{c} \\ \frac{y}{b} - \frac{x}{a} = c \end{cases}$$

(Exercise: check that these two planes are not parallel!) Any point $P = (x, y, z)$ on the line L has to satisfy both of these equations. Multiplying out, we get

$$\left(\frac{y}{b} + \frac{x}{a}\right) \left(\frac{y}{b} - \frac{x}{a}\right) = \frac{z}{c} \cdot c = z$$

so that any point $P = (x, y, z)$ on the line L must also satisfy the equation defining the hyperbolic paraboloid. Therefore, these lines are necessarily contained in the hyperbolic paraboloid, regardless of our choice of c .

Similarly, we can get *another* family of lines contained in the hyperbolic paraboloid by looking instead at the equations

$$\begin{cases} \frac{y}{b} + \frac{x}{a} = c \\ \frac{y}{b} - \frac{x}{a} = \frac{z}{c} \end{cases}$$

This is an example of a *doubly ruled surface*: for every point P on the hyperbolic paraboloid, there are two distinct lines passing through P that are contained in the surface. (Exercise: prove this! Most of the work was already done. What is left to do is check that, for every point P on the paraboloid, two of the lines listed above pass through P .)

1.5.4 Hyperboloids

There are two types of hyperboloids, given by the equations:

$$\text{(Hyperboloid of One Sheet)} \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$

$$\text{(Hyperboloid of Two Sheets)} \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = -1$$

The term *sheet* here refers to whether the hyperboloid splits into one piece or two pieces in space. You can see this readily in the following figures.

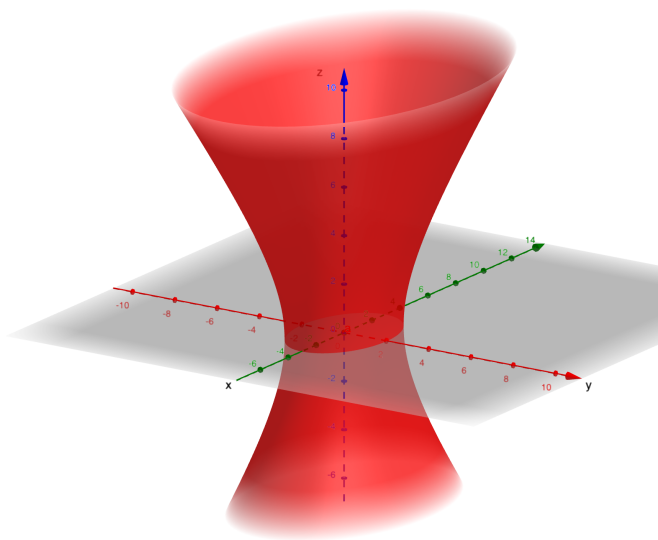


Figure 1.18: A hyperboloid of one sheet.

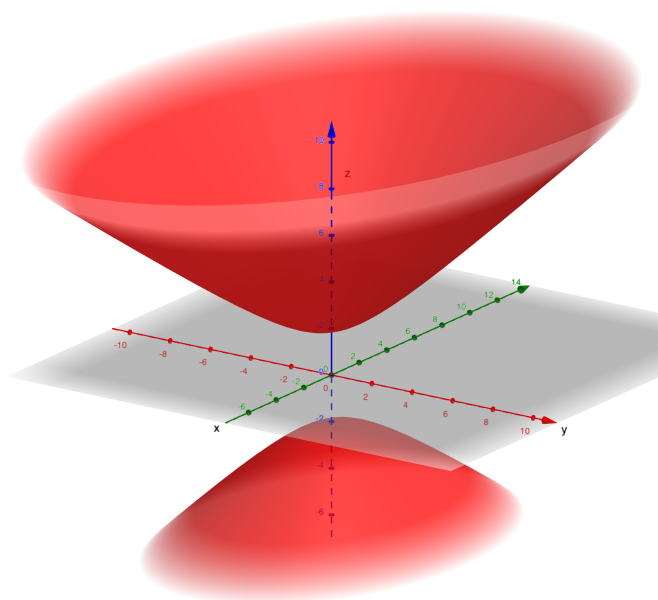


Figure 1.19: A hyperboloid of two sheets.

What do these surfaces look like “at infinity”? Suppose $a = b = 1$, and rewrite the above equations as

$$c^2(x^2 + y^2) = z^2 \pm c^2.$$

Notice that, in this equation, if $|z|$ is much, much larger than $|\pm c| = |c|$, then the

surface should look a lot like whatever surface is described by the equation

$$c^2(x^2 + y^2) = z^2.$$

This is because the $\pm c^2$ does not wiggle the value of z too much. This tells us that, away from the origin (where z is large) the hyperboloids look more and more like our next surface.

1.5.5 Cones

A (circular) cone is described by the equation $z^2 = c^2(x^2 + y^2)$.

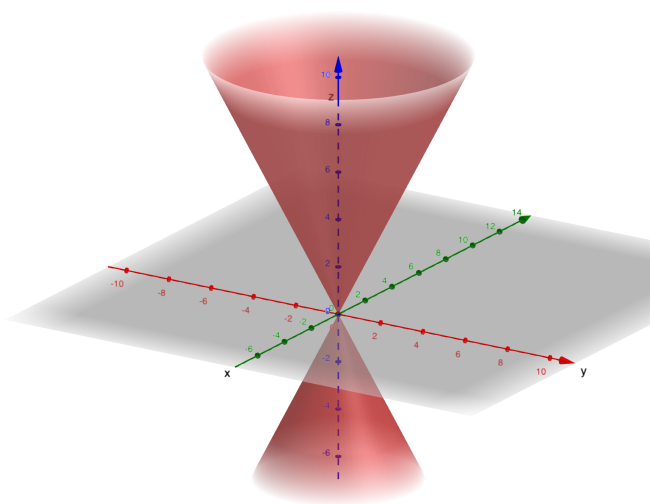


Figure 1.20: A circular cone.

Notice that the equation with the z variable in second-order covers both

$$z = +c\sqrt{x^2 + y^2} \quad \text{and} \quad z = -c\sqrt{x^2 + y^2},$$

simultaneously. If we only want to include one part of the cone, we would choose one of the above equations. With $c > 0$, the $+$ corresponds to the upper cone and the $-$ corresponds to the lower cone. There are also notions of an elliptic and parabolic cone, which we will not cover here. Try to use a graphing software like Geogebra to discover what these equations would look like!

1.5.6 Identifying Surfaces and Their Corresponding Equations

Often, we are presented with an image of some surface and would like to match it to a given equation. It can then be helpful to ask the following questions.

- Is the surface **bounded** or **unbounded**? In other words: are the variables x, y, z allowed to take on infinitely large values (meaning unbounded)? As an example: every sphere or ellipsoid is bounded, but the paraboloid surfaces we discussed are unbounded. Can you think of why this must be true?
- Are there any **intercepts** with the x - or y - or z -axes? For example, the cone intersects all three axes at the origin, because when $x = y = 0$, we necessarily have $z = 0$ (and similarly for any two variables simultaneously equalling zero). However, the hyperboloid of one sheet never intersects the z -axis. Try to figure out why this is true! Hint: remember that the z axis is given by the equations $x = y = 0$.
- Does the surface **intersect** the planes $x = 0$ or $y = 0$ or $z = 0$? What do these intersections look like? You can also check and see how the surface intersects the planes $x = c$ or $y = c$ or $z = c$ for various values of c . Be creative and try to pay attention to any special values appearing on the variables x, y, z for hints. These intersections with planes are called **traces** of the surface.
- Does the surface have any symmetries? For example, is it a surface of revolution? Is this reflected in the equations you are given? (Exercise: which of the surfaces in this section are surfaces of revolution?)

Worked Problems

1. Pair the accompanying figures to the following two equations:

$$z = -xy \quad \text{and} \quad z^2 = x^2 + y^2 - 1.$$

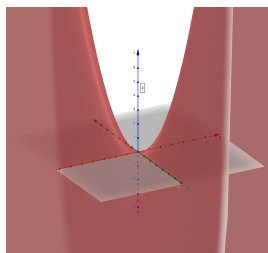


Figure 1.21

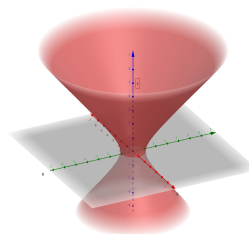


Figure 1.22

The correct answer is that Figure 1.21 matches $z = -xy$ and Figure 1.22 matches $z^2 = x^2 + y^2 - 1$. There are several ways we could show this:

- The origin $(0, 0, 0)$ belongs to the surface in Figure 1.21 but not in Figure 1.22. Notice that $x = y = z = 0$ is a solution to the equation $z = -xy$ but not the equation $x^2 + y^2 - 1 = z^2$.
- Figure 1.21 has an unrestricted range of values for x and y . However, Figure 1.22 shows that x and y are bounded away from the origin. This matches the fact that $x^2 + y^2 = z^2 + 1 \geq 1$ (since $z^2 \geq 0$).
- The intersection of Figure 1.22 with any plane $z = c$ is a circle. This matches the equation $x^2 + y^2 = c^2 + 1$ —an equation for a circle of radius $\sqrt{c^2 + 1}$. (This also gives Figure 1.22 a rotational symmetry about the z -axis, since circles are symmetric under rotation.)
- For another symmetry: make the change of variables $z \mapsto -z$. Then, $x^2 + y^2 - 1 = (-z)^2 = z^2$ but $-z = -(-xy) = xy$. In other words, only the equation $x^2 + y^2 - 1 = z^2$ is symmetric under the reflection $z \mapsto -z$. This matches Figure 1.22 (since if we flip the surface upside-down, we get the same surface).
- Take the plane passing through the origin and parallel to the sheet of paper you are reading. In the coordinate system indicated in Figures 1.21 and 1.22, this is the plane $x = -y$.

From the geometric representation, the intersection of this plane with the surface in Figure 1.21 is a parabola, whereas its intersection with the surface in Figure 1.22 is a hyperbola. Now let's see what happens if we try to compute the intersections with the same plane using the given equations. Making the substitution $x = -y$ in both equation gives:

$$z = -(-y)y = y^2 \quad \text{and} \quad 2x^2 - 1 = z^2.$$

These are the equations of a parabola and hyperbola (respectively). This, again, matches Figure 1.21 to $z = -xy$ and Figure 1.22 to $x^2 + y^2 - 1 = z^2$

Chapter 2

Functions of Several Variables

2.1 Graphs and Surfaces

We work with **functions** of n real variables. Recall: a function $f = f(x_1, \dots, x_n)$ is a rule which assigns to each $(x_1, \dots, x_n)$ in the domain of f (a subset of $\mathbb{R}^n$ – see below) a unique real number $f(x_1, \dots, x_n)$. We will use the following definitions and notations extensively.

- The **domain** $\mathcal{D}(f)$ is a subset of $\mathbb{R}^n$; if not specified explicitly, we assume that $\mathcal{D}(f)$ is the largest set upon which f is well defined.
- The **range** $\mathcal{R}(f)$ is the set of all values attained by f ,

$$\mathcal{R}(f) = \{f(x_1, \dots, x_n) : (x_1, \dots, x_n) \in \mathcal{D}(f)\}.$$

This is a set of real numbers. (Functions with values in other sets, for example complex-valued functions, can also be defined but will not be the subject of this course.)

- The **graph** of f is the set $\{(x_1, \dots, x_n, f(x_1, \dots, x_n)) : (x_1, \dots, x_n) \in \mathcal{D}(f)\}$. If the domain $\mathcal{D}(f) \subset \mathbb{R}^n$ (so that f is a function of n real variables) then the graph of f is a subset of $\mathbb{R}^{n+1}$.

Let us see some examples with computer-rendered images.

Example 2.1.1. *The function $f(x, y) = \sqrt{4 - x^2 - y^2}$ is well defined so long as $4 - x^2 - y^2 \geq 0$. So, we take the domain of f to be $\mathcal{D}(f) = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 4\}$.*

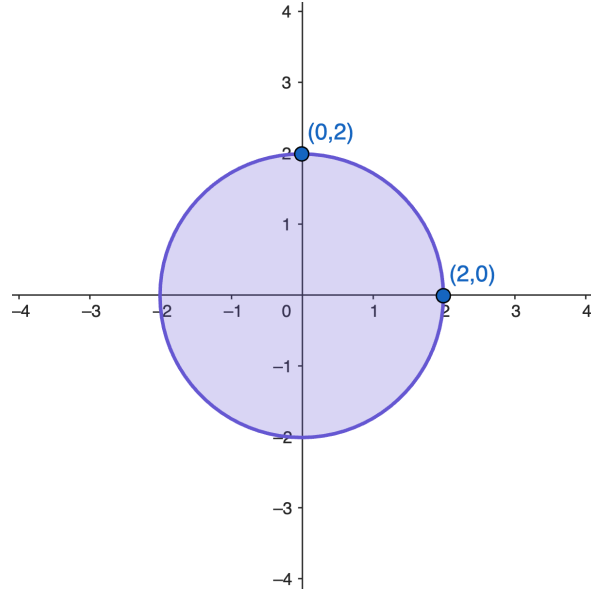


Figure 2.1: The domain of $f(x, y) = \sqrt{4 - x^2 - y^2}$.

The range of f is given by $\mathcal{R}(f) := [0, 2]$ and the graph of f is given by $\{(x, y, z) \in \mathbb{R}^3 : z = \sqrt{4 - x^2 - y^2}\}$ —a half-sphere in $\mathbb{R}^3$ of radius $r = 2$.

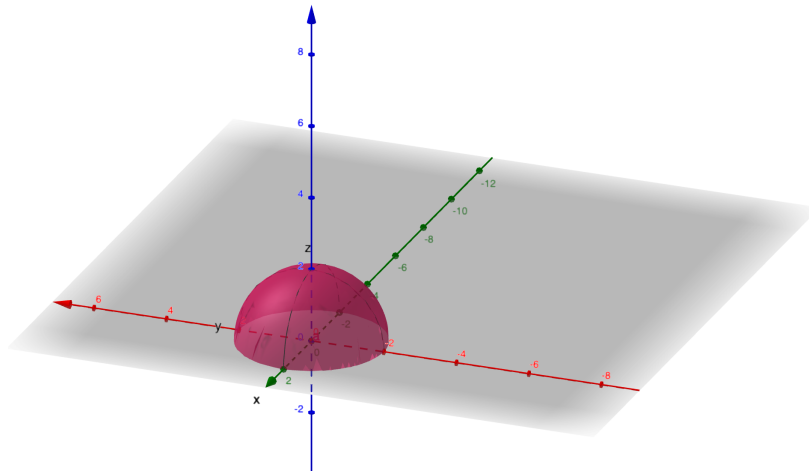


Figure 2.2: The graph of $f(x, y) = \sqrt{4 - x^2 - y^2}$ in $\mathbb{R}^3$.

Example 2.1.2. The function $f(x, y) = \frac{1}{x^2+y^2}$ is well defined whenever $(x, y) \neq (0, 0)$, so that the domain $\mathcal{D}(f) = \{(x, y) \in \mathbb{R}^2 : (x, y) \neq (0, 0)\}$. The range is given by $\mathcal{R}(f) = (0, \infty)$; to see this, take (x, y) close to $(0, 0)$ and see what happens.

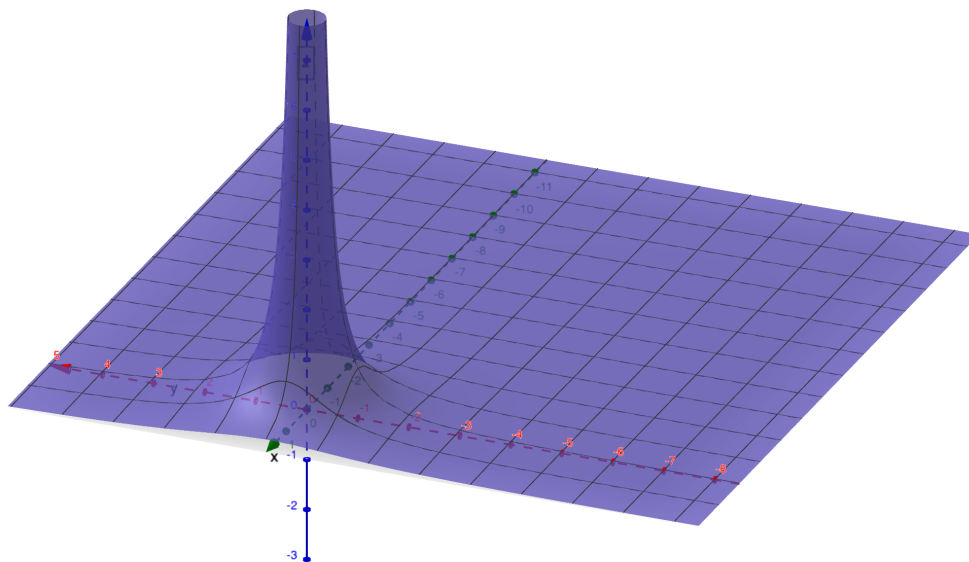


Figure 2.3: The graph of $f(x, y) = \frac{1}{x^2+y^2}$. This surface looks like the graph of $g(x) = \frac{1}{x^2}$ rotated about the z -axis.

2.1.1 Visualizing 3D graphs using level curves

Let $f(x, y)$ be a function of two variables. Consider the equation $f(x, y) = c$, where c is a real number. Often¹, this equation defines a curve in $\mathbb{R}^2$. These are the **level curves** of f . By drawing many such curves in $\mathbb{R}^2$, we can visualize the geometry of the graph of $f(x, y)$ (a surface in $\mathbb{R}^3$). This idea is similar to a contour map.

Example 2.1.3. The function $f(x, y) = \frac{1}{x^2+y^2}$ has level curves given by the equations $\frac{1}{x^2+y^2} = c$. These are the circles $x^2 + y^2 = \frac{1}{c}$. The curves are coded so that the dark

¹Not always – for example, if $f(x, y) = C$ is a constant function, then the set of (x, y) such that $f(x, y) = c$ is all of $\mathbb{R}^2$ if $c = C$, and empty otherwise. In situations like that, it makes more sense to talk about *level sets* rather than *level curves*. Conditions that guarantee that the equation $f(x, y) = c$ defines a curve will be discussed later in connection with the Implicit Function Theorem.

black level curve represents $f(x, y) = 1$. The dashed red represents $f(x, y) = C > 1$ and the solid light purple represents $f(x, y) < 1$. All values of C are evenly spaced (the common difference is .2)

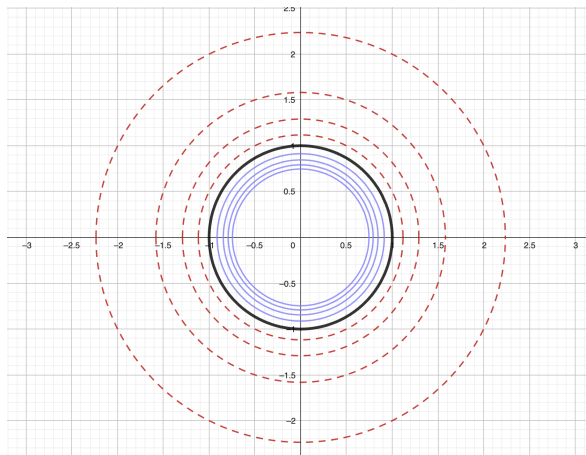


Figure 2.4: Some of the level curves of $f(x, y) = \frac{1}{x^2 + y^2}$.

Example 2.1.4. The level curves of the function $f(x, y) = \sqrt{4 - x^2 - y^2}$ are given by $\sqrt{4 - x^2 - y^2} = C$, or $x^2 + y^2 = 4 - C^2$. For $0 \leq C < 2$, these are circles as in the previous example, but the spacing is different.

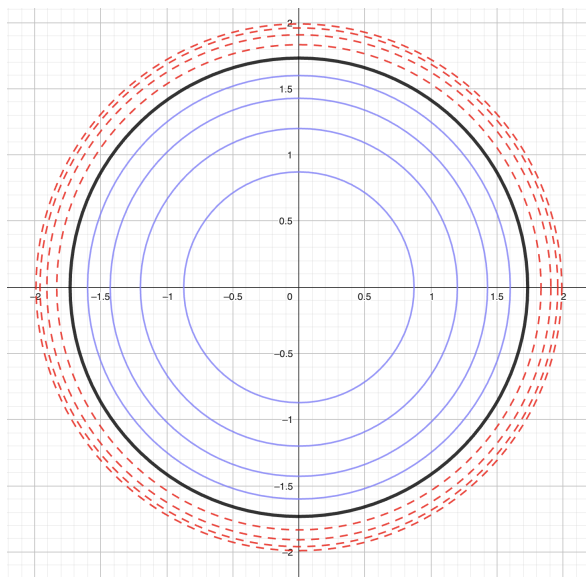


Figure 2.5: Some of the level curves of $f(x, y) = \sqrt{4 - x^2 - y^2}$.

Example 2.1.5. The graph of the function $f(x, y) = y^2 - x^2$ is a hyperbolic paraboloid. The level curves $y^2 - x^2 = c$ are hyperbolas in the xy -plane, so long as $c \neq 0$. However,

if $c = 0$, then $y^2 - x^2 = 0$ so $y = \pm x$. This gives a pair of lines intersecting the origin with slopes ± 1 .

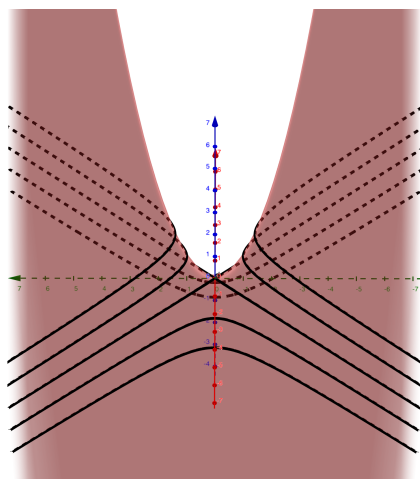


Figure 2.6: The hyperbolic paraboloid $z = y^2 - x^2$ along with cross-sections with planes $z = c$ for varying values of c .

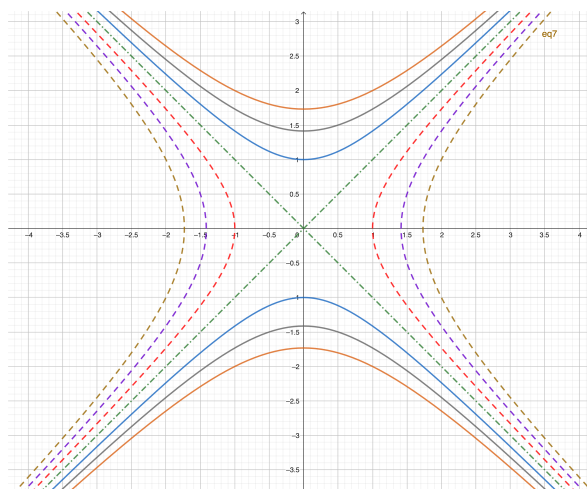


Figure 2.7: The level curves $y^2 - x^2 = c$. The dotted vs. solid lines carry no mathematical meaning—rather, they are to assist any readers who may not distinguish the colours!

2.1.2 Level surfaces for functions of 3 variables

For a function $f(x, y, z)$ of three variables, the graph

$$\{(x, y, z, f(x, y, z)) : (x, y, z) \in \mathcal{D}(f)\}$$

is a hypersurface in four dimensional space (that is, $\mathbb{R}^4$). Mathematically, this object is entirely well-defined! However, it is not possible to correctly visualize—much less draw—these hypersurfaces.

Despite this obstruction—that is, the confines of three dimensional awareness—we can still draw **level surfaces**. These are the surfaces defined by the equations $f(x, y, z) = c$ for representative choices of c in $\mathbb{R}$. Generically, for each value of c , this equation produces a surface in $\mathbb{R}^3$.

Example 2.1.6. *The function $f(x, y, z) = z^2 - x^2 - y^2$ has level surfaces given by the equations $z^2 - x^2 - y^2 = c$. When $c = 0$, this equation produces the cone $z^2 = x^2 + y^2$. When $c \neq 0$, we obtain hyperboloids (of one sheet when $c < 0$ and of two sheets when $c > 0$). The student is encouraged to review the definitions, equations and pictures for these surfaces from Section 1.5.*

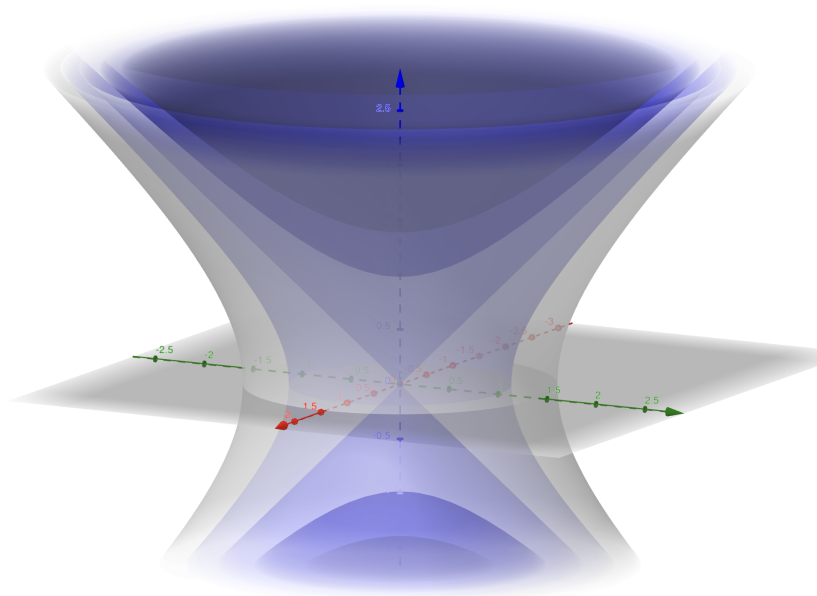


Figure 2.8: The level surfaces of $f(x, y, z) = z^2 - x^2 - y^2$.

2.1.3 Two final examples

We introduce two examples where the level curves display a “pathological” behaviour.

Example 2.1.7. *Find the level curves of the function $f(x, y) = \frac{x^2 - y^2}{x^2 + y^2}$, defined when $(x, y) \neq (0, 0)$.*

We compute its level curves as follows: suppose that $f(x, y) = c$, then

$$\frac{x^2 - y^2}{x^2 + y^2} = c, \text{ so that } x^2 - y^2 = c(x^2 + y^2),$$

$$(1 - c)x^2 = (1 + c)y^2.$$

When $c \neq -1$, we can solve for y^2 and therefore y :

$$y^2 = \frac{1 - c}{1 + c}x^2, \quad y = \pm \sqrt{\frac{1 - c}{1 + c}}x.$$

Notice that solutions only exist when $\frac{1-c}{1+c} \geq 0$, so that $-1 < c \leq 1$. For each $c \in (-1, 1)$, the above equation defines four half-lines meeting at (but not including) the point $(0, 0)$. When $c = 1$, we get $y = 0$ so there are only two half-lines. Since we assumed that $c \neq -1$, we now consider that case separately:

$$(1 + 1)x^2 = 0, \quad x^2 = 0, \quad x = 0,$$

which defines the y -axis with the $(0, 0)$ point removed (again, two half-lines).

Another way to solve for the level curves, in this question, is to change variables $x = r \cos \theta$ and $y = r \sin \theta$ for $r > 0$ and $\theta \in [0, 2\pi)$. This is known as the **polar coordinates** (you may know them from single-variable calculus). Using this,

$$f(x, y) = f(r, \theta) = \frac{r^2 \cos^2 \theta - r^2 \sin^2 \theta}{r^2} = \cos^2 \theta - \sin^2 \theta = \cos(2\theta).$$

Notice that, since $f(r, \theta) = \cos(2\theta)$, the range of f is $[-1, 1]$ (the same as the cosine function). For $c \in [-1, 1]$, the level curves $f(x, y) = c$ are defined by the equation

$$\cos(2\theta) = c$$

which, again, describes half-lines meeting at the origin.

All these half-lines corresponding to different values of c meet at the point $(0, 0)$. At the point $(0, 0)$ itself, the function is not defined (because we can't divide by 0). If we try to fill in the gap at $(0, 0)$ by using the values of f from nearby, it turns out that we cannot do that because $f(0, 0)$ would have multiple values! We will be able to say all this in a more rigorous way after we introduce limits and continuity.

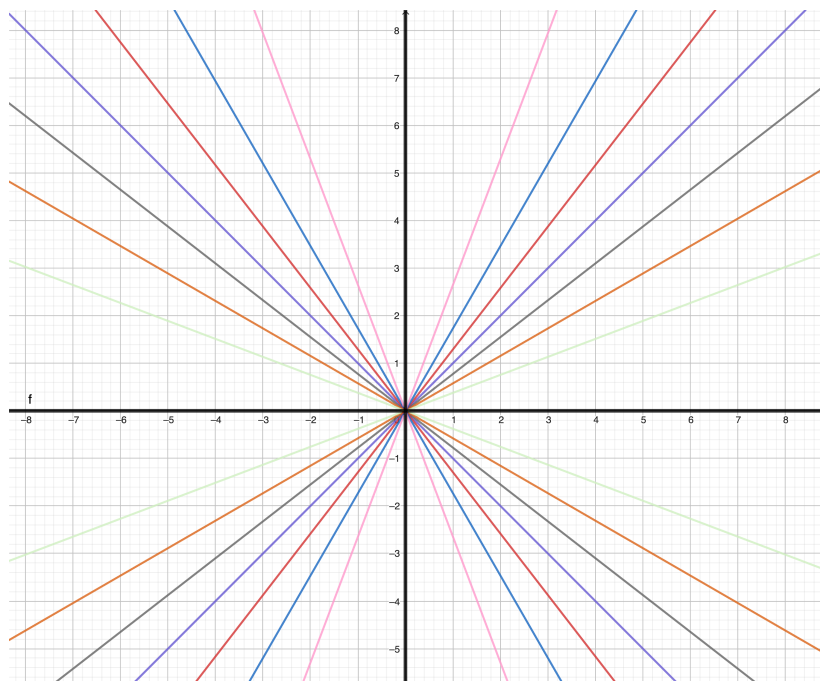


Figure 2.9: Some level curves of the function $f(x, y) = \frac{x^2 - y^2}{x^2 + y^2}$

Example 2.1.8. Find the level curves of the function $f(x, y) = \frac{x^4 - y^2}{x^4 + y^2}$, defined for $(x, y) \neq (0, 0)$.

The level curves are given by the equation

$$\frac{x^4 - y^2}{x^4 + y^2} = c \Leftrightarrow x^4 - y^2 = c(x^4 + y^2) \Leftrightarrow y = \pm \sqrt{\frac{1-c}{1+c}} x^2,$$

so long as $-1 < c \leq 1$. This restriction on c guarantees that $\frac{1-c}{1+c} \geq 0$. When $c = -1$, we instead obtain the equation $2x^4 = 0$, which describes the vertical line $x = 0$. To summarize:

- When $-1 < c < 1$, we get the two parabolas $y = \pm \sqrt{\frac{1-c}{1+c}} x^2$ with the origin $(0, 0)$ removed.
- When $c = -1$, we obtain the vertical line $x = 0$ with the origin removed.
- When $c = 1$, we obtain the horizontal line $y = 0$ with the origin removed.

Notice that, in all three situations, the level curves do not contain the origin $(0, 0)$. This is again because the domain of $f(x, y) = \frac{x^4 - y^2}{x^4 + y^2}$ never contained the origin! In this example, just like in the previous one, many level curves corresponding to different values of f converge to one point. Here, however, those curves are parabolas and not lines.

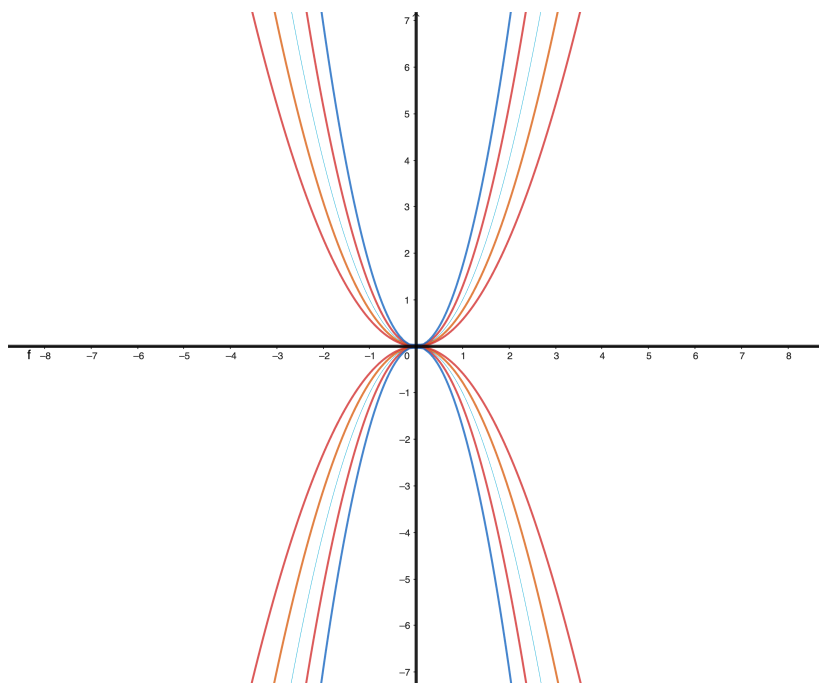


Figure 2.10: Some level curves of the function $f(x, y) = \frac{x^4 - y^2}{x^4 + y^2}$.

Practice Problems

1. Let $f(x, y) = \sqrt{y - x^2}$. Find the domain of f , and find the equations of its level curves.
2. Find the function $f(x, y, z)$ if for each constant D the level surface $f(x, y, z) = D$ is the plane perpendicular to $\mathbf{n} = -3\mathbf{i} + \mathbf{j} + 2\mathbf{k}$ and passing through the point $(0, 0, D^5)$.

2.2 Limits and Continuity

In this section, we work with functions of $f(x, y)$ of two variables (aka *bivariate* functions). Our goal is to introduce the definitions of **limits** and **continuity** for these functions. For functions on $\mathbb{R}^n$ with $n \geq 3$, the definitions and concepts are the same, you just need to add the extra variables. Working with functions of two variables allows us to investigate new multivariable phenomena while also keeping the notation as simple as possible.

Definition 2.2.1. If L is some real number and (a, b) is a point in $\mathbb{R}^2$, we say that f has the limit L at (a, b) and write

$$\lim_{(x,y) \rightarrow (a,b)} f(x, y) = L,$$

if

- every neighbourhood of (a, b) contains points of $\mathcal{D}(f)$ different from (a, b) ,
- for every $\epsilon > 0$ there is a $\delta > 0$ (which may depend upon ϵ) such that if $(x, y) \in \mathcal{D}(f)$ and $0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta$, then $|f(x, y) - L| < \epsilon$.

The first condition says that (x, y) can approach (a, b) while still remaining in the domain of f . The second condition says that, as (x, y) approaches (a, b) while remaining in the domain of f (the first condition ensures that this can happen), the values of $f(x, y)$ always approach L .

Some short remarks on limits of bivariate functions:

1. The point (a, b) in the definition of the limit need **not** be in $\mathcal{D}(f)$.
2. Even if (a, b) is in the domain of f , we do not require that $f(a, b) = L$ in the definition of a limit—more on this shortly.
3. If the function f has a limit L at the point (a, b) , then this limit is unique. For the limit to exist, the values of $f(x, y)$ have to approach **the same** value L regardless of how we approach (a, b) .
4. Arithmetic and functional operations (addition, multiplication, division, composition of functions, and so on) on limits of functions of two variables work the same way as for functions of one variable. See your textbook for more information.

Having defined limits, we can introduce continuity for functions of two real variables.

Definition 2.2.2. We say that $f(x, y)$ is **continuous** at (a, b) in $\mathbb{R}^2$ if

- the point (a, b) is in $\mathcal{D}(f)$;
- our function satisfies $\lim_{(x,y) \rightarrow (a,b)} f(x, y) = f(a, b)$

If f is continuous at each point (a, b) in its domain, we say that f is a **continuous function**.

Many common functions of two variables are continuous on their domains. Examples include polynomials, exponential functions, rational and trigonometric functions (on their domains of definition). Compositions of continuous functions are also continuous, so that we can use the common functions just mentioned to construct more complicated continuous functions.

Example 2.2.3. Consider the following functions:

1. The function $f(x, y) = x^3y^2 - \frac{1}{x}$ is well-defined for all (x, y) in $\mathbb{R}^2$ with $x \neq 0$. Moreover, you can verify that $f(x, y)$ is continuous at all points in its domain.
2. The function $f(x, y) = e^{x^2+4y^2}$ is well-defined and continuous at all points (x, y) in $\mathbb{R}^2$.
3. The function

$$f(x, y) = \begin{cases} \frac{1}{x^2+y^2} & \text{if } (x, y) \neq 0 \\ 0 & \text{if } (x, y) = 0 \end{cases}$$

is well-defined for all points (x, y) in $\mathbb{R}^2$. However, $f(x, y)$ is not continuous at $(a, b) = (0, 0)$ —in fact, $f(x, y)$ approaches infinity as $(x, y) \rightarrow (0, 0)$, so there is no value L that could satisfy the condition in Definition 1.1.

4. But if we just use $f(x, y) = \frac{1}{x^2+y^2}$ on its natural domain of definition (which is $\mathcal{D} = \mathbb{R}^2 \setminus \{(0, 0)\}$), then f is continuous on $\mathcal{D}$.

2.2.1 Evaluating limits using the Squeeze Theorem

The following *Squeeze Theorem* should look familiar from single-variable calculus. It continues to be true for functions of two variables.

Theorem 2.2.4. Suppose f, g, h are functions of two variables, defined on some common neighbourhood N of (a, b) except possibly at (a, b) . Further, suppose that

- there is some real number L such that

$$\lim_{(x,y) \rightarrow (a,b)} f(x, y) = \lim_{(x,y) \rightarrow (a,b)} h(x, y) = L,$$

- the function $g(x, y)$ lies between $f(x, y)$ and $h(x, y)$ for all $(x, y) \in N$ except possibly at (a, b) .

Then, $\lim_{(x,y) \rightarrow (a,b)} g(x, y) = L$.

Note that the second bullet point is a little bit vague, in that we do not specify whether $f(x, y) \leq g(x, y) \leq h(x, y)$ or $h(x, y) \leq g(x, y) \leq f(x, y)$. Either would work, and it could in fact happen that the first inequality holds for some points $(x, y) \in N$ and the second one holds for others. This is the case in Example 2.2.5 below.

The theorem could also be modified to allow cases where the domain of f does not contain an entire neighbourhood of (a, b) . For example, if $g(x, y)$ is only defined for (x, y) with $x > 0$ and we want to find its limit at $(0, 0)$, it's enough to find f and h so that f, g, h satisfy the assumptions of the Squeeze Theorem for $x > 0$ only.

Example 2.2.5. *Let us decide whether $f(x, y) = x \sin(\frac{1}{y})$, defined for (x, y) with $y \neq 0$, has a limit as $(x, y) \rightarrow (0, 0)$.*

The trick is to recall that

$$-1 \leq \sin\left(\frac{1}{y}\right) \leq 1, \text{ for every } y \neq 0.$$

This means that $f(x, y)$ always lies between x and $-x$, regardless of the value of y in $\mathbb{R} \setminus \{0\}$. Specifically, we have

$$\begin{aligned} -x &\leq f(x, y) \leq x \text{ for } x \geq 0, \\ x &\leq f(x, y) \leq -x \text{ for } x < 0. \end{aligned}$$

Since

$$\lim_{(x,y) \rightarrow (0,0)} (-x) = \lim_{(x,y) \rightarrow (0,0)} x = 0,$$

the Squeeze Theorem gives $\lim_{(x,y) \rightarrow (0,0)} f(x, y) = 0$.

2.2.2 Evaluating limits using the $\epsilon - \delta$ definition of a limit (Definition 1.1)

Example 2.2.6. *Let us prove using the $\epsilon - \delta$ definition of a limit that*

$$\lim_{(x,y) \rightarrow (0,0)} (x^2 + y^2) = 0.$$

Let $\epsilon > 0$. We need to find a $\delta = \delta(\epsilon) > 0$ such that

$$0 < \sqrt{x^2 + y^2} < \delta \implies |x^2 + y^2| < \epsilon.$$

Here, the notation $\delta = \delta(\epsilon)$ means that δ is allowed to depend (as a function) upon the given $\epsilon > 0$.

So, choose $\delta = \sqrt{\epsilon}$. Then $\delta > 0$ and

$$0 < \sqrt{x^2 + y^2} < \delta = \sqrt{\epsilon} \implies x^2 + y^2 < \epsilon.$$

Hence, by definition, $\lim_{(x,y) \rightarrow (0,0)} (x^2 + y^2) = 0$.

2.2.3 Functions that do not have limits

We now discuss some functions which do not have limits at certain points (a, b) in $\mathbb{R}^2$.

Example 2.2.7. Consider the function

$$f(x, y) = \frac{x^2 - y^2}{x^2 + y^2},$$

defined so long as $(x, y) \neq (0, 0)$. We encountered the level curves of this function in Section 2.1.

We claim that $f(x, y)$ does **not** have a limit as $(x, y) \rightarrow (0, 0)$, for the reason that the limits along different paths approaching $(0, 0)$ are different. Let us illustrate. If $(x, 0)$ and $(0, y)$ are points on the x and y axis respectively, then

$$f(x, 0) = \frac{x^2}{x^2} = 1, \quad f(0, y) = \frac{-y^2}{y^2} = -1.$$

So, evaluating the limit $(x, y) \rightarrow (0, 0)$ along the x - and y -axes produces distinct values. We could also revisit the level curves found in Section 2.1 to see that, for any value of c in $[-1, 1]$, there is at least one line L such that $f(x, y)$ approaches c as (x, y) approach $(0, 0)$ along L ! This multitude of limits from different directions means that the limit in the sense of Definition 2.2.1 does not exist.

Example 2.2.8. The function

$$f(x, y) = \frac{x^4 - y^2}{x^4 + y^2},$$

defined for $(x, y) \neq (0, 0)$, does not have a limit as $(x, y) \rightarrow (0, 0)$. As an exercise, prove this using a similar argument along the x - and y -axes.

Example 2.2.9. Let us modify the function from the last example. Consider the limit as $(x, y) \rightarrow (0, 0)$ of the function

$$g(x, y) = \begin{cases} \frac{x^4 - y^2}{x^4 + y^2} & \text{if } y \neq 0, \\ -1 & \text{if } y = 0. \end{cases}$$

Then, $g(x, 0) = g(0, y) = -1$ for any x, y in $\mathbb{R}$ (check this!). Even more, if we set $y = ax$ for some non-zero number a —so that $(x, y) = (x, ax)$ —we see that for $x \neq 0$,

$$g(x, ax) = \frac{x^4 - a^2x^2}{x^4 + a^2x^2} = \frac{x^2 - a^2}{x^2 + a^2},$$

so that $g(x, ax) \rightarrow -1$ as $x \rightarrow 0$. Geometrically: this means that along linear paths $\{(x, ax) : x \in \mathbb{R}\}$ through the origin, g approaches -1 as $x \rightarrow 0$. It might look like the bivariate limit should be -1 .

Yet, in spite of this, the limit $\lim_{(x,y) \rightarrow (0,0)} g(x,y)$ fails to exist. To see this, evaluate the limit along the parabolic curve $y = x^2$. Then, for $x \neq 0$,

$$g(x, x^2) = \frac{x^4 - (x^2)^2}{x^4 + (x^2)^2} = \frac{0}{2x^4} = 0.$$

In particular, $g(x, x^2) \rightarrow 0$ as $x \rightarrow 0$! Therefore the bivariate limit does not exist. Again, we could also find other parabolas that produce different value of the limit along path. To see how this is possible, we recommend revisiting the level curves in the last example in Section 2.1.

In conclusion: for a limit of $f(x,y)$ to exist at (a,b) , its value must be the same **for all possible paths** to (a,b) across the domain of f ! This means:

- If you are proving that a limit **does not** exist, it's enough to indicate two different paths of approach to (a,b) with different limits along those paths.
- It could also happen that the limit along a single path does not exist. In that case, we can see that the double limit does not exist just by considering that single path.
- If you are proving that a limit **exists**, it's never enough to consider just limits along paths. There are infinitely many possible paths of approach, and moreover, there are infinitely many *types of* paths that one could take. There is no way to check all of them. You have to use “affirmative” methods such as the rules for limits, the Squeeze Theorem, or the $\epsilon - \delta$ definition.

Remark 2.2.1. *There are other definitions of continuity that you may see in the literature, and this can lead to some nuance as to which functions are considered continuous on domains that have isolated points. For example, the topological definition of continuity is that the inverse images of all open sets are open. With this definition, a function defined only on the set of integer points $\mathcal{D} = \{(m,n) \in \mathbb{Z}^2 : m,n \text{ are integer}\}$ would always be continuous by default, with the limit condition vacuously true because there are no limits to be considered.*

Remark 2.2.2. *Some authors reserve the term “double limit” for the limit*

$$\lim_{x \rightarrow a, y \rightarrow b} f(x,y),$$

as opposed to the limit $\lim_{(x,y) \rightarrow (a,b)} f(x,y)$ that we have been considering. The difference is that the first limit “ignores” the values of f along the lines $x = a$ and $y = b$, not just at $(x,y) = (a,b)$. In practice, however, the first type of limits is often used in situations when f is undefined along the lines $x = a$ and $y = b$, in which case the second type of limits also ignores those two lines. For clarity, we recommend always writing out the limits using mathematical symbols and not just words.

Worked Problems

1. Let

$$f(x, y) = \begin{cases} \frac{x^3}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

Is f continuous at $(0, 0)$? Prove your answer.

Solution 1: Using the Squeeze Theorem. For all $(x, y) \neq (0, 0)$, we have

$$|f(x, y)| = \left| \frac{x^3}{x^2 + y^2} \right| = |x| \cdot \left| \frac{x^2}{x^2 + y^2} \right| \leq |x|.$$

This means that $f(x, y)$ lies between x and $-x$ for all $(x, y) \neq (0, 0)$. Since $\pm x \rightarrow 0$ as $(x, y) \rightarrow (0, 0)$, we also have $\lim_{(x, y) \rightarrow (0, 0)} f(x, y) = 0 = f(0, 0)$. Therefore f is continuous at $(0, 0)$.

Solution 2: Using the $\epsilon - \delta$ definition of a limit. Let $\epsilon > 0$. We want to find $\delta(\epsilon) > 0$ such that, if $0 < \sqrt{x^2 + y^2} < \delta$, then $\frac{|x^3|}{|x^2 + y^2|} < \epsilon$. First, we see as in Solution 1 that

$$\frac{|x^3|}{|x^2 + y^2|} \leq |x|, \text{ for all } (x, y) \neq (0, 0).$$

Now, remember that

$$|x| = \sqrt{x^2} \leq \sqrt{x^2 + y^2}, \text{ for all } x, y \text{ real.}$$

Hence, choosing $\delta = \epsilon$, we see that

$$\frac{|x^3|}{|x^2 + y^2|} \leq |x| \leq \sqrt{x^2 + y^2} < \delta = \epsilon,$$

which is what we wanted.

Note the strategy of the proof: to give an epsilon-delta proof for a complicated function $f(x, y)$, it helps to proceed at first as if you were going to use the Squeeze Theorem. If you can place $f(x, y)$ between functions $g(x, y)$ and $h(x, y)$ that are simpler but have the same limit (in this case we can use $g(x, y) = x$ and $h(x, y) = -x$), and find a δ that works for these functions, then the same δ will work for f .

2. Find, with proof, all values of $c \in \mathbb{R}$ such that the limit

$$\lim_{(x, y) \rightarrow (0, 0)} \frac{cx^2 + y^4}{x^2 + y^2}$$

exists.

If we let $x = 0$ and $y \rightarrow 0$, then the limit is

$$\lim_{y \rightarrow 0} \frac{y^4}{y^2} = \lim_{y \rightarrow 0} y^2 = 0.$$

Hence if the limit exists, it should be 0. But if we let $y = 0$ and $x \rightarrow 0$, we get

$$\lim_{x \rightarrow 0} \frac{cx^2}{x^2} = c,$$

so that the limit can only exist if $c = 0$.

We now prove that if $c = 0$ then the limit exists. We have

$$\left| \frac{y^4}{x^2 + y^2} \right| = |y|^2 \left| \frac{y^2}{\sqrt{x^2 + y^2}} \right| \leq |y|^2.$$

Since $|y|^2 \rightarrow 0$ as $(x, y) \rightarrow (0, 0)$, the limit exists and is equal to zero.

3. **Prove using the $\epsilon - \delta$ definition** that if f, g are both defined for all (x, y) in some neighbourhood of (a, b) , and if $\lim_{(x, y) \rightarrow (a, b)} f(x, y) = 2$ and $\lim_{(x, y) \rightarrow (a, b)} g(x, y) = 3$, then $\lim_{(x, y) \rightarrow (a, b)} f(x, y) + g(x, y) = 5$.

Let $\epsilon > 0$. We need to prove that there exists $\delta > 0$ such that for all x, y with

$$0 < \sqrt{(x - a)^2 + (x - b)^2} < \delta$$

we have

$$|f(x, y) + g(x, y) - 5| < \epsilon.$$

Since $\lim_{(x, y) \rightarrow (a, b)} f(x, y) = 2$, there exists $\delta_1 > 0$ such that for all x, y with $0 < \sqrt{(x - a)^2 + (x - b)^2} < \delta_1$ we have

$$|f(x, y) - 2| < \frac{\epsilon}{2}.$$

Similarly, since $\lim_{(x, y) \rightarrow (a, b)} g(x, y) = 3$, there exists $\delta_2 > 0$ such that for all x, y with $0 < \sqrt{(x - a)^2 + (x - b)^2} < \delta_2$ we have

$$|g(x, y) - 3| < \frac{\epsilon}{2}.$$

Let $\delta = \min(\delta_1, \delta_2)$. Then for all (x, y) such that $0 < \sqrt{(x - a)^2 + (x - b)^2} < \delta$, we have both $|f(x, y) - 2| < \epsilon/2$ and $|g(x, y) - 3| < \epsilon/2$. Therefore

$$\begin{aligned} |f(x, y) + g(x, y) - 5| &= |(f(x, y) - 2) + (g(x, y) - 3)| \\ &\leq |f(x, y) - 2| + |g(x, y) - 3| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon, \end{aligned}$$

where the second step follows from the triangle inequality. This proves our conclusion.

4. Let

$$f(x, y) = \begin{cases} 1 & \text{if } x + y > 0, \\ -1 & \text{if } x + y \leq 0. \end{cases}$$

Prove **using the ϵ - δ definition** that $f(x, y)$ is not continuous at $(0, 0)$.

Since $f(0, 0) = -1$, we have to prove the following: there exists $\epsilon > 0$ such that for every $\delta > 0$ there exist x, y with

$$0 < \sqrt{x^2 + y^2} < \delta \text{ and } |f(x, y) + 1| > \epsilon.$$

Let $\epsilon = 1$. For any $\delta > 0$, let $(x, y) = (\delta/2, 0)$. Then $\sqrt{x^2 + y^2} = \delta/2 < \delta$ and $|f(x, y) + 1| = 1 + \delta/2 > 1$, as claimed.

5. Suppose that a function $f(x, y)$ has the following form in polar coordinates:

$$f(r \cos \theta, r \sin \theta) = \begin{cases} r \tan \theta & \text{if } \theta \notin \{\frac{\pi}{2}, \frac{3\pi}{2}\}, \\ 0 & \text{if } \theta \in \{\frac{\pi}{2}, \frac{3\pi}{2}\}. \end{cases}$$

Is it true that $\lim_{(x,y) \rightarrow (0,0)} f(x, y) = 0$? Prove your answer. (In this question, an $\epsilon - \delta$ proof is not required.)

It is not true. To see this, consider the trajectory where $r \tan \theta = 1$, $0 \leq \theta < \pi/2$. Since $\tan \theta \rightarrow \infty$ as $\theta \rightarrow \pi/2$, we have $r \rightarrow 0$ but $f(r \cos \theta, r \sin \theta) = 1$ along that trajectory.

Practice Problems

1. Evaluate the following limits, or prove that they do not exist. In this question, you **do not have to** use the $\epsilon - \delta$ definition of limits.

(a) $\lim_{(x,y) \rightarrow (0,0)} \frac{\sin(xy)}{x^2 - y^2}$

(b) $\lim_{(x,y) \rightarrow (0,0)} \frac{xy^5 \cos(1/y)}{x^4 + y^4}$

(c) $\lim_{(x,y) \rightarrow (0,0)} \frac{xy^2}{x^4 + y^4}$

(d) $\lim_{(x,y) \rightarrow (0,0)} \frac{x^2 y^4}{x^2 + y^6}$

2. Evaluate the following limits, or prove that they do not exist. In this question, you **do not have to** use the $\epsilon - \delta$ definition of limits.

(a) $\lim_{(x,y) \rightarrow (0,0)} \frac{xy}{\sqrt{x^2 + y^2}}.$

(b) $\lim_{(x,y) \rightarrow (0,0)} \frac{(x - y)^2}{x^2 + y^2}$

(c) $\lim_{(x,y) \rightarrow (0,0)} \frac{x^3}{|x| + |y|}$

(d) $\lim_{(x,y) \rightarrow (0,0)} \frac{x + y}{\sqrt{x^2 + y^2}}.$

(e) $\lim_{(x,y) \rightarrow (0,0)} \frac{\sin^2 x \sin^2 y}{x^2 + y^2}$

3. Prove using the $\epsilon - \delta$ definition that $\lim_{(x,y) \rightarrow (0,0)} \frac{x^3 y^3}{(x^2 + y^2)^2} = 0$. (Given $\epsilon > 0$, find $\delta > 0$ such that...)

4. Prove using the $\epsilon - \delta$ definition that

$$\lim_{(x,y) \rightarrow (0,0)} \frac{x^3 y}{x^2 + y^4} = 0.$$

5. Give an $\epsilon - \delta$ proof that $\lim_{(x,y) \rightarrow (0,0)} \frac{x^2 + 4y^2}{|x| + |y|} = 0$. (Given $\epsilon > 0$, find $\delta > 0$... You might find it useful to consider three cases: when $x, y \neq 0$, when $x = 0$, and when $y = 0$.)

6. Prove using the $\epsilon - \delta$ definition that $\lim_{(x,y) \rightarrow (0,0)} \frac{2xy^3}{x^2 + y^2} = 0$. (Given $\epsilon > 0$, find $\delta > 0$ such that...)

7. Assume that the following is true for some function $f(x, y)$, defined for all x and y : for every $\epsilon > 0$ there is a $\delta > 0$ such that for some (x, y) such that $0 < \sqrt{(x - a)^2 + (y - b)^2} < \delta$ we have $|f(x, y) - L| < \epsilon$ (make sure to read this carefully!). Does this imply that $\lim_{(x,y) \rightarrow (a,b)} f(x, y) = L$? If yes, prove it. If not, give an example of a function $f(x, y)$ that satisfies the above condition but does not have a limit at (a, b) .

8. Prove using the $\epsilon - \delta$ definition that if $f(x, y)$ is a function such that

$$\lim_{x \rightarrow 0} f(x, 0) = 4, \quad \lim_{y \rightarrow 0} f(0, y) = 1,$$

then the limit $\lim_{(x,y) \rightarrow (0,0)} f(x, y)$ does not exist. (You need to prove that the “for every $\epsilon > 0$ there exists $\delta > 0$...” statement **cannot** be true for a function as above.)

2.3 Partial Derivatives

In this section, we work with functions $f(x_1, \dots, x_n)$ of n real variables.

Definition 2.3.1. The **partial derivative** f with respect to x_j is the function $f_j(x_1, \dots, x_n)$ of n real variables, defined via the formula

$$f_j(x_1, \dots, x_n) = \lim_{h \rightarrow 0} \frac{1}{h} \left(f(x_1, \dots, x_{j-1}, x_j + h, x_{j+1}, \dots, x_n) - f(x_1, \dots, x_n) \right)$$

In words, the partial derivative f_j is calculated by holding all variables x_i with $i \neq j$ fixed, and evaluating the (single-variable) derivative in the x_j variable. The simplest case is when f is a function of two variables.

Example 2.3.2. Suppose that $f(x, y)$ is a function of two real variables. Then, the partial derivatives $f_1(x, y)$ and $f_2(x, y)$ are

$$f_1(x, y) = \lim_{h \rightarrow 0} \frac{1}{h} (f(x + h, y) - f(x, y)),$$

$$f_2(x, y) = \lim_{h \rightarrow 0} \frac{1}{h} (f(x, y + h) - f(x, y)).$$

For the partial derivative f_j to exist, the limit in the definition above has to exist. Notice that these are *single-variable limits* because all variables except the x_j variable are held fixed.

It is important to realize that there are multiple ways of writing partial derivatives. For example, if $f = f(x_1, \dots, x_n)$, we may write

$$f_j(x_1, \dots, x_n) = \frac{\partial f}{\partial x_j}(x_1, \dots, x_n) = D_j f(x_1, \dots, x_n) = D_{x_j} f(x_1, \dots, x_n).$$

Often, if $f = f(x, y)$ is a function of 2 variables, you will see the notation

$$f_x(x, y) = f_1(x, y) = \frac{\partial f}{\partial x} = \text{etc.},$$

and similarly for functions $f(x, y, z)$ of 3 variables. To indicate the evaluation of the derivative at a given point (a, b) , we will use notation such as

$$f_x(a, b) = f_1(a, b) = \frac{\partial f}{\partial x}(a, b) = \frac{\partial f}{\partial x} \Big|_{(x,y)=(a,b)} = \text{etc.},$$

Most of the time, notation does not matter. However, there will be times later in this course when some choices of notation are preferable to others in terms of clarity or convenience. We will make note when such a situation arises.

Example 2.3.3. Consider the functions $f(x, y) = x^2y - 3 \cos x$ and $g(x, y, z) = xe^{yz^2}$. Then, the partial derivatives of f are

$$f_1(x, y) = \frac{\partial}{\partial x}(x^2y - 3 \cos x) = 2xy + 3 \sin x$$

$$f_2(x, y) = \frac{\partial}{\partial y}(x^2y - 3 \cos x) = x^2$$

and the partial derivatives of $g(x, y, z)$ are

$$\frac{\partial g}{\partial x}(x, y, z) = e^{yz^2}$$

$$\frac{\partial g}{\partial y}(x, y, z) = xe^{yz^2} \cdot z^2$$

$$\frac{\partial g}{\partial z}(x, y, z) = xe^{yz^2} \cdot (2yz).$$

Notice: to evaluate $\frac{\partial g}{\partial y}$ and $\frac{\partial g}{\partial z}$, we had to apply the Chain Rule to e^{yz^2} .

Since partial derivatives are defined as derivatives with respect to a single variable, the usual rules from single-variable calculus (sum, product, quotient, the Chain Rules) apply here as long as we are only differentiating in that one variable.

2.3.1 Partial Derivatives and Continuity

While partial derivatives follow the same single-variable differentiation rules, there are distinct differences between partial derivatives of multivariable functions and derivatives of single-variable functions.

One stark contrast: a function $f(x_1, \dots, x_n)$ with $n \geq 2$ can have some of its partial derivatives exist at a point P in $\mathbb{R}^n$, and yet still fail to be continuous at P . Why is this? Consider the case $n = 2$, so that $f = f(x, y)$. To calculate $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ at (a, b) in $\mathbb{R}^2$, we need only know the values $f(x, b)$ and $f(a, y)$ for x “close enough” to a and y “close enough” to b . Thus, the partial derivatives and their associated limits are defined along specific paths: the lines (x, b) and (a, y) in $\mathbb{R}^2$. However, the partial derivatives provide no information about the limit $\lim_{(x,y) \rightarrow (a,b)} f(x, y)$ along other paths. For example, the linear path $x - a = y - b$ is avoided by the limits defining f_1 and f_2 .

Let us consider an extended example to illustrate this idea.

Example 2.3.4. Let f be the piecewise-defined function

$$f(x, y) = \begin{cases} \frac{xy}{x^2+y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$$

We claim that

1. $f_1(0, 0) = f_2(0, 0) = 0$
2. $\lim_{(x,y) \rightarrow (0,0)} f(x, y)$ does not exist.

In particular, this means that f has both partial derivatives f_1 and f_2 at the origin, and yet it is not continuous there.

Let us prove each claim in the example.

Proof of Claim 1. We show that $f_1(0, 0) = 0$. The proof of $f_2(0, 0) = 0$ is almost identical and is left to the student.

First, notice that it is **not** sufficient to differentiate $f(x, y)$ away from the origin and then evaluate at $(0, 0)$. This is due to the fact that the formula for f away from the origin is not well-defined at the origin. Instead, we are forced to return to the definition of the partial derivative and the associated limit along the line $(x, y) = (x, 0)$.

However, for any point $(x, 0)$ in $\mathbb{R}^2$, we have

$$f(x, 0) = \begin{cases} \frac{0}{x^2+0^2} = 0, & x \neq 0 \\ 0, & x = 0, \end{cases}$$

so that we actually have $f(x, 0) = 0$ for **all** x real. Hence,

$$\frac{\partial f}{\partial x}(0, 0) = \frac{\partial}{\partial x}(0) \Big|_{x=0} = 0.$$

If you want to be pedantic, you can show this directly from the limit definition of $f_1(0, 0)$. Namely

$$f_1(0, 0) = \lim_{h \rightarrow 0^+} \frac{1}{h} (f(h, 0) - f(0, 0)) = \lim_{h \rightarrow 0^+} \frac{1}{h} (0 - 0) = \lim_{h \rightarrow 0^+} 0 = 0.$$

So, $f_1(0, 0) = 0$, as claimed. □

Proof of Claim 2. We show that $\lim_{(x,y) \rightarrow (0,0)} f(x, y)$ does not exist. In particular, $f(x, y)$ fails to be continuous at the origin.

To prove this, let us consider the limit $(x, y) \rightarrow (0, 0)$ along the linear path $(x, 0)$. From the first part of the formula for f , when $x \neq 0$, we have

$$f(x, 0) = \frac{0}{x^2 + 0^2} = 0 \rightarrow 0 \text{ as } x \rightarrow 0.$$

Now, consider the limit $(x, y) \rightarrow (0, 0)$ along the linear path $(x, y) = (x, x)$ for real x satisfying $x \neq 0$. Then

$$f(x, x) = \frac{x \cdot x}{x^2 + x^2} = \frac{1}{2} \rightarrow \frac{1}{2} \text{ as } x \rightarrow 0.$$

Since the limits along different trajectories do not match, the double limit does not exist. $\square$

Another geometric perspective on the previous example comes from examining the level curves of $f(x, y)$. Let us use the polar coordinates $(x, y) = (r \cos \theta, r \sin \theta)$; notice that $(x, y) = (0, 0)$ corresponds to $r = 0$ in this coordinate system. Then, for any $r > 0$, we have

$$f(x, y) = f(r \cos \theta, r \sin \theta) = \frac{(r \cos \theta)(r \sin \theta)}{r^2} = \cos \theta \sin \theta = \frac{1}{2} \sin(2\theta)$$

and $f(0, 0) = 0$. A diagram of the level curves is given below.

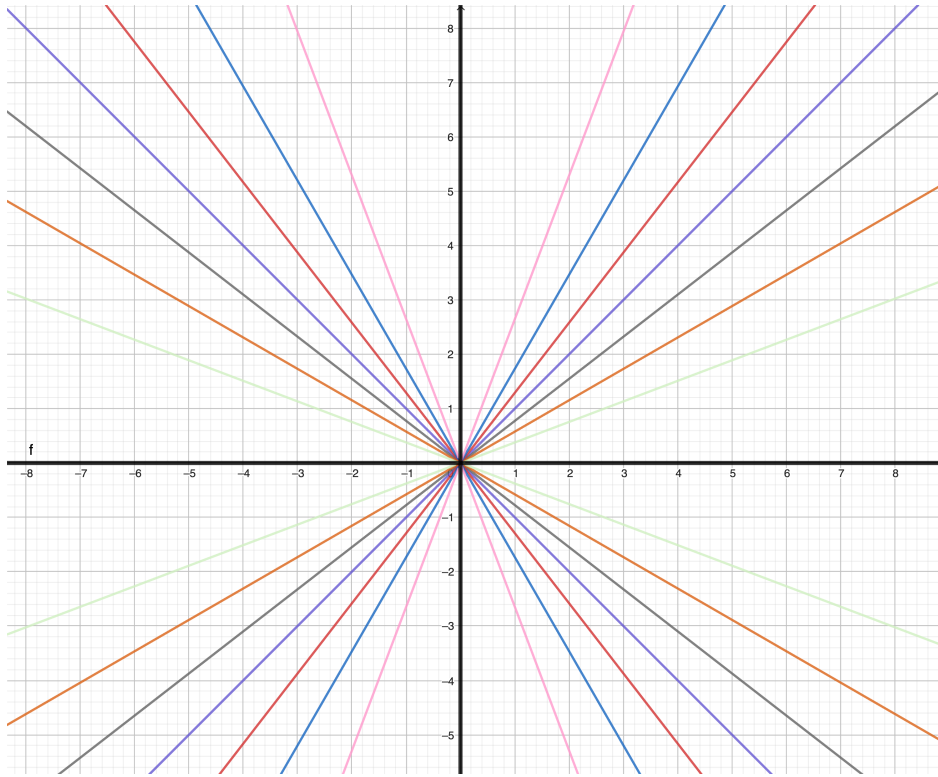


Figure 2.11: The level curves of the function f defined in Example 1.4 are lines through the origin.

The partial derivatives $f_1(0, 0)$ and $f_2(0, 0)$ are based only on the information for the lines $y = 0$ and $x = 0$ (respectively). These correspond to the bold black lines in the previous figure.

Worked Problems

1. Let $f(x, y) = \frac{x^3 - 2y^3}{x^2 + 4y^2}$ if $(x, y) \neq (0, 0)$, and $f(0, 0) = 0$. Calculate the partial derivatives of f at $(0, 0)$ using the definition of partial derivatives.

We have $f(x, 0) = \frac{x^3}{x^2} = x$ for $x \neq 0$, and $f(0, y) = \frac{-2y^3}{4y^2} = -\frac{y}{2}$ for $y \neq 0$. Hence

$$f_1(0, 0) = \lim_{h \rightarrow 0} \frac{f(h, 0) - f(0, 0)}{h} = \lim_{h \rightarrow 0} \frac{h - 0}{h} = 1,$$

$$f_2(0, 0) = \lim_{h \rightarrow 0} \frac{f(0, h) - f(0, 0)}{h} = \lim_{h \rightarrow 0} \frac{-\frac{h}{2} - 0}{h} = -\frac{1}{2}.$$

2. Let

$$f(x, y) = \begin{cases} \frac{xy^3}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

- (a) Does f have a limit at $(0, 0)$? Justify your answer.

Since $|x| \leq \sqrt{x^2 + y^2}$ and $|y| \leq \sqrt{x^2 + y^2}$, we have for all $(x, y) \neq (0, 0)$

$$\left| \frac{xy^3}{x^2 + y^2} \right| \leq \frac{(x^2 + y^2)^2}{x^2 + y^2} = x^2 + y^2.$$

Since $x^2 + y^2 \rightarrow 0$ as $(x, y) \rightarrow (0, 0)$, the function has limit 0 at that point.

- (b) Find $\frac{\partial f}{\partial x}(0, 0)$ and $\frac{\partial f}{\partial y}(0, 0)$, or else explain why they do not exist.

We have $f(x, 0) = 0$ for all x and $f(0, y) = 0$ for all y , so that $f_x(0, 0) = 0$ and $f_y(0, 0) = 0$.

Practice Problems

1. Let

$$f(x, y) = \begin{cases} \frac{\sin^3 x}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

Find the first order partial derivatives of f at $(0, 0)$ using the definition of partial derivatives.

2. Let

$$f(x, y) = \begin{cases} \frac{x^3 + xy - y^3}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

Find $\frac{\partial f}{\partial x}(0, 0)$ and $\frac{\partial f}{\partial y}(0, 0)$.

2.4 Tangent Planes

Recall, for a function $f(x, y)$ of two variables, the graph of f is the set $\{(x, y, f(x, y)) : (x, y) \in \mathcal{D}(f)\}$. This graph is a subset of $\mathbb{R}^3$. In this section, we will assume that the graph $z = f(x, y)$ is “nice enough” near some $(x, y) = (a, b)$, so that it has a tangent plane at the point $(a, b, f(a, b))$. This is an assumption that we will examine later. For now, we state the following without proof.

Theorem 2.4.1. *Assume that there is a neighbourhood $B_r(P)$ of a point $P = (a, b)$ such that $f(x, y)$ and both of its partials $f_1(x, y)$, $f_2(x, y)$ are continuous on $B_r(P)$. Then the graph $z = f(x, y)$ has a tangent plane at $(a, b, f(a, b))$.*

This covers many common functions (polynomials, exponentials, trig functions...) whose partial derivatives are continuous on their domain of definition, but excludes “pathological” examples such as Example 2.3.4 in Section 2.3.

2.4.1 Equation for the Tangent Plane

To determine an equation for the tangent plane at $(a, b, f(a, b))$, we need:

- a point that lies in the plane,
- a vector $\mathbf{n}$ perpendicular to the plane.

The point $(a, b, f(a, b))$ lies in the tangent plane. Now we need to find the normal vector. Our strategy for this is to find two vectors $\mathbf{T}_1$, $\mathbf{T}_2$ that are parallel to the tangent plane, then take their cross product to find $\mathbf{n}$.

We can find $\mathbf{T}_1$ and $\mathbf{T}_2$ as follows. If the tangent plane exists, it has to contain two tangent lines to the curves $(x, b, f(x, b))$ and $(a, y, f(a, y))$ where (a, b) is the (fixed) given point and x and y are variable.

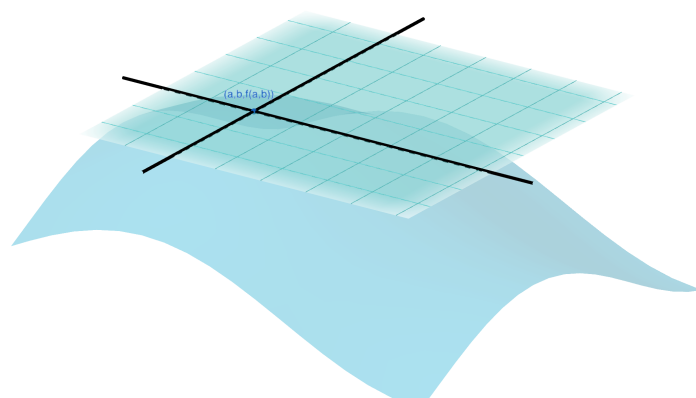


Figure 2.12: The graph of a function $z = f(x, y)$, its tangent plane, and the tangent lines described here.

Suppose we take a cross-section in the plane $y = b$. Then we obtain the following.

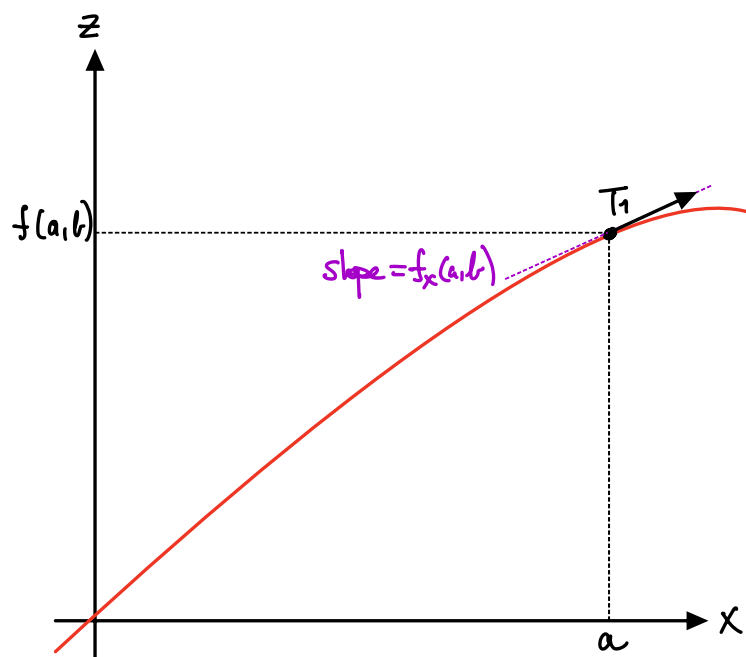


Figure 2.13: An intersection of the previous surface with the $y = b$ plane.

In the plane $y = b$, the tangent line is parallel to the $y = 0$ plane, and has slope $f_x(a, b)$ in the $y = b$ plane. In $\mathbb{R}^3$, the direction vector of this tangent line is given by the formula

$$\mathbf{T}_1 = \mathbf{i} + f_x(a, b)\mathbf{k}.$$

Notice that $\mathbf{T}_1$ has no $\mathbf{j}$ component because $\mathbf{T}_1$ is parallel to the xz -plane. Similarly,

the tangent line in the plane $x = a$ has the direction vector

$$\mathbf{T}_2 = \mathbf{j} + f_y(a, b)\mathbf{k}.$$

We now find the normal vector. The vectors $\mathbf{T}_1$ and $\mathbf{T}_2$ are both parallel to the tangent plane at $(a, b, f(a, b))$. Hence, the vector

$$\mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & f_x(a, b) \\ 0 & 1 & f_y(a, b) \end{vmatrix} = -f_x(a, b)\mathbf{i} - f_y(a, b)\mathbf{j} + \mathbf{k}$$

is perpendicular to the tangent plane. The equation for the tangent plane then follows from the calculation,

$$\begin{aligned} \mathbf{n} \cdot \langle x - a, y - b, z - f(a, b) \rangle &= 0 \\ \Leftrightarrow -f_x(a, b)(x - a) - f_y(a, b)(y - b) + (z - f(a, b)) &= 0 \end{aligned}$$

A simplification leads to the following formula.

Proposition 2.4.2. *Suppose that $z = f(x, y)$ admits a tangent plane at the point $(a, b, f(a, b))$ in $\mathbb{R}^3$. Then, the equation of this tangent plane is given by*

$$z = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b) \quad (2.4.1)$$

Beware! This equation is only valid **if** the tangent plane exists at the given point $(x, y, z) = (a, b, f(a, b))$ on our surface $z = f(x, y)$.

The vector $\mathbf{n} = \mathbf{T}_1 \times \mathbf{T}_2$ is the **normal vector** to the surface at $(a, b, f(a, b))$, and the line passing through $(a, b, f(a, b))$ and parallel to $\mathbf{n}$ is sometimes called the **normal line** to the surface at that point.

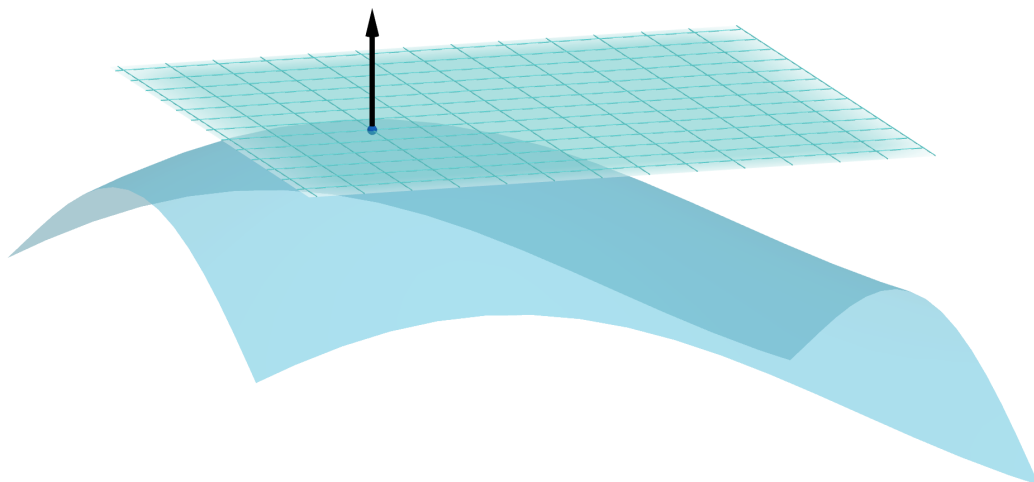


Figure 2.14: The vector $\mathbf{n}$ at the point $(a, b, f(a, b))$ on the tangent plane.

2.4.2 Tangent Planes May Not Exist

It is entirely possible that the tangent plane to $z = f(x, y)$ at a point $(a, b, f(a, b))$ might not exist. This may happen *even in situations where we could apply the tangent plane equation* (2.4.1). Below is an example of a surface $z = f(x, y)$ where $f_x(0, 0)$ and $f_y(0, 0)$ both exist and yet there is no good approximating tangent plane for the surface at $(0, 0, f(0, 0))$.

Example 2.4.3. Consider the graph of $f(x, y)$ where f is given by

$$f(x, y) = \begin{cases} \frac{xy}{x^2+y^2}, & (x, y) \neq 0 \\ 0, & (x, y) = 0 \end{cases}$$

We previously calculated that $f_x(0, 0) = f_y(0, 0) = 0$ (and, clearly, $f(0, 0) = 0$). Hence, were a tangent plane to exist at $(a, b, f(a, b)) = (0, 0, 0)$, it would necessarily have equation

$$z = 0 + 0(x - 0) + 0(y - 0) = 0.$$

But there is a problem! Notice that, for $t \neq 0$, $f(t, t) = 1/2$. Hence, the graph of $f(x, y)$ contains the punctured line $L_0 = \{(t, t, 1/2) : t \in \mathbb{R} \setminus \{0\}\}$. This line is parallel to the “false” tangent plane $\{(x, y, z) : z = 0\}$ and lies at a distance $1/2$ from it. In particular, the false tangent plane $\{z = 0\}$ fails to give a good approximation to the graph $\{(x, y, f(x, y)) : (x, y, z) \in \mathbb{R}^3\}$ along the punctured line L_0 .

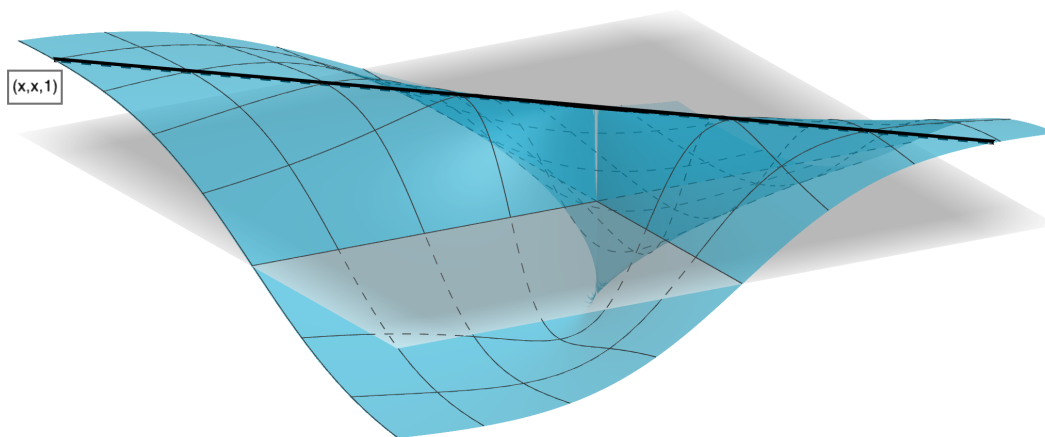


Figure 2.15: The surface $z = f(x, y)$ overlaid with the line $\{(t, t, 1/2) : t \in \mathbb{R}\}$ and the plane $\{(x, y, z) : z = 0\}$.

Worked Problems

In the questions below, we will take it for granted that the tangent planes at the given points exist. This is provided to us by Theorem 2.4.1.

1. Find the equation of the tangent plane to the graph $f(x, y) = xy^2$ at $(x, y) = (2, 1)$.

We have $f(2, 1) = 2 \cdot 1 = 2$, and

$$f_x(x, y) = y^2 \implies f_x(2, 1) = 1$$

$$f_y(x, y) = 2xy \implies f_y(2, 1) = 4$$

The tangent plane has normal vector $\mathbf{n} = \langle -1, -4, 1 \rangle$ and equation

$$z = 2 + 1(x - 2) + 4(y - 1) = x + 4y - 4.$$

2. Find the vector parametric equation of the normal line to the surface $z = xy^2$ at $(x, y) = (2, 1)$.

We already have,

$$\mathbf{n} = \langle -f_x(2, 1), -f_y(2, 1), 1 \rangle = \langle -1, -4, 1 \rangle.$$

Moreover, the line necessarily passes through the point $(2, 1, f(2, 1)) = (2, 1, 2)$. The equation of this line is therefore,

$$\mathbf{r}(t) = \langle 2, 1, 2 \rangle + t\langle -1, -4, 1 \rangle$$

3. Find all points on the surface $z = x^4 + 2y^2$ where the plane tangent to the surface is parallel to the plane $x + y + z = 0$.

We have $\frac{\partial z}{\partial x} = 4x^3$ and $\frac{\partial z}{\partial y} = 4y$. The normal vector to the surface at an arbitrary point (x, y, z) is given by,

$$\mathbf{n}_1 = -4x^3\mathbf{i} - 4y\mathbf{j} + \mathbf{k}$$

We want this vector to be parallel to the vector normal to the plane $x + y + z = 0$, which is the vector $\mathbf{n}_2 = \mathbf{i} + \mathbf{j} + \mathbf{k}$. Notice that the coefficient of $\mathbf{k}$ in $\mathbf{n}_1$ and $\mathbf{n}_2$ is the same (equal to 1). So, if we want $\mathbf{n}_1$ and $\mathbf{n}_2$ to be parallel, we must have $\mathbf{n}_1 = \mathbf{n}_2$. In particular, by equating $\mathbf{i}$ and $\mathbf{j}$ components

$$x^3 = -\frac{1}{4}, \quad -4y = 1 \implies x = -\sqrt[3]{\frac{1}{4}}, \quad y = -\frac{1}{4}.$$

Let us determine $z = f(x, y) = x^4 + 2y^2$, which is,

$$z = \left(-\sqrt[3]{\frac{1}{4}}\right)^4 + 2\left(-\frac{1}{4}\right)^2 = \left(\frac{1}{4}\right)^{4/3} + \frac{1}{8}.$$

So, the only point $(x, y, z) = (x, y, f(x, y))$ on the surface with tangent plane parallel to the plane $x + y + z = 0$ is,

$$(x, y, z) = \left(-\sqrt[3]{\frac{1}{4}}, -\frac{1}{4}, \left(\frac{1}{4}\right)^{4/3} + \frac{1}{8} \right)$$

4. Find all points on the paraboloid $z = x^2 + 4y^2$ where the tangent plane is parallel to the plane passing through the points $(0, 0, 0)$, $(6, 1, 4)$, $(1, -1, 2)$. (For each such point, provide its (x, y, z) coordinates.)

Two planes are parallel if and only if their normal vectors are parallel. Therefore, we need to find the normal vector $\mathbf{n}$ to the plane containing the three given points, and then we need to find the points on the paraboloid where the normal vector $\mathbf{N}$ to the paraboloid is a multiple of $\mathbf{n}$.

Let $O = (0, 0, 0)$, $P = (6, 1, 4)$ and $Q = (1, -1, 2)$. We can get a normal vector to the plane by taking the cross product of $\vec{OP} = \langle 6, 1, 4 \rangle$ and $\vec{OQ} = \langle 1, -1, 2 \rangle$:

$$\mathbf{n} = \vec{OP} \times \vec{OQ} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 6 & 1 & 4 \\ 1 & -1 & 2 \end{vmatrix} = \langle 6, -8, -7 \rangle.$$

Next, we find a vector normal to the paraboloid. Let $f(x, y) = x^2 + 4y^2$, so that the graph of f is the paraboloid in question. Given $(a, b) \in \mathbb{R}^2$, a normal vector to the tangent plane is given by $\mathbf{N}(a, b) = \langle -f_x(a, b), -f_y(a, b), 1 \rangle = \langle -2a, -8b, 1 \rangle$.

Note that $\mathbf{N}(a, b)$ is parallel to $\mathbf{n}$ exactly when $\mathbf{N}(a, b) = \lambda \mathbf{n}$ for some $\lambda \in \mathbb{R}$. In terms of components, this means that

$$\begin{cases} -2a &= 6\lambda \\ -8b &= -8\lambda \\ 1 &= -7\lambda \end{cases}$$

From the last equation we get $\lambda = -\frac{1}{7}$, and then the other two equations imply that $a = \frac{3}{7}$ and $b = -\frac{1}{7}$. Hence the only point on the paraboloid where the tangent plane is parallel to the plane through O, P, Q is $(\frac{3}{7}, -\frac{1}{7}, \frac{13}{49})$, where we found the z -coordinate from the equation of the paraboloid.

Practice Problems

- Find the equation of the tangent plane to the graph of the function $f(x, y) = ye^{xy^2}$ at the point $(0, 2)$.

2. Find the equation of the tangent plane to the surface $z = 5x^2y - y^2$ at the point where $x = -1$, $y = 3$.
3. Find all points $(x, y, f(x, y))$ where the tangent plane to the graph of the function $f(x, y) = x^2 + 3xy - 2y^2$ is parallel to the plane $x + 10y - z = 3$. For each such point, find the equation of the tangent plane.
4. Find all points $P = (1, b, c)$ such that P lies on the paraboloid $z = x^2 + y^2$ and the tangent plane to this paraboloid at P passes through the point $(2, 4, 18)$.
5. Find all values of (a, b) such that the tangent plane to the surface $z = 4x^2 - y^2$ at $(a, b, 4a^2 - b^2)$ is parallel to the line with parametric equations $x = t + 1$, $y = 2t$, $z = -4t + 9$.

2.5 Higher Order Derivatives

If a function $f(x_1, \dots, x_n)$ has a partial derivative $\frac{\partial f}{\partial x_j}(x_1, \dots, x_n)$, we can then try to find the partial derivatives of $\frac{\partial f}{\partial x_j}$, such as $\frac{\partial}{\partial x_i}(\frac{\partial f}{\partial x_j})$ for any $1 \leq i \leq n$. As a simple example:

$$f(x, y) = x^2y^3 \Rightarrow \frac{\partial f}{\partial x}(x, y) = 2xy^3 \Rightarrow \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial x} \right) (x, y) = \frac{\partial}{\partial y}(2xy^3) = 6xy^2.$$

Notation: there are several ways to denote higher order partials. For example, if $z = f(x, y)$, we can write

$$\frac{\partial}{\partial x} \left(\frac{\partial z}{\partial y} \right) = \frac{\partial^2 z}{\partial x \partial y} = f_{21}(x, y) = f_{yx}(x, y).$$

Notice how the order of differentiation is indicated in each type of notation. For example,

$$\underbrace{\frac{\partial^3 f}{\partial z \partial x \partial y} = \frac{\partial}{\partial z} \left[\frac{\partial}{\partial x} \left(\frac{\partial f}{\partial y} \right) \right]}_{\text{differentiate first in } y \text{ then } x \text{ then } z} \qquad \underbrace{f_{31} = (f_3)_1}_{\text{differentiate in } x_3 \text{ then } x_1}$$

The symbol ∂^n for $n = 1, 2, 3, \dots$ appearing in the numerator indicates the **order** of the derivative—the total number of times we differentiate. For iterated derivatives with respect to the same variable, we will use

$$f_{11} = f_{xx} = \frac{\partial^2 f}{\partial x^2}, \quad f_{yyyx} = \frac{\partial^4 f}{\partial x \partial y^3}, \quad \text{and so on.}$$

These are the conventions that we will use, but other textbooks and resources might reverse the order of subscripts, for example $f_{31} = (f_1)_3$ instead of $f_{31} = (f_3)_1$. In many cases, the order of differentiation does not matter—we discuss this in the following subsection. In those cases where it does matter, it is better to use notation such as $(f_1)_3$ or $\frac{\partial}{\partial x}(\frac{\partial z}{\partial y})$ that indicates the order of differentiation explicitly.

The following example illustrates a scenario where $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$.

Example 2.5.1. Find all second order partial derivatives of $f(x, y) = x^2 + x \sin y$.

We have

$$\frac{\partial f}{\partial x} = 2x + \sin y \qquad \frac{\partial f}{\partial y} = x \cos y$$

so that

$$\begin{aligned} \frac{\partial^2 f}{\partial x^2} &= \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial x} \right) = 2 & \frac{\partial^2 f}{\partial y \partial x} &= \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial x} \right) = \cos y \\ \frac{\partial^2 f}{\partial x \partial y} &= \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial y} \right) = \cos y & \frac{\partial^2 f}{\partial y^2} &= \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial y} \right) = -x \sin y \end{aligned}$$

2.5.1 Equality of Mixed Partial

The previous example illustrates the following more general rule.

Theorem 2.5.2 (Equality of Mixed Partial). Suppose two mixed n -th order partial derivatives of $f(x_1, \dots, x_n)$ involve the same differentiations but in different order. If:

- these partials are continuous at a point P in $\mathbb{R}^n$; and,
- the function f and all partials of f of order less than n are continuous on a neighbourhood of P ,

then those partial derivatives are equal at P .

Here are a few comments regarding the assumptions of the theorem.

1. Typically, we will apply the theorem to functions which are linear combinations or compositions of rational functions, trigonometric functions, exponentials, logarithms, etc. All partial derivatives of such functions are also combinations or compositions of similar functions; therefore, they are continuous at all points where they are defined. In such cases the assumptions of the theorem are satisfied for all

P such that the n -th order partials of f are defined on some neighbourhood of P . For example, if

$$f(x, y, z) = e^{x/y^2} \sin \left(\frac{x^3 + 5y^2 + z^2}{x - z} \right),$$

we know from the theorem (even without computing any derivatives!) that

$$f_{xyz} = f_{zyx} = f_{yzx} = \cdots$$

at all points (x, y, z) where $y \neq 0$ and $x \neq z$.

2. It is possible to set up a function (using multiple formulas) where the assumptions of the theorem fail and the mixed partials are *not* equal. For example, let

$$f(x, y) = \begin{cases} \frac{xy(x^2 - y^2)}{x^2 + y^2}, & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

Then, a calculation shows $f_{12}(0, 0) \neq f_{21}(0, 0)$. See the practice problems for more details.

2.5.2 Idea of the Proof

Let us consider the special case of functions $f(x, y)$ of two variables and try to show that $f_{xy} = f_{yx}$ under the assumptions of the theorem. Let

$$Q(x, y, \Delta x, \Delta y) = f(x + \Delta x, y + \Delta y) - f(x, y + \Delta y) - f(x + \Delta x, y) + f(x, y).$$

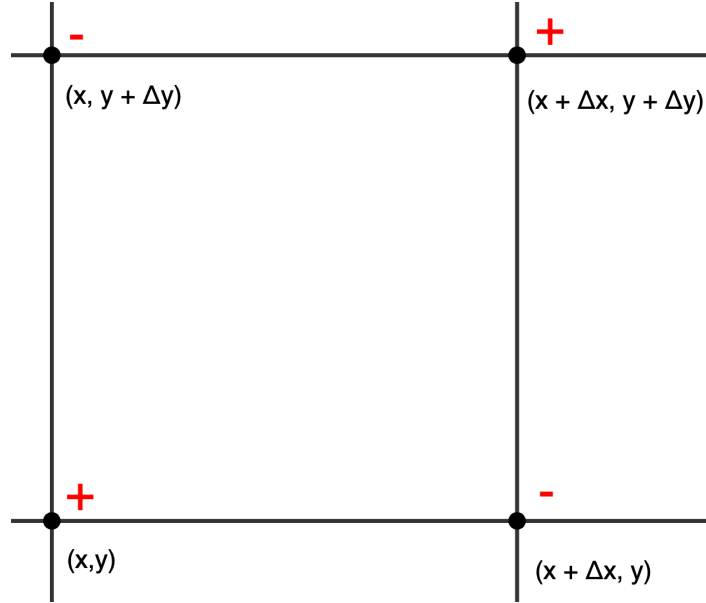


Figure 2.16: The function Q adds up the values of f at the four vertices of a rectangle, with alternating $\pm$ signs as indicated.

Assume (for the moment) that, in addition to the other assumptions of the theorem, the following limit in two variables exists:

$$\lim_{(\Delta x, \Delta y) \rightarrow (0,0)} \frac{Q(x, y, \Delta x, \Delta y)}{\Delta x \Delta y} = L, \text{ for some real number } L. \quad (2.5.1)$$

We claim that if (2.5.1) holds, then both $f_{xy}(x, y)$ and $f_{yx}(x, y)$ are equal to L . To prove this, let us show that

$$f_{yx}(x, y) = L; \quad (2.5.2)$$

the calculation for f_{xy} is similar.

First, recall that

$$f_y(x, y) = \lim_{\Delta y \rightarrow 0} \frac{1}{\Delta y} [f(x, y + \Delta y) - f(x, y)]$$

so that

$$f_y(x + \Delta x, y) = \lim_{\Delta y \rightarrow 0} \frac{1}{\Delta y} [f(x + \Delta x, y + \Delta y) - f(x + \Delta x, y)].$$

Then,

$$\begin{aligned} f_{yx}(x, y) &= \lim_{\Delta x \rightarrow 0} \frac{1}{\Delta x} [f_y(x + \Delta x, y) - f_y(x, y)] \\ &= \lim_{\Delta x \rightarrow 0} \frac{1}{\Delta x} \left[\lim_{\Delta y \rightarrow 0} \frac{1}{\Delta y} [f(x + \Delta x, y + \Delta y) - f(x + \Delta x, y)] \right. \\ &\quad \left. - \lim_{\Delta y \rightarrow 0} \frac{1}{\Delta y} [f(x, y + \Delta y) - f(x, y)] \right] \\ &= \lim_{\Delta x \rightarrow 0} \frac{1}{\Delta x} \left[\lim_{\Delta y \rightarrow 0} \frac{1}{\Delta y} Q(x, y, \Delta x, \Delta y) \right] \end{aligned}$$

Now, if we assume equation (2.5.1) so that the two-variable limit exists, then the last equality implies (2.5.2).

In reality, we do not yet know whether the limit (2.5.1) exists. This is why the above is just the idea of the proof and not the complete argument. However, this can be fixed by using the Mean Value Theorem instead. The assumption that the partials f_x, f_y are continuous on a neighbourhood of P is needed to justify the application of the Mean Value Theorem to these partials, and the assumption that f_{xy} and f_{yx} are continuous at P is needed when we take the limit at P . See your textbook for details.

2.5.3 Partial differential equations: a quick introduction

A *Partial Differential Equation* (PDE) is an identity involving the partial derivatives of some function. This is a large area of study (with separate courses dedicated to it). In this small subsection, we give two simple examples.

Example 2.5.3. Verify that $z = \frac{2x-y}{x+y}$ satisfies the PDE $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 0$.

A calculation (do it!) shows that,

$$\frac{\partial z}{\partial x} = \frac{3y}{(x+y)^2}, \quad \frac{\partial z}{\partial y} = -\frac{3x}{(x+y)^2}.$$

Thus

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = x\left(\frac{3y}{(x+y)^2}\right) + y\left(\frac{-3x}{(x+y)^2}\right) = 0.$$

(As an additional exercise, you can check that $z = \frac{ax+by}{x+y}$ satisfies the same equation for any fixed $a, b \in \mathbb{R}$.)

Example 2.5.4. If f and g are twice-differentiable functions of one variable, prove that $w(x, t) = f(x + ct) + g(x - ct)$ (where $c \neq 0$ is some constant) satisfies the **wave equation**

$$\frac{\partial^2 w}{\partial t^2} = c^2 \frac{\partial^2 w}{\partial x^2}.$$

A calculation (do it!) shows that

$$w_{xx} = f''(x + ct) + g''(x - ct), \quad w_{tt} = c^2 f''(x + ct) + c^2 g''(x - ct),$$

which shows that $w_{xx} = c^2 w_{tt}$.

Practice Problems

Warning: the calculations in Questions 2 and 3 below are relatively long.

1. Let $f(x, y) = x \sin(xy^2)$.

(a) Find all first and second order partial derivatives of f .

(b) Find the equation of the tangent plane to the surface $z = f(x, y)$ at the point where $x = \pi$, $y = 1$.

2. A function $f(x_1, \dots, x_n)$ of n -many variables is said to be **harmonic** if it satisfies the equation,

$$\frac{\partial^2 f}{\partial x_1^2} + \cdots + \frac{\partial^2 f}{\partial x_n^2} = 0.$$

Demonstrate that the following functions are harmonic:

- (a) $f(x, y) = \frac{x}{x^2+y^2}$ for $(x, y) \neq 0$;
- (b) $f(x, y, z) = \frac{x}{(x^2+y^2+z^2)^{3/2}}$ for $(x, y, z) \neq 0$ (a longer calculation);
- (c) $f(x_1, \dots, x_n) = \frac{1}{(x_1^2+\dots+x_n^2)^{(n/2)-1}}$ for $(x_1, \dots, x_n) \neq (0, \dots, 0)$ and $n \geq 3$.

3. Let

$$F(x, y) = \begin{cases} \frac{xy(x^2 - y^2)}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

Demonstrate carefully that both $F_{12}(0, 0)$ and $F_{21}(0, 0)$ exist and are not equal. Why does this not violate Theorem 2.5.2?

Hint: we advise that you proceed by obtaining the following in order:

- $F_1(0, y)$ for all $y \in \mathbb{R}$ and $F_2(x, 0)$ for all $x \in \mathbb{R}$,
- $F_{12}(0, 0)$ and $F_{21}(0, 0)$.

If done correctly, you will obtain that $F_{21}(0, 0) = 1$ and $F_{12}(0, 0) = -1$.

Remark 2.5.1. Problem 3 is a very common example of a function $F(x, y)$ for which $F_{11} \neq F_{22}$. You will likely see it everywhere if you look online. As far as we can say, the earliest attribution of this belongs to Italian mathematician Giuseppe Peano (b. 1858).

2.6 The Chain Rule

In this section, we focus on applying the Chain Rule, with proofs postponed until later. All formulas below are valid, so long as we assume all functions involved are C^1 according to the definition we now make.

Definition 2.6.1. A function $f(x_1, \dots, x_n)$ is C^k on an open set S if f and all its partial derivatives of order less than or equal to k are continuous functions on S . In particular, f is C^1 on S if f and its first-order derivatives $f_1, \dots, f_n$ are continuous on S .

We also say that f is C^∞ on S if f and all of its partial derivatives of any order are continuous on S .

For example, polynomials, rational functions, trigonometric functions, exponentials and all compositions of said functions are C^∞ on their domains of definition.

2.6.1 The Chain Rule for Functions of Two Variables

We start by recalling the Chain Rule for single variable functions—this is what we want to generalize.

Theorem 2.6.2. *Suppose that $h(x) = f(g(x))$ is the composition of two functions f and g of one real variable. Further suppose, for some x in $\mathbb{R}$, both $g'(x)$ and $f'(g(x))$ exist. Then $h'(x)$ exists and satisfies*

$$\frac{d}{dx}f(g(x)) = f'(g(x)) \cdot g'(x).$$

Written another way, if we let $g(x) = y$ and $f(g(x)) = z$, then

$$\frac{dz}{dx} = \frac{dz}{dy} \frac{dy}{dx}.$$

The situation changes when we have a function of many variables. For example, if $f(x, y)$ is a function of two variables where $x = x(t)$ and $y = y(t)$ are single variable functions depending on a third variable t .

Example 2.6.3. *Let (x, y) denote the longitude and latitude of a point on a map and let $f(x, y)$ denote the altitude at the point (x, y) . Suppose that a hiker is walking along a path so that their coordinates (longitude and latitude) at time t are $(x(t), y(t))$. Then the function $f(x, y) = f(x(t), y(t))$ represents the hiker's altitude at time t .*

The Chain Rule for functions $f(x(t), y(t))$ is as follows.

Theorem 2.6.4. *Assume that $x(t)$, $y(t)$, and $f(x(t), y(t))$ are all C^1 . Then*

$$\frac{d}{dt}f(x(t), y(t)) = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt}.$$

We use the notation $\frac{d}{dt}$ instead of partial notation $\frac{\partial}{\partial t}$ because $f(x(t), y(t))$ is a function of one variable.

We will see some examples of the Chain Rule applied to functions $f(x(t), y(t))$ in the Worked Examples section.

2.6.2 The Chain Rule for Multiple Variables

If we have more variables, the Chain Rule tracks the dependence of all “intermediate” variables on all “final” variables. For example (again, assuming all functions are C^1):

- If $f(x, y)$ is a function of two variables and $x = x(s, t)$ and $y = y(s, t)$ are also functions of two variables, then

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial s} \quad \text{and} \quad \frac{\partial f}{\partial t} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial t}.$$

(This is a situation we will encounter later when we consider change of variables for multivariate functions.)

- If $f(x, y, z)$ is a function of three variables, and x, y, z are all functions of two variables (s, t) , then

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial s} + \frac{\partial f}{\partial z} \frac{\partial z}{\partial s},$$

and the partial derivative with respect to s is similar.

An important special case is the following. Suppose $f(x, y, z, t)$ is a function of four real variables and we also have $x = x(t)$, $y = y(t)$ and $z = z(t)$ given as functions of t . Then, assuming all involved functions are C^1 , the Chain Rule says

$$\frac{d}{dt} f(x(t), y(t), z(t), t) = \frac{\partial f}{\partial x} x'(t) + \frac{\partial f}{\partial y} y'(t) + \frac{\partial f}{\partial z} z'(t) + \frac{\partial f}{\partial t}.$$

This is one of the cases where notation does matter. The $\frac{d}{dt}$ on the left indicates the total derivative of f with respect to t . It takes into account all the ways in which f depends on t , including through the intermediate dependent variables x, y, z . The derivative $\frac{\partial f}{\partial t}$ is the same as f_4 , indicating differentiation of $f(x, y, z, t)$ in the fourth variable. You can think of it as being multiplied by a factor of $\frac{dt}{dt} = 1$.

Example 2.6.5. Let $f(x, y, z, t) = x + 2y + 3z + t^2$ and

$$x(t) = \sin t, \quad y(t) = \cos t, \quad z(t) = e^t.$$

Then $\frac{\partial f}{\partial t} = f_4 = 2t$, but

$$\frac{d}{dt} f(x(t), y(t), z(t), t) = \frac{d}{dt} (\sin t + 2 \cos t + 3e^t + t^2) = \cos t - 2 \sin t + 3e^t + 2t.$$

Again, as in the case of functions $f(x(t), y(t))$, functions $f(x, y, z, t)$ can represent real-world phenomena.

Example 2.6.6. If (x, y, z) are the spatial coordinates in a room and t represents the time variable, we can define a function $f(x, y, z, t)$ as the temperature at the point (x, y, z) in the room at time t . Thinking of a drone flying around the room, with coordinates $(x(t), y(t), z(t))$ at time t , the function $f(x(t), y(t), z(t), t)$ represents the temperature measured by the drone at time t .

We can also consider the higher order partials. For this, we need to assume that our functions are C^k for some $k > 1$, and iterate the Chain Rule as needed.

Worked Problems

1. Let $f(x, y) = \sin(3x + y)$ with $x(t) = 2t + 2$ and $y(t) = t^2$. Use the Chain Rule to find $\frac{d}{dt}f(x(t), y(t))$.

We have

$$\frac{\partial f}{\partial x} = 3 \cos(3x + y) \quad , \quad \frac{\partial f}{\partial y} = \cos(3x + y)$$

$$\frac{dx}{dt} = 2 \quad , \quad \frac{dy}{dt} = 2t$$

Thus,

$$\begin{aligned} \frac{d}{dt}f(x(t), y(t)) &= \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt} \\ &= 3 \cos(3x(t) + y(t)) \cdot 2 + \cos(3x(t) + y(t)) \cdot 2t \\ &= (6 + 2t) \cos(t^2 + 6t + 6). \end{aligned}$$

In this example, we could just have plugged in

$$f(x(t), y(t)) = \sin(3(2t + 2) + t^2) = \sin(t^2 + 6t + 6),$$

and differentiated in t . We will now see an example where this would not work.

2. At time $t = 0$, a hiker is at a point (x_0, y_0) , moving along a trajectory $(x(t), y(t))$ with $x'(0) = 2$ and $y'(0) = -1$. The altitude at (x, y) is given by a function $f(x, y)$ with $\frac{\partial f}{\partial x}(x_0, y_0) = 0.02$ and $\frac{\partial f}{\partial y}(x_0, y_0) = 0.05$. How quickly does the hiker's altitude change?

We solve this using the Chain Rule. That is,

$$\begin{aligned} \frac{d}{dt}f(x(t), y(t))|_{t=0} &= \frac{\partial f}{\partial x}(x_0, y_0) x'(0) + \frac{\partial f}{\partial y}(x_0, y_0) y'(0) \\ &= 0.02 \cdot 2 + 0.05 \cdot (-1) \\ &= -0.01. \end{aligned}$$

3. Let $f(x, y)$ be a C^1 function such that $f_1(0, 0) = 7$ and $f_2(0, 0) = 3$. Find $\frac{\partial f}{\partial t}$ at $(s, t) = (0, 0)$ if $x = 4s - t$ and $y = 2s + 5t$.

When $(s, t) = (0, 0)$, one readily checks that $(x(s, t), y(s, t)) = (0, 0)$, so that

$$\begin{aligned} \frac{\partial f}{\partial t}|_{(s,t)=(0,0)} &= f_1(0, 0) \frac{\partial x}{\partial t} + f_2(0, 0) \frac{\partial y}{\partial t} = 7 \cdot (-1) + 3 \cdot 5 \\ &= 8 \end{aligned}$$

Practice Problems

1. Let $z = f(x, y)$, where $f(x, y)$ is C^1 for all x, y . Suppose that $f_x(\sqrt{3}, 1) = 3$ and $f_y(\sqrt{3}, 1) = 5$. If we switch to polar coordinates r, θ so that $x = r \cos \theta$, $y = r \sin \theta$, what is $\frac{\partial z}{\partial \theta}$ at $(r, \theta) = (2, \pi/6)$?

(Hint: $\sin(\pi/6) = 1/2$, $\cos(\pi/6) = \sqrt{3}/2$.)

2. Suppose that the temperature at a point with coordinates (x, y, z) and at time t is given by $T(x, y, z, t) = 20 + 3x - 2y + 4z^2 + xye^{0.03t}$. At time $t = 3$, a drone is at the point $(5, 2, 1)$ and flies along a trajectory such that $x'(3) = 7$, $y'(3) = 2$, $z'(3) = -1$. What is the rate of change of the temperature measured by the drone at $t = 3$?

3. Let $z = f(u, v)$, where $f(u, v)$ is C^1 for all u, v . Let $u = xy$, $v = y/x$.

(a) Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ in terms of f_1 and f_2 .

(b) If $f_1(-6, -\frac{3}{2}) = 4$ and $f_2(-6, -\frac{3}{2}) = -1$, find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ at $(x, y) = (-2, 3)$.

4. Suppose that a function $f(x, y)$ is defined everywhere in the plane and has continuous partial derivatives of all orders. Assume that $f(2, 3) = 2$, $f_1(2, 3) = -1$, $f_2(2, 3) = 4$, $f_{11}(2, 3) = 0$, $f_{12}(2, 3) = f_{21}(2, 3) = 3$, $f_{22}(2, 3) = 1$. Find $\frac{\partial}{\partial x} f(2x^2, x + y)$ and $\frac{\partial^2}{\partial x^2} f(2x^2, x + y)$ at $(x, y) = (1, 2)$. (Notice that f_1 does not denote the same derivative as $\partial f / \partial x$ in this problem! Same for f_2, f_{11} , and so on.)

Chapter 3

Differentiability

3.1 Linear Approximation and Differentiability

Recall that for a single-variable function $f(x)$ differentiable at $x = a$, we have the linear approximation

$$f(x) \approx L(x) := f(a) + f'(a)(x - a).$$

This is a good approximation in the sense that

$$|f(x) - L(x)| = o(|x - a|),$$

where the “little o ” notation means

$$\lim_{x \rightarrow a} \frac{|f(x) - L(x)|}{|x - a|} = 0.$$

We now develop a similar notion of linearization for multi-variable functions.

3.1.1 Linearization for Functions of Two Variables

For a function $f(x, y)$ whose partial derivatives f_x and f_y exist at a point (a, b) , we want to approximate the values of $f(x, y)$ near $(x, y) = (a, b)$ by the linear function

$$L(x, y) := f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b). \quad (3.1.1)$$

This is the function from the tangent plane equation: we said earlier that if the graph $z = f(x, y)$ has a non-vertical tangent plane at the point $(a, b, f(a, b))$, then the equation of this tangent plane is given by $z = L(x, y)$. We did not say previously what exactly it means that this plane is “tangent” to the graph, or what conditions we need to assume about $f(x, y)$. This is what we will do now.

- We define the linearization of $f(x, y)$ at (a, b) to be the function $L(x, y)$ as in (3.1.1). For the formula to make sense, we need both partials f_1 and f_2 to exist at (a, b) , but this does not yet mean that $L(x, y)$ is a good approximation to anything.
- In Definition 3.1.1, we state precisely what it means for $L(x, y)$ to be a good approximation to $f(x, y)$ near (a, b) . The term we will use for this is differentiability. This is a stronger condition than just the existence of $L(x, y)$.
- We will then look at conditions that guarantee that f is differentiable at a point. An easy-to-use criterion is that if f is C^1 on an open set S , it is also differentiable at every point of S (Theorem 3.1.2). For functions that are not C^1 , we may need to check the differentiability condition directly, and we will see examples of that in the next section.
- We will also see a few examples of using the linear approximation to find approximate values of functions.

Definition 3.1.1. We say that $f(x, y)$ is **differentiable** at (a, b) if the approximation $L(x, y)$ defined in (3.1.1) is good, in the sense that

$$|f(x, y) - L(x, y)| = o(\sqrt{(x - a)^2 + (y - b)^2}) \quad (3.1.2)$$

This time, the little- o notation of (3.1.2) means that

$$\lim_{(x, y) \rightarrow (a, b)} \frac{|f(x, y) - L(x, y)|}{\sqrt{(x - a)^2 + (y - b)^2}} = 0,$$

or equivalently

$$\lim_{(\Delta x, \Delta y) \rightarrow (0, 0)} \frac{|f(a + \Delta x, b + \Delta y) - L(a + \Delta x, b + \Delta y)|}{\sqrt{(\Delta x)^2 + (\Delta y)^2}} = 0$$

where $\Delta x = x - a$ and $\Delta y = y - b$.

We used the absolute value in the expression $|f(x, y) - L(x, y)|$ to emphasize that this limit gives a bound on the size of the difference. However, the condition

$$\lim_{(x, y) \rightarrow (a, b)} \frac{f(x, y) - L(x, y)}{\sqrt{(x - a)^2 + (y - b)^2}} = 0$$

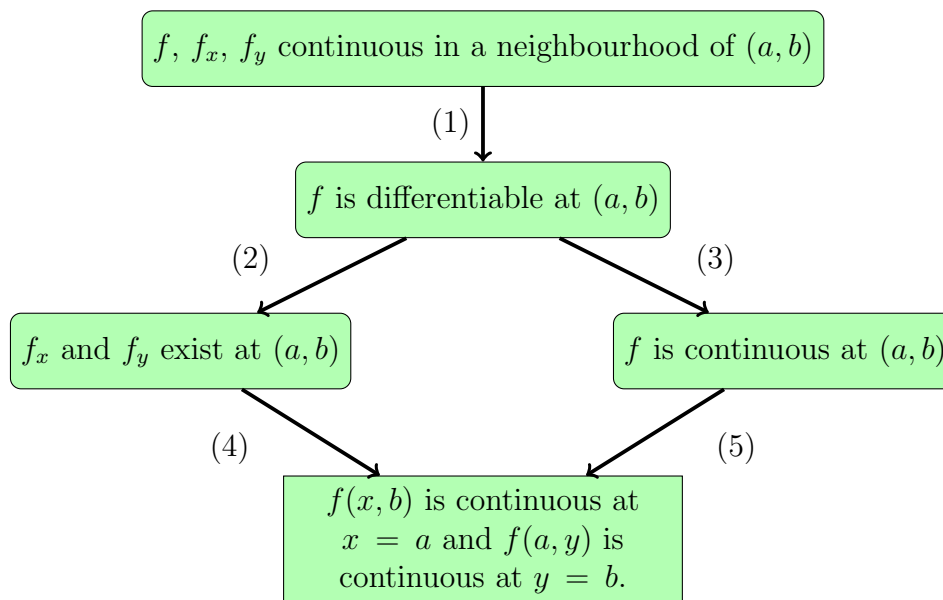
without absolute values, is equivalent since

$$\lim_{(x, y) \rightarrow (a, b)} F(x, y) = 0 \quad \Leftrightarrow \quad \lim_{(x, y) \rightarrow (a, b)} |F(x, y)| = 0.$$

How does our definition of differentiability for $f(x, y)$ compare with our C^1 condition (that is, that f and all its first-order partial derivatives are continuous on some open set)? This question is answered in the following theorem, stated here without proof.

Theorem 3.1.2. *If $f(x, y)$ is C^1 on a neighbourhood of (a, b) , it is differentiable at (a, b) .*

The next diagram illustrates the relationships between the various properties of f related to continuity and differentiability. The implications cannot be reversed; for example, $f(x, y)$ can be differentiable at (a, b) *without being* C^1 !



We list the justifications for the implications in the previous diagram.

- (1) This implication is Theorem 3.1.2.
- (2) We need the functions f_x and f_y to exist in order to define $L(x, y)$ in the first place. Also, if f is well-approximated—in the sense of (3.1.2)—by a linear function $L(x, y)$, then f_x and f_y have to exist and $L(x, y)$ has to be given by (3.1.1). We discussed this argument in the section on Tangent Planes. We encourage you to revisit this argument now: if we fix $y = b$ and consider the values of x near a , what does the condition (3.1.2) say about $f(x, b)$?
- (3) This is discussed below in Theorem 3.1.3.
- (4) This follows from single variable calculus applied to the (single variable) functions $f(x, b)$ and $f(a, y)$.
- (5) If $\lim_{(x, y) \rightarrow (a, b)} f(x, y) = f(a, b)$ then in particular $\lim_{x \rightarrow a} f(x, b) = f(a, b)$. The similar statement for b and y is true.

Theorem 3.1.3. *If $f(x, y)$ is differentiable at (a, b) , then it is continuous at (a, b) .*

Proof. We want to show that

$$\lim_{(x,y) \rightarrow (a,b)} f(x,y) = f(a,b),$$

under the assumption that f is differentiable at (a,b) . Since the linearization $L(x,y)$ is a “good approximation” of $f(x,y)$ at (a,b) , in the sense that

$$f(x,y) = L(x,y) + o(\sqrt{(x-a)^2 + (y-b)^2})$$

and $L(x,y)$ is continuous at (a,b) , we expect that the same should be true for $f(x,y)$. However, we should write a rigorous proof of this fact! We will do that now.

Let us define $R(x,y) = f(x,y) - L(x,y)$, the difference between f and its linearization. Then, since f is differentiable at (a,b) , we have

$$\lim_{(x,y) \rightarrow (a,b)} \frac{R(x,y)}{\sqrt{(x-a)^2 + (y-b)^2}} = 0.$$

So, let $\epsilon = 1$ (say). Then, there exists some $\delta = \delta(\epsilon) > 0$ such that for all (x,y) with $0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta$ we have

$$\frac{|R(x,y)|}{\sqrt{(x-a)^2 + (y-b)^2}} < 1, \text{ so that } 0 \leq |R(x,y)| < \sqrt{(x-a)^2 + (y-b)^2},$$

for the same values of (x,y) . Since $\lim_{(x,y) \rightarrow (a,b)} \sqrt{(x-a)^2 + (y-b)^2} = 0$, the above inequality (together with the Squeeze Theorem) implies

$$\lim_{(x,y) \rightarrow (a,b)} R(x,y) = 0.$$

But, using the definition of $R(x,y)$, this gives

$$\begin{aligned} \lim_{(x,y) \rightarrow (a,b)} f(x,y) &= \lim_{(x,y) \rightarrow (a,b)} L(x,y) + \lim_{(x,y) \rightarrow (a,b)} R(x,y) \\ &= f(a,b) + 0 \\ &= f(a,b). \end{aligned}$$

□

Example 3.1.4. *Let*

$$f(x,y) = \begin{cases} x^2 + y^2 & \text{if at least one of } x, y \text{ is rational,} \\ 0 & \text{otherwise.} \end{cases}$$

Then $f(x,y)$ is differentiable at $(0,0)$ and its linear approximation at that point is $L(x,y) = 0$. However, f is definitely not C^1 ! In fact, f is neither continuous nor differentiable at any point other than $(0,0)$. Think about how you might try to prove this. We will have a worked example of this type in the next section.

3.1.2 Differentials

Linear approximation can be interpreted in terms of differentials similar to those in single-variable calculus. Letting $z = f(x, y)$, we write

$$dz = df = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy,$$

so that, for fixed values of (x, y) , we interpret this as an expression for dz as a linear function of dx and dy . We will use this idea in applications of differentials to linear approximation and error estimates in the Worked Problems Section. We strongly encourage you to read through these examples.

Worked Problems

1. Find the linear approximation to $f(x, y) = xe^{3x-2y}$ at $(a, b) = (2, 3)$. Use it to evaluate the approximate value of $f(2.05, 3.01)$.

We have,

$$f_x(x, y) = e^{3x-2y} + 3xe^{3x-2y}, \quad f_y(x, y) = -2xe^{3x-2y}$$

so that,

$$f(2, 3) = 2, \quad f_x(2, 3) = 7, \quad f_y(2, 3) = -4,$$

which gives a linearization for $f(x, y)$ at $(2, 3)$ as

$$L(x, y) = 2 + 7(x - 2) - 4(y - 3).$$

Hence, our approximation gives

$$f(2.05, 3.01) \approx L(2.05, 3.01) = 2 + 7 \cdot 0.05 - 4 \cdot 0.01 = 2.31.$$

2. The radius and height of a cylindrical container are measured with accuracy 1%. What is the maximal percentage error in the calculated (based on the measurements) volume of the container?

We have $V = \pi r^2 h$. The associated differential is given by

$$dV = (2\pi hr)dr + (\pi r^2)dh.$$

For our measurement to have within 1% accuracy, this means that

$$|dr| \leq \frac{1}{100}r, \quad |dh| \leq \frac{1}{100}h.$$

Consequently,

$$|dV| \leq \left| 2\pi hr \cdot \frac{1}{100}r \right| + \left| \pi r^2 \cdot \frac{1}{100}h \right| = \frac{2\pi r^2 h}{100} + \frac{\pi r^2 h}{100} = \frac{3}{100}V.$$

Therefore, the maximal percentage error is 3%.

Practice Problems

1. Let $f(x, y) = ye^{xy^2}$.
 - (a) Find the equation of the tangent plane to the graph of the function $f(x, y)$ at the point $(x, y) = (0, 2)$.
 - (b) Use linear approximation at $(0, 2)$ to give an approximate value of $f(-0.2, 2.01)$.
 - (c) Find $f_{12}(x, y)$.
2. Let $z = \frac{\sin(x)}{\cos(y)} = \sin(x) \sec(y)$.
 - (a) Find the general expression for the differential of $z(x, y)$ for x and y in the interval $[0, \frac{\pi}{2})$.
 - (b) Now, utilizing the differential calculated in Part (a), approximate the value of the function at $(a, b) = (0.1, 0.3)$ using a nearby point.
 - (c) Rather than utilizing the the linearization formula from Lecture Notes 3.1, utilize first order Taylor approximation for $\sin$ and $\cos$ to give another estimation of the function at $(0.1, 0.3)$. Do your approximations agree?
3. Suppose that $f(x, y)$ is a differentiable function of x and y , and that $x = x(s, t)$ and $y = y(s, t)$ are themselves differentiable functions of s, t .
 - (a) Using the Multivariable Chain Rule (see *Lecture Notes 2.6*), find a general formula for the linearization of $f(x(s, t), y(s, t))$ for (s, t) near a point (a, b) in terms of the first-order partial derivatives of the functions f, x and y .

Definition 3.1.5. A function $g(x, y)$ of two-variables is called **Lipschitz** if there exists a number $C > 0$ such that, given any pair of points (x_0, y_0) and (x_1, y_1) in $\mathbb{R}^2$, one has

$$|g(x_1, y_1) - g(x_0, y_0)| \leq C\sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2}.$$

The smallest value of $C > 0$ for which the above inequality holds is known as the **Lipschitz constant** of g .

- (b) Suppose that $f(x, y)$, $x(s, t)$ and $y(s, t)$ are all functions whose first-order partial derivatives are bounded by some number $M > 0$. By **bounded**, we mean that:

$$\left| \frac{\partial f}{\partial x}(x, y) \right|, \left| \frac{\partial f}{\partial y}(x, y) \right| \leq M, \text{ for every pair } (x, y) \in \mathbb{R}^2,$$

$$\left| \frac{\partial x}{\partial s}(s, t) \right|, \left| \frac{\partial x}{\partial t}(s, t) \right|, \leq M, \text{ for every pair } (s, t) \in \mathbb{R}^2.$$

$$\left| \frac{\partial y}{\partial s}(s, t) \right|, \left| \frac{\partial y}{\partial t}(s, t) \right|, \leq M, \text{ for every pair } (s, t) \in \mathbb{R}^2.$$

Prove that for any $(a, b) \in \mathbb{R}^2$, the linearization of $f(x(s, t), y(s, t))$ for (s, t) near (a, b) is Lipschitz, with Lipschitz constant $C \leq 4M^2$.

Hint: this is tricky! You will need four tools to prove this. They are (in order of likely use): the formula you obtained from Part (a.), the triangle inequality (used twice), the bound on the first-order partial derivatives, and the inequality

$$|U| + |V| \leq 2 \max\{|U|, |V|\} \leq 2\sqrt{|U|^2 + |V|^2},$$

which is valid for any two real numbers U, V (prove this to yourself!)

Remark 3.1.1. We will discuss proofs involving differentiability in more detail in the following Section. We also postpone Practice Problems asking you to prove the differentiability of given functions $f(x, y)$ to the next part of the notes.

3.2 More on Differentiability

3.2.1 Nondifferentiability Happens

We start with the example

$$f(x, y) = \begin{cases} \frac{x^3}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0). \end{cases}$$

Let us try to determine whether f is differentiable at $(0, 0)$.

Some preliminary remarks:

1. First notice that the function $f(x, y)$ is continuous at the origin. We already looked at this example earlier (see Worked Problems in Section 2.2), but let us recap it briefly. We have

$$|f(x, y)| = \left| \frac{x^3}{x^2 + y^2} \right| = |x| \left| \frac{x^2}{x^2 + y^2} \right| \leq |x| \rightarrow 0, \text{ as } (x, y) \rightarrow (0, 0),$$

so that by the Squeeze Theorem,

$$\lim_{(x, y) \rightarrow (0, 0)} f(x, y) = 0 = f(0, 0).$$

However, this does not tell us whether f is differentiable at 0.

2. We could try to use the Theorem saying that if f_x and f_y are continuous in a neighbourhood of $(0, 0)$, then f is differentiable at $(0, 0)$. Yet:

- It is possible for f to be differentiable at $(0, 0)$ even if f_x or f_y are not continuous in a neighbourhood of $(0, 0)$, so this might not be a conclusive test.
- Checking that f_x and f_y are continuous in a neighbourhood of $(0, 0)$, in cases like this (functions defined by a piecewise formula), is more difficult than just checking the definition of differentiability at $(0, 0)$.

Returning to the example, we need to determine whether

$$\lim_{(x,y) \rightarrow (0,0)} \frac{f(x,y) - L(x,y)}{\sqrt{x^2 + y^2}} = 0,$$

where $L(x, y)$ is the linear approximation

$$L(x, y) = f(0, 0) + f_x(0, 0)x + f_y(0, 0)y.$$

Clearly, $f(0, 0) = 0$. Using the definition of partial derivatives, we obtain

$$f_x(0, 0) = \lim_{h \rightarrow 0} \frac{1}{h} [f(h, 0) - f(0, 0)] = \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{h^3}{h^2} - 0 \right] = 1$$

$$f_y(0, 0) = \lim_{h \rightarrow 0} \frac{1}{h} [f(0, h) - f(0, 0)] = \lim_{h \rightarrow 0} \frac{1}{h} [0 - 0] = 0$$

so that

$$L(x, y) = x.$$

So, for $(x, y) \neq (0, 0)$ we have

$$\frac{f(x, y) - L(x, y)}{\sqrt{x^2 + y^2}} = \frac{\frac{x^3}{x^2 + y^2} - x}{\sqrt{x^2 + y^2}} = -\frac{xy^2}{(x^2 + y^2)^{3/2}}.$$

For differentiability, this needs to have a limit of zero as $(x, y) \rightarrow (0, 0)$ along all possible paths. But let us consider what happens when we set $y = x$ and let $(x, x) \rightarrow (0, 0)$. Then,

$$\frac{f(x, x) - L(x, x)}{\sqrt{x^2 + x^2}} = \frac{x \cdot x^2}{(2x^2)^{3/2}} = -\frac{x^3}{2^{3/2}|x|^3} = \begin{cases} -2^{-3/2}, & \text{if } x > 0 \\ 2^{-3/2}, & \text{if } x < 0 \end{cases}$$

Therefore the limit is not equal to zero—even worse, the limit does not exist. So, our function cannot be differentiable at the origin.

3.2.2 Proving Differentiability

We rework the same argument to prove that

$$g(x, y) = \begin{cases} \frac{x^4}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

is, in fact, differentiable at the origin.

Since $f(x, 0) = \frac{x^4}{x^2} = x^2$ for $x \neq 0$, $f(0, y) = \frac{0}{y^2} = 0$ for $y \neq 0$, and $f(0, 0) = 0$, we get

$$\begin{aligned} f_1(0, 0) &= \lim_{x \rightarrow 0} \frac{f(x, 0) - f(0, 0)}{x} = \lim_{x \rightarrow 0} \frac{x^2}{x} = 0, \\ f_2(0, 0) &= \lim_{y \rightarrow 0} \frac{f(0, y) - f(0, 0)}{y} = \lim_{y \rightarrow 0} \frac{0}{y} = 0. \end{aligned}$$

Hence the linear approximation to f at $(0, 0)$ is $L(x, y) = 0$. For f to be differentiable at $(0, 0)$, we need

$$\lim_{(x, y) \rightarrow (0, 0)} \frac{f(x, y) - L(x, y)}{\sqrt{x^2 + y^2}} = \lim_{(x, y) \rightarrow (0, 0)} \frac{x^4}{(x^2 + y^2)^{3/2}} = 0.$$

Since $\sqrt{x^2 + y^2} \geq |x|$, we have

$$0 \leq \left| \frac{x^4}{(x^2 + y^2)^{3/2}} \right| \leq \frac{x^4}{|x|^3} = |x|$$

so that the limit is 0 by the Squeeze Theorem, and the function is differentiable.

3.2.3 A Proof of the Chain Rule Using Differentiability

We now give a proof of the following special case of the Chain Rule.

Theorem 3.2.1. *Assume that $z = f(x, y)$ is differentiable at (a, b) . Further, suppose $x(t_0) = a$ and $y(t_0) = b$ and that both $x'(t_0)$ and $y'(t_0)$ exist. Then we have*

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt} \quad (3.2.1)$$

Remark 3.2.1. *Notice that the assumptions of Theorem 3.2.1 are actually weaker than those of the corresponding theorem in Section 2.6! There, we assumed for simplicity that all functions are C^1 so that we could focus on the statements and applications of the Chain Rule. That was also before we introduced the rigorous concept of differentiability for multivariate functions. Now that we have that concept, we can use it instead of the C^1 assumption.*

The fact that it is enough here for f to be differentiable at (a, b) —and not necessarily C^1 in its neighbourhood—will be important in the next section when we compute directional derivatives of differentiable functions.

Proof of Theorem 3.2.1. We approximate f near (a, b) by the linear function

$$L(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b).$$

Differentiability means that the error $R(x, y) = f(x, y) - L(x, y)$ satisfies

$$\lim_{(x, y) \rightarrow (0, 0)} \frac{R(x, y)}{\sqrt{(x - a)^2 + (y - b)^2}} = 0.$$

If we could actually replace $f(x, y)$ by $L(x, y)$, then the Chain Rule would be very easy to prove. This is because

$$\begin{aligned} \frac{d}{dt} [L(x(t), y(t))] &= \frac{d}{dt} [f(a, b) + f_x(a, b)(x(t) - a) + f_y(a, b)(y(t) - b)] \\ &= f_x(a, b)x'(t) + f_y(a, b)y'(t) \end{aligned}$$

We cannot *replace* f by L . However, we can approximate $\frac{df}{dt}$ by $\frac{dL}{dt}$, by writing

$$\frac{d}{dt} f = \frac{d}{dt} [L + \underbrace{(f - L)}_R] = \frac{dL}{dt} + \frac{dR}{dt}.$$

So, to prove (3.2.1), it suffices to show that

$$\left. \frac{dR}{dt} \right|_{t=t_0} = 0. \quad (3.2.2)$$

We do this now. Let

$$x = a + \Delta x, \quad y = b + \Delta y, \quad t = t_0 + \Delta t.$$

We have

$$\left. \frac{dR(x(t), y(t))}{dt} \right|_{t=t_0} = \lim_{\Delta t \rightarrow 0} \frac{1}{\Delta t} [R(x(t), y(t)) - \underbrace{R(x(t_0), y(t_0))}_{\text{zero since } f(a, b) = L(a, b)}].$$

Now, recalling that $x = a + \Delta x$ and $y = b + \Delta y$,

$$\frac{R(x(t), y(t))}{\Delta t} = \underbrace{\left(\frac{R(a + \Delta x, b + \Delta y)}{\sqrt{(\Delta x)^2 + (\Delta y)^2}} \right)}_{I_1(\Delta x, \Delta y)} \cdot \underbrace{\left(\frac{\sqrt{(\Delta x)^2 + (\Delta y)^2}}{\Delta t} \right)}_{I_2(\Delta x, \Delta y)}$$

The assumption that $f(x, y)$ is differentiable at (a, b) implies

$$\lim_{(\Delta x, \Delta y) \rightarrow (0,0)} I_1(\Delta x, \Delta y) = 0.$$

The function I_2 can be rewritten as

$$I_2(\Delta x, \Delta y) = \pm \sqrt{\left(\frac{\Delta x}{\Delta t}\right)^2 + \left(\frac{\Delta y}{\Delta t}\right)^2}.$$

(The $\pm$ sign depends on whether Δt is positive or negative.) This has the limit

$$I_2 \rightarrow \pm \sqrt{[x'(t_0)]^2 + [y'(t_0)]^2}, \text{ as } \Delta t \rightarrow 0^\pm.$$

In particular, the function $I_2(\Delta x, \Delta y)$ must remain bounded by some constant C on a neighbourhood of $t = t_0$. Putting these two statements for I_1 and I_2 together, we obtain (3.2.2). $\square$

Practice Problems

1. (In this question, you have to use the definitions of partial derivatives and differentiability.) Let

$$f(x, y) = \begin{cases} \frac{x^3 - \sin^3 x}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

- (a) Find the first order partial derivatives of f at $(0, 0)$.
 (b) Is f differentiable at $(0, 0)$? Prove your answer.
2. (a) Prove that there exist positive constants c, C (independent of x, y) such that

$$c(x^2 + y^2) \leq x^2 - xy + 4y^2 \leq C(x^2 + y^2).$$

- (b) Let

$$f(x, y) = \begin{cases} \frac{xy^3}{x^2 - xy + 4y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

Prove that f is differentiable at $(0, 0)$. (Hint: You have to use the definition of differentiability. You may also have to use the estimate from (a).)

3. Let

$$f(x, y) = \begin{cases} \frac{x^3 - y^3 + 2xy}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

(a) Find the partial derivatives $f_1(0, 0)$, $f_2(0, 0)$.

(b) Is $f(x, y)$ differentiable at $(0, 0)$? Prove your answer.

4. Let

$$f(x, y) = \begin{cases} \frac{(x - e^x + 1)y^2}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

Is f differentiable at $(0, 0)$? Prove your answer.

5. Let

$$f(x, y) = \begin{cases} x^2 \sin(1/y) & \text{if } y \neq 0 \\ 0 & \text{if } y = 0 \end{cases}$$

Prove that:

(a) f is differentiable at $(0, 0)$ (you will have to use the definition of differentiability).

(b) f is not continuous in any neighbourhood of $(0, 0)$. (For every neighbourhood of $(0, 0)$, prove that there is a point in that neighbourhood where f is not continuous.)

6. Let

$$f(x, y) = \begin{cases} xy & \text{if } x, y \text{ are both rational,} \\ -xy & \text{if at least one of } x, y \text{ is irrational.} \end{cases}$$

(a) Find all points where f is continuous. Prove your answer **using the epsilon-delta definition** of continuity. (You may have to consider several cases separately.)

(b) Find all points where f is differentiable. Prove your answer. (You may have to consider several cases separately, and you will have to use the definition of differentiability at least once. Epsilon-delta proofs are not required in this part, but may be useful depending on how you solve the question.)

3.3 Gradients and Directional Derivatives

We now focus on the following definition.

Definition 3.3.1. The gradient of a function $f(x_1, \dots, x_n)$ is given by

$$\begin{aligned}\nabla f(x_1, \dots, x_n) &= f_1(x_1, \dots, x_n)\mathbf{e}_1 + \cdots + f_n(x_1, \dots, x_n)\mathbf{e}_n \\ &= \frac{\partial f}{\partial x_1}\mathbf{e}_1 + \cdots + \frac{\partial f}{\partial x_n}\mathbf{e}_n.\end{aligned}$$

If f is a function $f: \mathbb{R}^n \rightarrow \mathbb{R}$, the gradient ∇f is a vector-valued function that maps points in $\mathbb{R}^n$ to vectors in $\mathbb{R}^n$. We occasionally use the notation $\text{grad } f$ to denote the gradient.

For a function $f(x, y)$ of two variables, the gradient is written as

$$\nabla f(x, y) = f_1(x, y)\mathbf{i} + f_2(x, y)\mathbf{j} = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}.$$

We can also consider the gradient as an *operation* on functions, similar to taking derivatives like $\frac{d}{dt}$ or $\frac{\partial}{\partial x_j}$. We write:

$$\begin{aligned}(\text{two dimensions}) \quad \nabla &:= \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} \\ (n \text{ dimensions}) \quad \nabla &:= \mathbf{e}_1\frac{\partial}{\partial x_1} + \cdots + \mathbf{e}_n\frac{\partial}{\partial x_n}\end{aligned}$$

The notation here means that we apply ∇ to f by evaluating the partial derivatives of f and then multiplying them by the corresponding basis vectors. Usually, when multiplying a vector by a scalar function, we write the scalar function first (for instance, $g(x)\mathbf{i}$ rather than $\mathbf{i}g(x)$). Here, we reverse this convention to prioritize clarity: we want to make sure that $\nabla f = (\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y})f$ is understood as $\mathbf{i}\frac{\partial f}{\partial x} + \mathbf{j}\frac{\partial f}{\partial y}$, as opposed to taking the partial derivatives of the $\mathbf{i}$ and $\mathbf{j}$ vectors and then multiplying by f .

Using the gradient operation allows us to:

1. Find directional derivatives;
2. Find directions of steepest ascent or descent;
3. Find normal vectors to level surfaces or curves of functions.

We now examine each of these three ideas. Examples appear in the Worked Problems below.

3.3.1 Directional Derivatives for Functions of Two Variables

Recall that a unit vector $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j}$ in $\mathbb{R}^2$ is any vector which has length one, so that $u_1^2 + u_2^2 = 1$. For any unit vector $\mathbf{u}$, we define the **directional derivative** in the direction of $\mathbf{u}$ at (a, b) as the one-sided limit

$$D_{\mathbf{u}}f(a, b) := \lim_{h \rightarrow 0^+} \frac{1}{h} [f(a + hu_1, b + hu_2) - f(a, b)], \quad (3.3.1)$$

assuming that this limit exists. This can be interpreted as a single-variable derivative in the following sense: if the function $f(a + tu_1, b + tu_2)$ is differentiable in t at $t = 0$, then

$$D_{\mathbf{u}}f(a, b) = \left. \frac{d}{dt} f(a + tu_1, b + tu_2) \right|_{t=0}. \quad (3.3.2)$$

(In fact, many sources use (3.3.2) as the definition of the directional derivative.) However, the one-sided limit in (3.3.1) may exist even when the full derivative in (3.3.2) does not. This happens for example when $f(x, y) = \sqrt{x^2 + y^2}$ and $(a, b) = (0, 0)$ (exercise: check this!).

If f happens to be differentiable at (a, b) , we have a particularly simple expression for $D_{\mathbf{u}}$.

Theorem 3.3.2. *Suppose that $f(x, y)$ is differentiable at (a, b) . Then, for every unit vector $\mathbf{u} = \langle u_1, u_2 \rangle$, one has*

$$D_{\mathbf{u}}f(a, b) = f_1(a, b)u_1 + f_2(a, b)u_2 = \nabla f(a, b) \cdot \mathbf{u}.$$

This generalizes to differentiable functions of several variables: if $f(x_1, \dots, x_n)$ is differentiable at $(a_1, \dots, a_n)$, then

$$D_{\mathbf{u}}f(a_1, \dots, a_n) = \nabla f(a_1, \dots, a_n) \cdot \mathbf{u}.$$

Proof. Evaluate the derivative (3.3.2) using the Chain Rule. □

Two special cases are of particular interest. First, we will look at what happens when we try to maximize or minimize the directional derivative. Next, we will consider the relationship between directional derivatives and the level curves or surfaces of the function.

3.3.2 Direction of Steepest Ascent or Descent

Let $f(x, y)$ be differentiable at some point (a, b) , and let $\mathbf{u}$ (a unit vector in $\mathbb{R}^2$) represent a direction. Then

$$D_{\mathbf{u}}f(a, b) = \nabla f(a, b) \cdot \mathbf{u} = |\nabla f(a, b)| |\mathbf{u}| \cos \theta,$$

where θ is the angle between the unit vector $\mathbf{u}$ and the gradient vector $\nabla f(a, b)$. Since $|\mathbf{u}| = 1$, we get

$$D_{\mathbf{u}}f(a, b) = |\nabla f(a, b)| \cos \theta. \quad (3.3.3)$$

Suppose that we fix (a, b) , and try to choose the direction $\mathbf{u}$ so that the directional derivative is either maximized or minimized. Let us think about what the identity (3.3.3) tells us here.

- The direction of steepest ascent is the direction in which the rate of increase of f is maximized. In other words, we are looking for a unit vector $\mathbf{u}$ that maximizes $D_{\mathbf{u}}f(a, b)$. Since $|\cos \theta| \leq 1$, the directional derivative $D_{\mathbf{u}}f(a, b)$ is maximized when $\cos \theta = 1$. This means that $\theta = 0$, so that $\mathbf{u}$ points in the same direction as $\nabla f(a, b)$. In other words,

$$\mathbf{u} := \frac{\nabla f}{|\nabla f|}$$

always gives the direction of maximal increase. The rate of steepest ascent is given by

$$D_{\frac{\nabla f}{|\nabla f|}} f := |\nabla f|.$$

- Similarly, $D_{\mathbf{u}}f(a, b)$ is minimized when $\cos \theta = -1$. This implies that $\theta = \pi$, so that $\mathbf{u}$ and $\nabla f(a, b)$ point in opposite directions. Thus, the direction of steepest descent is always given by $\mathbf{u} = -\frac{\nabla f}{|\nabla f|}$, the reverse direction of $\nabla f(a, b)$!

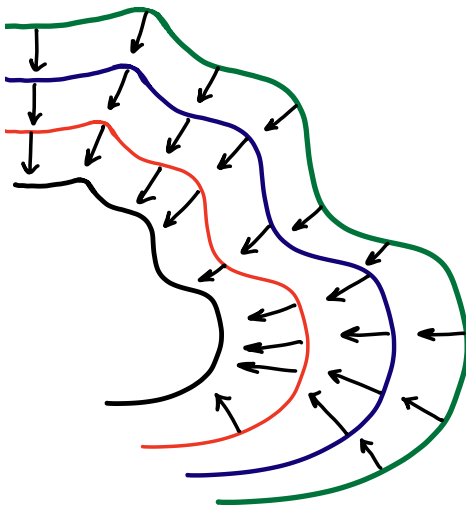


Figure 3.1: The level curves of some surface $z = f(x, y)$. The arrows in the picture indicate the directions of steepest ascent, and so point in the same direction as ∇f .

To illustrate this graphically, we assume that $f(x, y)$ is a C^1 function on an open domain, and represent $f(x, y)$ as a “topographical map” of the surface $z = f(x, y)$ in $\mathbb{R}^3$. The arrows indicate the directions where, intuitively, we would expect the climb to be the steepest. The above computations show that they should also match the direction of the gradient. It is not a coincidence that they are perpendicular to the level curves; we will investigate this in the next subsection.

3.3.3 Normal Vectors and Gradient

Theorem 3.3.3. *Suppose that $f(\mathbf{x})$ is C^1 on some neighbourhood of $\mathbf{a}$, and that $\nabla f(\mathbf{a}) \neq \mathbf{0}$. Then the vector $\nabla f(\mathbf{a})$ is normal to the level surface of $f(\mathbf{x})$ which passes through $\mathbf{a}$.*

Sketch of Proof for Functions of Two Variables. We start with some $f(x, y)$ satisfying the assumptions of the theorem at $\mathbf{a} = (a, b) \in \mathbb{R}^2$, so that f is C^1 on some neighbourhood of (a, b) and $\nabla f(a, b) \neq \langle 0, 0 \rangle$. We also let $c = f(a, b)$, so that the level curve of $f(x, y)$ passing through (a, b) is given by the equation $f(x, y) = c$.

We need to show that, at (a, b) , the vector $\nabla f(a, b)$ is normal to the level curve $f(x, y) = c$. To accomplish this, we parametrize the level curve $f(x, y) = c$ by a differentiable vector-valued function

$$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j},$$

where $x(0) = a$ and $y(0) = b$, so that $\mathbf{r}(0) = a\mathbf{i} + b\mathbf{j}$. (In this sketch, we will just *assume* that this can be done. See Remark 3.3.1 below.) Then the vector $\mathbf{v} = \langle x'(0), y'(0) \rangle$ is tangent to that level curve. Thinking about $\mathbf{r}(t)$ as the position vector of a point moving along the level curve, the vector $\mathbf{v}$ represents the velocity of that object at $t = 0$.

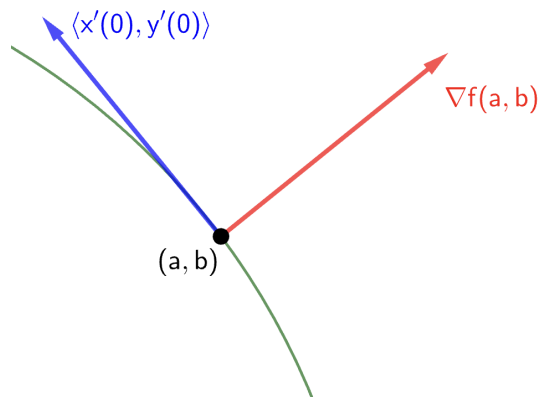


Figure 3.2: The two vectors appearing in the proof sketch.

Since $\mathbf{r}(t)$ parametrizes the level curve $f(x, y) = c$, we know that $f(x(t), y(t)) = c$ for all values of t in some neighbourhood of $t_0 = 0$. Differentiating and using the Chain Rule in such a neighbourhood, we get

$$\begin{aligned} 0 &= \frac{d}{dt}f(x(t), y(t))|_{t=0} = \frac{\partial f}{\partial x}(a, b) x'(0) + \frac{\partial f}{\partial y}(a, b) y'(0) \\ &= \left\langle \frac{\partial f}{\partial x}(a, b), \frac{\partial f}{\partial y}(a, b) \right\rangle \cdot \langle x'(0), y'(0) \rangle \\ &= \nabla f(a, b) \cdot \mathbf{v}. \end{aligned}$$

This means that $\nabla f(a, b)$ is perpendicular to $\mathbf{v}$. Therefore $\nabla f(a, b)$ is also normal to the level curve $f(x, y) = c$ at the point (a, b) .

Alternatively, we can invoke directional derivatives. Thinking again about $\mathbf{r}(t)$ as a trajectory of a point, and considering that the value of $f(\mathbf{r}(t))$ remains constant along that trajectory, we expect that $D_{\mathbf{u}}f(a, b) = 0$ if $\mathbf{u}$ is a unit vector tangent to the curve. But

$$D_{\mathbf{u}}f(a, b) = 0 \text{ if } \cos \theta = 0,$$

where θ is the angle between $\nabla f(a, b)$ and $\mathbf{u}$. This implies that $\theta = \pi/2$, so that we again see that $\nabla f(a, b)$ is perpendicular to the level curve $f(x, y) = c$ at (a, b) . $\square$

Remark 3.3.1. *The fact the level curve $f(x, y) = c$ has a nice differentiable parametrization $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$ is not given! In fact, at this point, we do not even have any justification for saying that the set*

$$\{(x, y) \in \mathbb{R}^2 : f(x, y) = c\}$$

is a curve. Yet, as it happens, the hypotheses of Theorem 3.3.3 are indeed strong enough to guarantee that the graph of $f(x, y) = c$ is in fact a curve and also admits a suitable parametrization near $(x, y) = (a, b)$. Here, suitable means a C^1 parametrization so that one can apply the Chain Rule as in the previous proof. We will address this question a little bit later in connection with the Implicit Function Theorem.

In dimensions $n \geq 3$, the proof is similar if we consider curves lying on the level surface $f(\mathbf{x}) = f(\mathbf{a})$ and their tangent vectors. We give a figure below, but omit the computations.

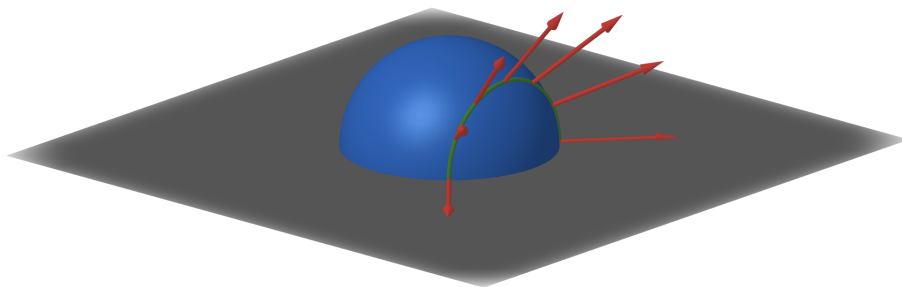


Figure 3.3: The surface defined by the equation $z = \sqrt{1 - x^2 - y^2}$ overlain with the curve obtained by taking the intersection with the plane $y = \frac{1}{2}$. Several vectors normal to the surface and along the curve are shown in red.

3.3.4 Non-differentiable Functions

We conclude by reminding the reader that, while formula for the directional derivative $D_{\mathbf{u}}f(a, b)$ simplifies greatly if we assume that $f(x, y)$ is differentiable at (a, b) , the directional derivative may still exist even if f fails to be differentiable. In this situation, we must appeal directly to the definition of the directional derivative. So, without the assumption of differentiability at (a, b) , the formula $D_{\mathbf{u}}f(a, b) = \nabla f(a, b) \cdot \mathbf{u}$ may or may not hold. Several examples are given in the Worked Problems below.

Worked Problems

1. Let $f(x, y) = x^2y - xe^y$. Find $D_{\mathbf{u}}f(1, 0)$ if $\mathbf{u}$ is the unit vector in the direction of $\mathbf{v} = \langle 3, -1 \rangle$.

The function $f(x, y)$ is differentiable (why?) for all (x, y) in the plane. So, we first calculate the gradient,

$$\nabla f(x, y) = \langle 2xy - e^y, x^2 - xe^y \rangle,$$

so that $\nabla f(1, 0) = \langle 0 - 1, 1 - 1 \rangle = \langle -1, 0 \rangle$. Next, we normalize $\mathbf{v}$ to a unit vector by writing,

$$\mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{1}{\sqrt{10}} \langle 3, -1 \rangle.$$

Finally, we can use the formula in Theorem 3.3.2 to write

$$D_{\mathbf{u}}f(1, 0) = \langle -1, 0 \rangle \cdot \frac{1}{\sqrt{10}} \langle 3, -1 \rangle = -\frac{3}{\sqrt{10}}$$

2. (In this problem, we show that if f is a differentiable function, then two directional derivatives in non-parallel directions determine the gradient of f .)

Suppose f is differentiable at (a, b) and that $D_{\mathbf{u}}f(a, b) = 3$, $D_{\mathbf{v}}f(a, b) = 2$, where

$$\mathbf{u} = \frac{1}{\sqrt{5}}\langle 2, 1 \rangle \text{ and } \mathbf{v} = \langle -1, 0 \rangle.$$

Determine $\nabla f(a, b)$.

We are given that

$$\begin{aligned} 3 = D_{\mathbf{u}}f(a, b) &= \frac{2}{\sqrt{5}}f_x(a, b) + \frac{1}{\sqrt{5}}f_y(a, b) \\ 2 = D_{\mathbf{v}}f(a, b) &= -f_x(a, b) + 0 \end{aligned}$$

which is a system of two linear equations in the unknown variables $f_x(a, b)$ and $f_y(a, b)$. Solving this system of equations gives

$$f_x(a, b) = -2, \quad f_y(a, b) = 4 + 3\sqrt{5},$$

so that $\nabla f(a, b) = \langle -2, 4 + 3\sqrt{5} \rangle$.

3. (In this problem, we find the equation of a tangent plane to a surface by calculating the gradient)

The point $P = (2, 3, 1)$ lies on the surface $x^2 - y^2 + 4z^2 = -1$. Find the equation of the tangent plane at that point.

Let $f(x, y, z) = x^2 - y^2 + 4z^2$. Then

$$\nabla f(x, y, z) = \langle 2x, -2y, 8z \rangle \Rightarrow \nabla f(2, 3, 1) = \langle 4, -6, 8 \rangle.$$

We use this vector as our normal vector at P . The equation of the tangent plane is then

$$4(x - 2) - 6(y - 3) + 8(z - 1) = 0$$

which simplifies to

$$2x - 3y + 4z = -1.$$

4. (In this question, we show that the formula for computing directional derivatives from the gradient does not necessarily hold if the function is not differentiable.)

Show that, for the function

$$f(x, y) = \begin{cases} \frac{x^2y}{x^4+y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$$

the formula $D_{\mathbf{u}}f = \nabla f \cdot \mathbf{u}$ fails to hold at the origin.

Let $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j}$ be some unit vector. Using the definition of the directional derivative, we calculate

$$\begin{aligned} D_{\mathbf{u}}f(0,0) &= \lim_{h \rightarrow 0^+} \frac{1}{h} \left[f(hu_1, hu_2) - f(0,0) \right] \\ &= \lim_{h \rightarrow 0^+} \frac{1}{h} \left[\frac{h^3 u_1^2 u_2}{h^4 u_1^4 + h^2 u_2^2} - 0 \right] \\ &= \lim_{h \rightarrow 0^+} \frac{u_1^2 u_2}{h^2 u_1^4 + u_2^2}. \end{aligned}$$

Evaluating the final limit as $h \rightarrow 0^+$, we have

$$D_{\mathbf{u}}f(0,0) = \begin{cases} \frac{u_1^2}{u_2}, & u_2 \neq 0 \\ 0, & u_2 = 0 \end{cases} \quad (3.3.4)$$

Let's now see what happens if we try to calculate $D_{\mathbf{u}}f(0,0)$ using the gradient formula. Since $f(x,0) = f(0,y) = 0$ for all x and y , we know that $\nabla f(0,0) = \mathbf{0}$. Hence, for all unit vectors $\mathbf{u}$, we have $\nabla f(0,0) \cdot \mathbf{u} = 0$, which disagrees with our calculated value of $D_{\mathbf{u}}f(0,0)$ in (3.3.4)!

In this example, the formula in (3.3.4) is correct, and the gradient formula is not. The reason for it is that f is not differentiable at $(0,0)$, so that the assumptions needed for the gradient formula are not satisfied.

Practice Problems

- Find all points (x, y, z) on the surface $xy + z^2 = 20$ where the tangent plane to that surface is perpendicular to the vector $2\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}$.
- Let $f(x, y) = x \sin(xy^2)$.
 - Find the equation of the tangent plane to the surface $z = f(x, y)$ at the point where $x = \pi$, $y = 1$.
 - Find the directional derivative of f at $(\pi, 1)$ in the direction of the vector $\mathbf{v} = \mathbf{i} - 3\mathbf{j}$.
 - Find a vector $\mathbf{w}$ in the xy -plane tangent to the level curve of f at $(\pi, 1)$. (The answer might not be unique. Any non-zero tangent vector will suffice.)

3. Let $f(x, y) = xe^{x-2y}$.

(a) Find the directional derivative of f at $(x, y) = (2, 1)$ in the direction of the vector $3\mathbf{i} - \mathbf{j}$.

(b) Find a unit vector $\mathbf{u}$ such that $D_{\mathbf{u}}f(2, 1) = 0$.

4. Let $f(x, y)$ be a differentiable function such that $f(2, 5) = 3$, $f_x(2, 5) = 4$, and $D_{\mathbf{u}}f(2, 5) = 1$, where $\mathbf{u}$ is the unit vector in the direction of $\mathbf{i} - \sqrt{3}\mathbf{j}$. Find the equation of the plane tangent to the graph of $z = f(x, y)$ at $(x, y) = (2, 5)$.

5. The tangent plane to the graph of $z = f(x, y)$ at $(x, y) = (3, 4)$ has the equation $x - 6y - 2z = 1$.

(a) Find the directional derivative $D_{\mathbf{u}}f(3, 4)$ if $\mathbf{u} = \frac{\sqrt{3}}{2}\mathbf{i} - \frac{1}{2}\mathbf{j}$.

(b) Is there a unit vector $\mathbf{v}$ such that $D_{\mathbf{v}}f(3, 4) = 4$? If yes, find it. If no, explain why.

6. The surfaces $4x^2 + y^2 = z^2$ and $xy = z + 1$ intersect in a curve that contains the point $P = (2, 3, 5)$. Find the equation of the line tangent to that curve at P .

7. Let

$$g(x, y) = \begin{cases} \frac{x^2y}{x^6 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

Find the directional derivative $D_{\mathbf{u}}g(0, 0)$, if $\mathbf{u}$ is the unit vector in the direction of $3\mathbf{i} + 4\mathbf{j}$. (This function is not differentiable or even continuous at $(0, 0)$, so you have to use the definition of the directional derivative.)

8. Let

$$f(x, y) = \begin{cases} \frac{3xy^2 - y^3}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

(a) Is f differentiable at $(0, 0)$? Prove your answer.

(b) Find the directional derivative $D_{\mathbf{u}}f(0, 0)$ for a general unit vector $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j}$. If it does not exist, explain why.

9. Let

$$f(x, y) = \begin{cases} \frac{x^5}{x^4 + y^4} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

Find the general formula for $D_{\mathbf{u}}f(0, 0)$, where $\mathbf{u} = (u_1, u_2)$ is a unit vector, in terms of u_1, u_2 .

10. Suppose that $f(x, y)$ is a function defined for all (x, y) in a neighbourhood of (a, b) . Suppose also that

$$D_{\mathbf{u}}f(a, b) = 2, \quad D_{\mathbf{v}}f(a, b) = 3,$$

where

$$\mathbf{u} = \frac{\sqrt{3}}{2}\mathbf{i} + \frac{1}{2}\mathbf{j}, \quad \mathbf{v} = \frac{\sqrt{3}}{2}\mathbf{i} - \frac{1}{2}\mathbf{j},$$

and that $f_1(a, b) = -1$. Prove that f is not differentiable at (a, b) .

11. Is there a differentiable function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ such that $D_{\mathbf{u}}f(0, 0) = u_1^2 - u_2^2$ for every unit vector $\mathbf{u} = (u_1, u_2)$? If yes, find it. If no, explain why.

3.4 Implicit Functions

The Implicit Function Theorem is often stated in textbooks in a general form that uses vector-valued functions. In this section, we focus on two important special cases that cover most of the applications we need, and mention vector-valued functions briefly at the end. We assume that all functions involved are C^1 on their domain of definition, and ask the following two questions.

1. Given an equation $F(x, y) = 0$, does this define y as a function of x ? Similarly, given that $F(x, y, z) = 0$, does this define z as a function of x and y ? (Of course, we could keep asking this for functions of n variables).
2. Given a system of equations

$$\begin{cases} u = f(x, y) \\ v = g(x, y) \end{cases}$$

can we solve for x and y as functions of u and v ? A common example of this is change of variables—we will use this for example in multivariable integration.

In both situations, we are especially interested in whether the resulting functions are differentiable—and, if so, is there a formula for calculating their derivative?

3.4.1 Single Equation with Implicitly Defined Function

Consider the equation $F(x, y) = 0$ in two variables. Recall that we are attempting to determine whether the variable y can be written as a function of x based on this equation. So, suppose we had $F(x, y(x)) = 0$. We can differentiate the equation using the Chain Rule to obtain

$$0 = \frac{d}{dx}F(x, y(x)) = F_1(x, y(x)) + F_2(x, y(x)) \frac{dy}{dx}.$$

If we make the assumptions that F is a C^1 function and $F_2 \neq 0$, we may solve the previous equation for $\frac{dy}{dx}$. We obtain

$$\frac{dy}{dx} = -\frac{F_1}{F_2} \quad (3.4.1)$$

which gives the variable y as an implicit function of the variable x . You may have seen this version of implicit differentiation in single-variable calculus.

Example 3.4.1. *Given that $F(x, y) = x^2 + y^2 - 4 = 0$ (which is an equation relating the variables x and y), let us determine $\frac{dy}{dx}$. From equation (3.4.1), we know that*

$$\frac{dy}{dx} = -\frac{F_1(x, y)}{F_2(x, y)} = -\frac{2x}{2y} = -\frac{x}{y},$$

which is well-defined so long as $F_2(x, y) = y \neq 0$.

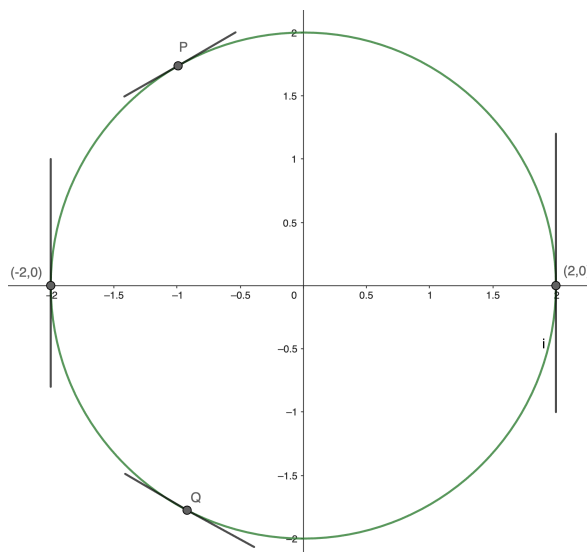


Figure 3.4: The graph of the equation $F(x, y) = 0$ in the above example.

Notice that $\frac{dy}{dx}$ can be a function of both x and y . In the above example, the slopes of the curve (which is a circle) at the points P and Q are different even though their

x -coordinates are the same. Moreover, at the points where $y = 0$, y is not well defined as a function of x . So, our method for determining $\frac{dy}{dx}$ fails here.

For the equation $F(x, y, z) = 0$, the method is largely similar to what we just saw. Suppose $z = z(x, y)$ is some function of x and y which satisfies the equation. Then we can find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ by differentiating implicitly the equation $F(x, y, z(x, y)) = 0$ and using the Chain Rule. Since $F(x, y, z(x, y))$ depends upon x in both the x and $z(x, y)$ variables, the Chain Rule gives

$$0 = \frac{\partial}{\partial x} [F(x, y, z(x, y))] = F_1(x, y, z) + F_3(x, y, z(x, y)) \cdot \frac{\partial z}{\partial x},$$

which, after solving for $\frac{\partial z}{\partial x}$, gives

$$\frac{\partial z}{\partial x} = -\frac{F_1(x, y, z)}{F_3(x, y, z)}, \quad \forall (x, y, z) \text{ with } F_3(x, y, z) \neq 0. \quad (3.4.2)$$

where, again, we have assumed that F is C^1 . Similar reasoning gives,

$$\frac{\partial z}{\partial y} = -\frac{F_2(x, y, z)}{F_3(x, y, z)}, \quad \forall (x, y, z) \text{ with } F_3(x, y, z) \neq 0. \quad (3.4.3)$$

We will see some examples illustrating the usage of (3.4.2) and (3.4.3) in the Worked Problems Section.

3.4.2 Systems of Equations

Now suppose that we are given the system of equations

$$\begin{cases} u = f(x, y) \\ v = g(x, y) \end{cases}$$

Can we solve for x and y as differentiable functions of u and v ? If so, what are the formulae for the derivatives of x and y in terms of u and v ?

Writing $x = x(u, v)$ and $y = y(u, v)$, and assuming that x and y are differentiable, we differentiate in u and use the Chain Rule to write

$$\begin{aligned} 1 &= \frac{\partial}{\partial u} u = \frac{\partial}{\partial u} [f(x(u, v), y(u, v))] = f_1(x, y) \frac{\partial x}{\partial u} + f_2(x, y) \frac{\partial y}{\partial u} \\ 0 &= \frac{\partial}{\partial u} v = \frac{\partial}{\partial u} [g(x(u, v), y(u, v))] = g_1(x, y) \frac{\partial x}{\partial u} + g_2(x, y) \frac{\partial y}{\partial u} \end{aligned}$$

Re-writing this equality in matrix form, we have

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} f_1 & f_2 \\ g_1 & g_2 \end{pmatrix} \begin{pmatrix} x_u \\ y_u \end{pmatrix} \quad (3.4.4)$$

where $x_u = \frac{\partial x}{\partial u}$ and $y_u = \frac{\partial y}{\partial u}$. We solve the system using Cramer's rule, which gives

$$\frac{\partial x}{\partial u} = \frac{\begin{vmatrix} 1 & f_2 \\ 0 & g_2 \end{vmatrix}}{\begin{vmatrix} f_1 & f_2 \\ g_1 & g_2 \end{vmatrix}} = \frac{g_2}{\begin{vmatrix} f_1 & f_2 \\ g_1 & g_2 \end{vmatrix}} \quad \frac{\partial y}{\partial u} = \frac{\begin{vmatrix} f_1 & 1 \\ g_1 & 0 \end{vmatrix}}{\begin{vmatrix} f_1 & f_2 \\ g_1 & g_2 \end{vmatrix}} = -\frac{g_1}{\begin{vmatrix} f_1 & f_2 \\ g_1 & g_2 \end{vmatrix}}$$

Similarly, if we differentiate in v , we arrive at the system of equations

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} f_1 & f_2 \\ g_1 & g_2 \end{pmatrix} \begin{pmatrix} x_v \\ y_v \end{pmatrix} \quad (3.4.5)$$

We use Cramer's Rule to solve this system of equations:

$$\frac{\partial x}{\partial u} = \frac{\begin{vmatrix} 0 & f_2 \\ 1 & g_2 \end{vmatrix}}{\begin{vmatrix} f_1 & f_2 \\ g_1 & g_2 \end{vmatrix}} = \frac{-f_2}{\begin{vmatrix} f_1 & f_2 \\ g_1 & g_2 \end{vmatrix}} \quad \frac{\partial y}{\partial u} = \frac{\begin{vmatrix} f_1 & 0 \\ g_1 & 1 \end{vmatrix}}{\begin{vmatrix} f_1 & f_2 \\ g_1 & g_2 \end{vmatrix}} = \frac{f_1}{\begin{vmatrix} f_1 & f_2 \\ g_1 & g_2 \end{vmatrix}}$$

(Cramer's Rule should be explained in your textbook, but you can also use any other method you like. The point is that you need to solve the system of linear equations given above.)

Notice that, in order to construct the partial derivatives of $x(u, v)$ and $y(u, v)$, we need the matrix of partial derivatives

$$\begin{pmatrix} f_1 & f_2 \\ g_1 & g_2 \end{pmatrix}$$

to be invertible. Equivalently, the determinant of this matrix (which we call the Jacobian) and write

$$\frac{\partial(f, g)}{\partial(x, y)} = \frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} f_1 & f_2 \\ g_1 & g_2 \end{vmatrix}$$

must be non-zero. We see examples in the Worked Problem Section below.

3.4.3 Differentiation in Higher Dimensions

This approach for determining implicit functions may seem a bit *ad hoc*, meaning that we seemingly worked backward from the conclusion in some cases. In this section, we seek to remedy this a bit by providing a common framework for Questions 1 and 2. We will then sketch how our implicit differentiation formulas could be justified rigorously.

Our common framework is based on vector-valued functions $\mathbf{f} : \mathbb{R}^n \rightarrow \mathbb{R}^m$. (Instead of a number, the output of the function is an m -dimensional vector.) For short, we often write $\mathbf{f}(\mathbf{x}) = \mathbf{y}$ where $\mathbf{x} = \langle x_1, \dots, x_n \rangle$, $\mathbf{y} = \langle y_1, \dots, y_m \rangle$, and each y_j satisfies

$$y_j = f_j(x_1, \dots, x_n).$$

Note: this is one situation where subscript notation does not indicate differentiation. Rather, f_j represents the j -th component function of $\mathbf{f} = \langle f_1, \dots, f_m \rangle$.

Our concepts of differentiability and linear approximation extend to this setting, as long as we remember that the linear approximation also needs to be vector-valued. Specifically, we want to use the following linear approximation near $\mathbf{x} = \mathbf{a}$,

$$\mathbf{f}(\mathbf{x}) \approx \mathbf{L}(\mathbf{x}) := \mathbf{f}(\mathbf{a}) + D\mathbf{f}(\mathbf{a})(\mathbf{x} - \mathbf{a}).$$

Here, $D\mathbf{f}(\mathbf{a})$ denotes the **Derivative Matrix** (sometimes also called the **Jacobian Matrix**) of $\mathbf{f}$, defined as

$$D\mathbf{f}(\mathbf{a}) = \begin{pmatrix} \frac{\partial y_1}{\partial x_1} & \cdots & \frac{\partial y_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial y_m}{\partial x_1} & & \frac{\partial y_m}{\partial x_n} \end{pmatrix},$$

where all partial derivatives are evaluated at $\mathbf{x} = \mathbf{a}$. We say that $\mathbf{f}$ is differentiable at $\mathbf{a}$ if $\mathbf{R}(\mathbf{x}) = \mathbf{f}(\mathbf{x}) - \mathbf{L}(\mathbf{x})$ satisfies

$$\lim_{\mathbf{x} \rightarrow \mathbf{a}} \frac{|\mathbf{R}(\mathbf{x})|}{|\mathbf{x} - \mathbf{a}|} = 0.$$

We also have a vector-valued version of the Chain Rule: if $\mathbf{f} : \mathbb{R}^n \rightarrow \mathbb{R}^m$ and $\mathbf{g} : \mathbb{R}^m \rightarrow \mathbb{R}^l$ are vector-valued functions whose all components are C^1 , then,

$$D(\mathbf{g} \circ \mathbf{f})(\mathbf{x}) = \underbrace{D\mathbf{g}(\mathbf{f}(\mathbf{x}))}_{l \times m \text{ matrix}} \underbrace{D\mathbf{f}(\mathbf{x})}_{m \times n \text{ matrix}}.$$

Notice that, in the vector-valued case, we replace scalar multiplication with matrix multiplication.

The general form of the Implicit Function Theorem is, then, as follows.

Theorem 3.4.2. *Let $\mathbf{F}(\mathbf{x}, \mathbf{y})$ be a vector-valued function from a set $S \subset \mathbb{R}^{n+k}$ to $\mathbb{R}^k$, where $\mathbf{x} = (x_1, \dots, x_n)$ and $\mathbf{y} = (y_1, \dots, y_k)$. Assume that:*

- $\mathbf{F}(\mathbf{a}, \mathbf{b}) = 0$ for some $(\mathbf{a}, \mathbf{b}) \in S$,
- $\mathbf{F}$ is C^1 in a neighbourhood of $(\mathbf{a}, \mathbf{b})$, meaning that its components $F_1, \dots, F_k$ are all C^1 in that neighbourhood,
- the $k \times k$ matrix

$$D_{\mathbf{y}}\mathbf{F}(\mathbf{a}, \mathbf{b}) = \begin{pmatrix} \frac{\partial F_1}{\partial y_1} & \cdots & \frac{\partial F_1}{\partial y_k} \\ \vdots & & \vdots \\ \frac{\partial F_k}{\partial y_1} & & \frac{\partial F_k}{\partial y_k} \end{pmatrix}$$

is invertible.

Then there is a (possibly smaller) neighbourhood of $(\mathbf{a}, \mathbf{b})$ where we can solve the equation $\mathbf{F}(\mathbf{x}, \mathbf{y}) = 0$ for $\mathbf{y}$ as a C^1 vector-valued function of $\mathbf{x}$. Moreover, the derivatives $\partial y_i / \partial x_j$ can be found by differentiating implicitly, and then solving, the system of equations $\mathbf{F}(\mathbf{x}, \mathbf{y}(\mathbf{x})) = 0$.

The proof of the theorem is a bit technical (see your textbook for more details), but the general idea is to use linear approximation. If the function $\mathbf{F}$ were actually linear, we could just set up a system of linear equations for $\mathbf{y}$ and then solve it. The difficulty is in extending the argument to functions that are C^1 : they are no longer linear, but they are still well approximated by linear functions near every point, and those linear functions do not change too much as we move from one point to another.

We now return to our Questions 1 and 2, with a view to using linear approximation to justify the calculations we did.

Question 1: First, suppose again that $F(x, y, z) = 0$ is the equation relating the variables x , y and z . (This is just a real-valued function, so vector-valued functions are not needed for this part.) We first solve the case when the function F is linear in its variables, so that our equation has the form

$$F(x, y, z) = Ax + By + Cz + D = 0.$$

Assuming $C \neq 0$, we solve for z to find

$$z = \frac{1}{C}(-Ax - By - D) \implies \frac{\partial z}{\partial x} = -\frac{A}{C}, \quad \frac{\partial z}{\partial y} = -\frac{B}{C}.$$

However, this corresponds to equations (3.4.2) and (3.4.3)! Why so? Well, when $F(x, y, z) = Ax + By + Cz + D$ we have

$$F_1 = A, F_2 = B, F_3 = C \implies \frac{\partial z}{\partial x} = -\frac{F_1}{F_3} \text{ and } \frac{\partial z}{\partial y} = -\frac{F_2}{F_3}.$$

For general C^1 functions $F(x, y, z)$ —that is, functions which are not assumed to be linear—we may instead approximate by a linear function $L(x, y, z)$. The condition that $F(x, y, z)$ is C^1 guarantees that this linearization works as a suitable approximation and that the approximation errors do not spoil the picture.

Question 2: For systems of equations, we write

$$\mathbf{F}(x, y) = \begin{pmatrix} f(x, y) \\ g(x, y) \end{pmatrix}$$

which is a vector-valued function. The assumption that $f(x, y)$ and $g(x, y)$ are C^1 allows us to use the linear approximation

$$\mathbf{F}(x, y) \approx \mathbf{F}(a, b) + D\mathbf{F}(a, b) \begin{pmatrix} x - a \\ y - b \end{pmatrix}, \text{ where } D\mathbf{F} := \begin{pmatrix} f_1 & f_2 \\ g_1 & g_2 \end{pmatrix}$$

If, in this situation, the functions $f(x, y)$ and $g(x, y)$ were linear in their variables, we could simply solve the linear system

$$\begin{cases} u = f(x, y) \\ v = g(x, y) \end{cases}$$

for x and y , then differentiate the corresponding solution. A formal proof of the Implicit Function Theorem uses this linearization as an approximation and sets up a linear system of equations for the derivatives. The crux of the proof, then, is proving that the error introduced by this linearization is controllable—a fact which, again, is guaranteed by the assumption that f and g are C^1 .

3.4.4 Implicitly defined curves and surfaces

This is a good time to return to the question of when an equation $f(x, y) = c$ defines a curve in the plane, or when an equation $f(x, y, z) = C$ defines a surface in $\mathbb{R}^3$.

We will look at the equation $f(x, y) = c$ first. Let $F(x, y) = f(x, y) - c$, so that the equation $f(x, y) = c$ is equivalent to $F(x, y) = 0$. Assume that f is C^1 on some neighbourhood of $(x, y) = (a, b)$, and that $\nabla f(a, b) \neq \mathbf{0}$. Then at least one of the partials $f_1(a, b), f_2(a, b)$ is non-zero. Assume that $f_2(a, b) \neq 0$. Then also $F_2(a, b) \neq 0$, and we can apply the Implicit Function Theorem to say that there is a neighbourhood of (a, b) where we can solve for $y = y(x)$ as a C^1 function. This means that, in that neighbourhood, the graph $f(x, y) = c$ is a curve with a C^1 parametrization $(x, y(x))$. If we assume that $f_1(a, b) \neq 0$ instead, then the same is true with x and y interchanged.

For equations $f(x, y, z) = C$ in 3 variables, the situation is similar. Assume that f is C^1 on some neighbourhood of $(x, y, z) = (a, b, c)$, and that $\nabla f(a, b, c) \neq \mathbf{0}$. Without loss of generality, we may assume that $f_3(a, b, c) \neq 0$. Then, again by the Implicit Function Theorem, the equation $f(x, y, z) = C$ has a C^1 solution $z = z(x, y)$ in some neighbourhood of (a, b, c) , describing a surface with a nice parametrization.

This is why the C^1 assumption in Theorem 3.3.3 implies that the level set $f(\mathbf{x}) = f(\mathbf{a})$ is, at least locally, a surface with a C^1 parametrization (a curve if $n = 2$). If we knew this up front (for some other reason), then it would be enough to assume that f is just differentiable at $\mathbf{a}$, and not necessarily C^1 on its neighbourhood.

Worked Problems

1. A certain quadric surface is given by the equation $y^2 = 2x^2 + z^2 - 1$. Find the value of $\frac{\partial z}{\partial x}$ at the point $(2, 4, 3)$ on the surface.

The surface is given by the equation $F(x, y, z) = 0$, where $F(x, y, z)$ is the function

$F(x, y, z) = 2x^2 - y^2 + z^2 - 1$. Differentiating implicitly in x gives

$$\frac{\partial z}{\partial x}\bigg|_{(x,y,z)=(2,4,3)} = -\frac{F_x}{F_z}\bigg|_{(x,y,z)=(2,4,3)} = -\frac{4x}{2z}\bigg|_{(x,y,z)=(2,4,3)} = -\frac{4}{3}.$$

In this question, an alternative method would be to solve for the variable z explicitly, so that

$$z^2 = y^2 - 2x^2 + 1 \Rightarrow z = \sqrt{y^2 - 2x^2 + 1},$$

and then differentiate in x . (Note: we chose the $+$ square-root because at $(2, 4, 3)$, we know that $z = 3 > 0$.) This method is not available in Question 2 below.

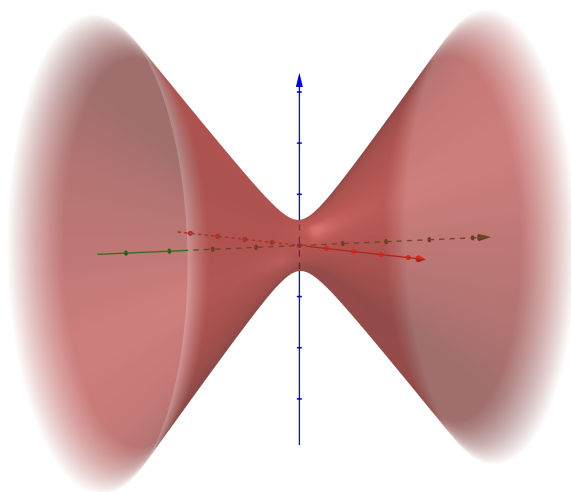


Figure 3.5: The surface given by the equation $F(x, y, z) = 0$.

2. Suppose x, y and z satisfy the equation $x + x \sin y + y \sin z = 2z \cos x$. Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ at the point $(0, \pi, \frac{\pi}{2})$.

This time, we cannot really solve for z in any explicit form, so that implicit differentiation is the only method we can use. First differentiate the equation in x , under the assumption that $z := z(x, y)$, which gives

$$\frac{\partial}{\partial x}(x + x \sin y + y \sin z) = \frac{\partial}{\partial x}(2z \cos x)$$

so that

$$1 + \sin y + y \cos z \cdot \frac{\partial z}{\partial x} = 2 \frac{\partial z}{\partial x} \cos x - 2z \sin x.$$

Now use that $(x, y, z) = (0, \pi, \frac{\pi}{2})$ to obtain

$$1 = 2 \frac{\partial z}{\partial x} \implies \frac{\partial z}{\partial x} = \frac{1}{2}.$$

A similar differentiation in y gives

$$0 + x \cos y + \sin z + y \cos z \frac{\partial z}{\partial y} = 2 \frac{\partial z}{\partial y} \cos x.$$

Evaluating at $(x, y, z) = (0, \pi, \frac{\pi}{2})$ gives $\frac{\partial z}{\partial y} = \frac{1}{2}$.

3. Suppose we have the system of equations

$$\begin{cases} u = 2x^2 + y \\ v = y^2 - 4x \end{cases}$$

which implicitly defines x and y as functions of u and v . Find $\frac{\partial x}{\partial u}$ and $\frac{\partial y}{\partial u}$ wherever possible.

Differentiate the system of equations in u , keeping in mind that $x = x(u, v)$ and $y = y(u, v)$, and applying the Chain Rule accordingly. This gives

$$\begin{aligned} 1 &= \frac{\partial u}{\partial u} = \frac{\partial}{\partial u}(2x^2 + y) = 4x \frac{\partial x}{\partial u} + \frac{\partial y}{\partial u} \\ 0 &= \frac{\partial v}{\partial u} = \frac{\partial}{\partial u}(y^2 - 4x) = 2y \frac{\partial y}{\partial u} - 4 \frac{\partial x}{\partial u} \end{aligned}$$

We simplify the second equation to $0 = -2\frac{\partial x}{\partial u} + y\frac{\partial y}{\partial u}$, and then solve using Cramer's rule:

$$\begin{aligned} \frac{\partial x}{\partial u} &= \frac{\begin{vmatrix} 1 & 1 \\ 0 & y \end{vmatrix}}{\begin{vmatrix} 4x & 1 \\ -2 & y \end{vmatrix}} = \frac{y}{4xy + 2} \\ \frac{\partial y}{\partial u} &= \frac{\begin{vmatrix} 4x & 1 \\ -2 & 0 \end{vmatrix}}{\begin{vmatrix} 4x & 1 \\ -2 & y \end{vmatrix}} = \frac{2}{4xy + 2} = \frac{1}{2xy + 1} \end{aligned}$$

Notice that both formulae are valid so long as $2xy + 1 \neq 0$, so that $xy \neq -1/2$ suffices.

Practice Problems

1. A surface is given by the equation $e^x(y+1) - e^y z = (e^z - 1)x$.

(a) Prove that the point $P = (1, 0, 1)$ lies on the surface.

(b) Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ at P .

2. Suppose that z is a function of x, y such that $\sin(xz) - yz = 0$. Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ where possible.

3. Suppose that the functions $x(u, v)$ and $y(u, v)$ solve the equations

$$\begin{cases} u &= 3x + 2e^y \\ v &= e^x - y \end{cases}$$

in some neighbourhood of the point where $(x, y) = (1, 0)$. Find $\frac{\partial x}{\partial u}$ and $\frac{\partial y}{\partial u}$ where possible, and evaluate them at $(x, y) = (1, 0)$.

4. The equations

$$\begin{aligned} u &= x^3 - y^2 \\ v &= 2xy^3 \end{aligned}$$

define x, y implicitly as functions of u, v near $x = -1, y = 1$. Find $\frac{\partial x}{\partial u}$ and $\frac{\partial y}{\partial u}$ where possible, and evaluate them at $x = -1, y = 1$.

5. The quantities u, v are defined as functions of x, y so that they satisfy the equations $x = u \cos(u + v)$, $y = v \sin u$. Find $\frac{\partial u}{\partial x}, \frac{\partial v}{\partial x}$ at the point where $u = \pi/3, v = 0$. (Hint: $\sin(\pi/3) = \frac{\sqrt{3}}{2}, \cos(\pi/3) = \frac{1}{2}$.)

3.5 Taylor Polynomials

In this section, we review the notion of Taylor polynomials for functions of one real variable before generalizing this idea to functions of n -many variables.

3.5.1 A review of Taylor polynomials for one variable functions

If $f(x)$ is a single-variable function defined in a neighbourhood of $x = a$, and if $f'(a)$, $f''(a), \dots, f^{(k)}(a)$ exist, we define its k -th order Taylor polynomial at a as

$$p_k(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \dots + \frac{f^{(k)}(a)}{k!}(x - a)^k$$

Then $p_k(x)$ is the unique k -th order polynomial such that

$$f(a) = p_k(a), f'(a) = p'_k(a), \dots, f^{(k)}(a) = p_k^{(k)}(a).$$

Let $R_k(x) = f(x) - p_k(x)$ be the remainder of f . This remainder satisfies:

- If f is k times differentiable in a neighbourhood of a , then

$$\lim_{x \rightarrow a} \frac{R_k(x)}{(x - a)^k} = 0. \quad (3.5.1)$$

(For $k = 1$, this is the condition from the definition of differentiability.)

- If f is C^{k+1} in a neighbourhood of a , then

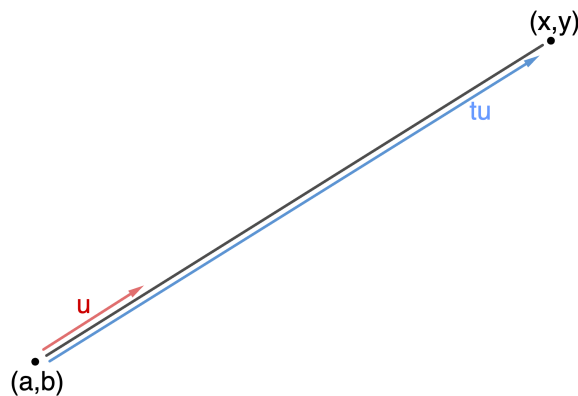
$$R_k(x) = \frac{f^{(k+1)}(u)}{(k+1)!}(x - a)^{k+1}, \quad (3.5.2)$$

for some u between a and x .

Geometrically, for functions of one variable:

- The linear equation $y = p_1(x)$ describes the tangent line at $x = a$.
- The quadratic equation $y = p_2(x)$ describes the parabola that provides the best (among parabolas) approximation to $y = f(x)$ near $x = a$. Remember: this idea is used in the second derivative test for critical points.

(See your single-variable calculus textbook for proofs and more details.) We now seek an analogue of these ideas for functions of n variables. The general case is accomplished in most textbooks. Here, we work out the second order Taylor approximation to $f(x, y)$ near (a, b) .



3.5.2 Second order Taylor approximation

Assume $f(x, y)$ is C^2 in a neighbourhood of (a, b) , and consider the line

$$(x(t), y(t)) = (a + tu_1, b + tu_2), \quad t \in \mathbb{R},$$

for some fixed unit vector $\mathbf{u} = \langle u_1, u_2 \rangle$. For this part of the argument, we think of the direction $\mathbf{u}$ as fixed and t as our variable.

Consider the function

$$g(t) = f(x(t), y(t)) = f(a + tu_1, b + tu_2). \quad (3.5.3)$$

Write its second order Taylor polynomial at $t = 0$:

$$p_2(t) = g(0) + g'(0) \cdot t + \frac{g''(0)}{2} \cdot t^2.$$

Now, $g(0) = f(a, b)$, and

$$g'(0) = \left. \frac{d}{dt} [f(a + tu_1, b + tu_2)] \right|_{t=0} = D_{\mathbf{u}}f(a, b) = \nabla f(a, b) \cdot \mathbf{u}.$$

Also, using the Chain Rule and then setting $t = 0$, we have

$$\begin{aligned} g''(0) &= \left. \frac{d}{dt} [D_{\mathbf{u}}f(a + tu_1, b + tu_2)] \right|_{t=0} \\ &= \left. \frac{d}{dt} [f_1(a + tu_1, b + tu_2)u_1 + f_2(a + tu_1, b + tu_2)u_2] \right|_{t=0} \\ &= f_{11}(a, b)u_1^2 + f_{12}(a, b)u_1u_2 + f_{21}(a, b)u_2u_1 + f_{22}(a, b)u_2^2. \end{aligned}$$

You have probably noticed that $f_{21} = f_{12}$ in the above equation if f is C^2 , and we will go back to that in a moment. It is useful, though, to start by rewriting the last line in matrix form. Define the Hessian matrix of $f(x, y)$ as

$$\mathcal{H}f = \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix}.$$

With this notation, the last line in our computation of $g''(0)$ is equal to

$$\langle x - a, y - b \rangle \cdot \mathcal{H}f(a, b) \begin{pmatrix} x - a \\ y - b \end{pmatrix}$$

This matrix formula has the advantage of generalizing to higher dimensions (we will do that below). Additionally, if f is C^2 on some domain, then $f_{12} = f_{21}$ on that domain as noted above, so that $\mathcal{H}f$ is symmetric.

We now return to our formula for $p_2(t)$, and substitute the above expressions for $g(0)$, $g'(0)$, $g''(0)$. We get

$$\begin{aligned} p_2(t) &= f(a, b) + [f_1(a, b)u_1 + f_2(a, b)u_2]t \\ &\quad + \frac{1}{2}[f_{11}(a, b)u_1^2 + f_{12}(a, b)u_1u_2 + f_{21}(a, b)u_2u_1 + f_{22}(a, b)u_2^2]t^2. \end{aligned}$$

Finally, converting back to the x, y variables using $u_1t = x - a$ and $u_2t = y - a$, this becomes

$$\begin{aligned} p_2(x, y) &= f(a, b) + f_1(a, b)(x - a) + f_2(a, b)(y - b) \\ &\quad + \frac{1}{2}\left[f_{11}(a, b)(x - a)^2 + f_{12}(a, b)(y - b)(x - a) \right. \\ &\quad \left. + f_{21}(a, b)(x - a)(y - b) + f_{22}(a, b)(y - b)^2\right] \end{aligned}$$

This is our Taylor polynomial. In matrix form,

$$p_2 = f(a, b) + \nabla f(a, b) \cdot \langle x - a, y - b \rangle + \frac{1}{2} \langle x - a, y - b \rangle \cdot \mathcal{H}f(a, b) \begin{pmatrix} x - a \\ y - b \end{pmatrix}$$

Then $p_2(x, y)$ is the unique second order polynomial in x and y such that $p_2(a, b) = f(a, b)$ and all partials of p_2 of order ≤ 2 match those of f . With the equality of mixed partials taken into account, we have

$$\begin{aligned} p_2(x, y) &= f(a, b) + f_1(a, b)(x - a) + f_2(a, b)(y - b) \\ &\quad + \frac{1}{2}f_{11}(a, b)(x - a)^2 + f_{12}(a, b)(x - a)(y - b) + \frac{1}{2}f_{22}(a, b)(y - b)^2. \end{aligned}$$

3.5.3 Remainder estimates

We would like $p_2(x, y)$ to be a good approximation to $f(x, y)$ near (a, b) . In fact, we would like $p_2(x, y)$ to be a better approximation to $f(x, y)$ than $L(x, y)$ was. Geometrically, the linear approximation $z = L(x, y)$ represents the best fit to $z = f(x, y)$ by a plane. The hope is that, for a curved surface $z = f(x, y)$, we may be able to get a

better fit using curved quadric surfaces. Quantitatively, we would like the second order error term $R_2(x, y) := f(x, y) - p_2(x, y)$ to satisfy

$$R_2(x, y) = o((x - a)^2 + (y - b)^2), \quad (3.5.4)$$

the analogue of (3.5.1) in 2 dimensions. For comparison, recall that the error term in linear approximation, $R(x, y) = f(x, y) - L(x, y)$, only satisfied

$$R_2(x, y) = o\left(\sqrt{(x - a)^2 + (y - b)^2}\right),$$

which is worse than (3.5.4).

The estimate (3.5.4) is in fact true if $f(x, y)$ is C^2 on a neighbourhood of (a, b) , and can be proved by calculating the one-dimensional remainder estimates for the intermediate function $g(t)$ defined in (3.5.1). If $f(x, y)$ is C^3 on a neighbourhood of (a, b) , then there is also an estimate analogous to (3.5.2). See your textbook for details.

As an example, consider that the second order Taylor polynomial $p_2(x, y)$ of

$$f(x, y) = (x^2 + y^2)e^{-(x^2 + y^2)},$$

at $(a, b) = (0, 0)$ is given by $p_2(x, y) = x^2 + y^2$. This follows from the calculations:

$$f_x(0, 0) = f_y(0, 0) = f_{xy}(0, 0) = f_{yx}(0, 0) = 0$$

$$f_{xx}(0, 0) = f_{yy}(0, 0) = 2.$$

(You should check this for yourself!) In polar coordinates,

$$f(r, \theta) = r^2 e^{-r^2}, \quad p_2(r, \theta) = r^2,$$

so that

$$|R(r, \theta)| = |f(r, \theta) - p_2(r, \theta)| = r^2 |1 - e^{-r^2}| = o(r^2).$$

To get the little-o estimate at the end, we used that $1 - e^{-r^2} \rightarrow 0$ as $r \rightarrow 0$. This shows that $R_2(x, y)$ satisfies (3.5.4). In fact, in this case we can do even better: since

$$\lim_{r \rightarrow 0} \frac{r^2(1 - e^{-r^2})}{r^3} = \lim_{r \rightarrow 0} \frac{1 - e^{-r^2}}{r} = \lim_{r \rightarrow 0} \frac{-2re^{-r^2}}{1} = 0$$

by L'Hôpital's Rule, we actually have $|R(r, \theta)| = o(r^3)$. This means that, close to $(0, 0)$, the Taylor polynomial $x^2 + y^2$ is an excellent approximation to $f(x, y)$. However, the approximation becomes worse as you move away from the origin. See Figures 3.6 and 3.7.

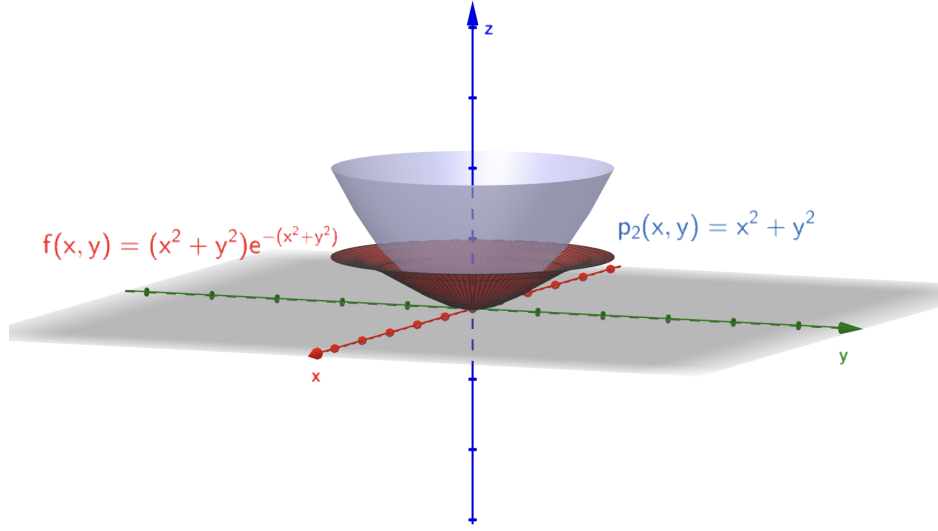


Figure 3.6: We see that, near the point $(a, b) = (0, 0)$, the surface p_2 is a good approximation of f . The above shows $f(r, \theta)$ and $p_2(r, \theta)$ for values of r satisfying $|r| \leq 1$.

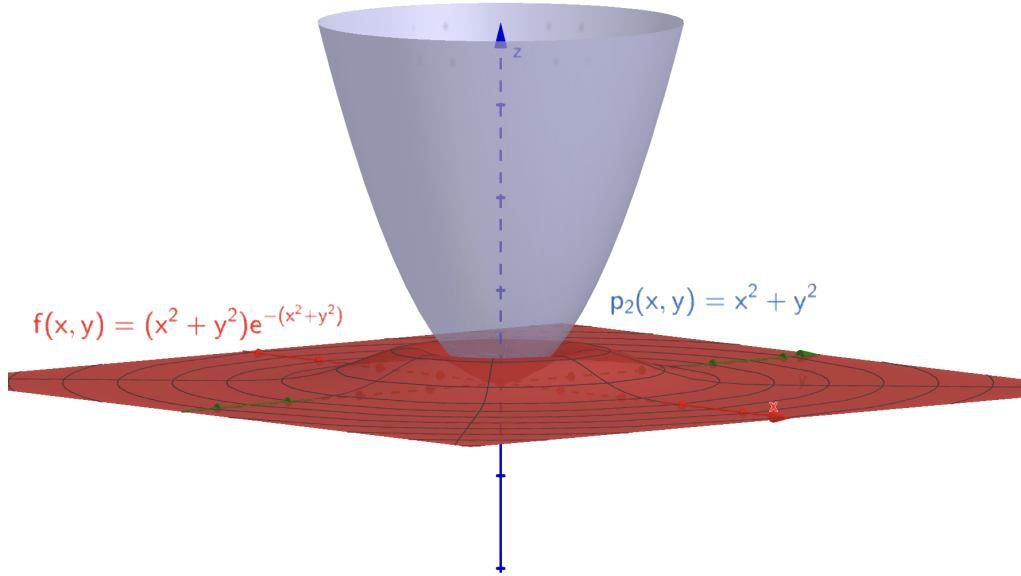


Figure 3.7: As $r \rightarrow \infty$, we see that p_2 “grows” quadratically, while $f(x, y)$ flattens out like e^{-r^2} . Contour lines are added to make the flatness of the surface $f(x, y)$ more easily seen.

3.5.4 Taylor approximation in many dimensions

Now suppose $f = f(x_1, \dots, x_n)$ is C^2 in a neighbourhood of $\mathbf{a} = (a_1, \dots, a_n)$. We then have the Taylor approximation $f(\mathbf{x}) \approx p_2(\mathbf{x})$, where

$$p_2(\mathbf{x}) := f(\mathbf{a}) + \nabla f(\mathbf{a}) \cdot (\mathbf{x} - \mathbf{a}) + \frac{1}{2}(\mathbf{x} - \mathbf{a}) \cdot \mathcal{H} f(\mathbf{a})(\mathbf{x} - \mathbf{a})^T, \quad (3.5.5)$$

where T denotes the matrix transpose and $\mathcal{H}f(\mathbf{a})$ is the $n \times n$ Hessian matrix

$$\mathcal{H}f = \begin{pmatrix} f_{11} & f_{12} & \cdots & f_{1n} \\ f_{21} & f_{22} & \cdots & f_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ f_{n1} & f_{n2} & \cdots & f_{nn} \end{pmatrix}$$

with all derivatives evaluated at $\mathbf{a}$. The function $p_2(\mathbf{x}) = p_2(x_1, \dots, x_n)$ is a polynomial in n variables of degree at most 2, so that its graph is a quadric surface in $n + 1$ dimensions.

Worked Problems

1. If $f(x, y) = x^4y + 2xy^3 - y^4$, find its second order Taylor approximation $p_2(x, y)$ at $(a, b) = (1, 2)$.

At $(a, b) = (1, 2)$, we have $f(1, 2) = 2$. Moreover

$$\begin{aligned} f_x(x, y) &= 4x^3y + 2y^3 \Rightarrow f_x(1, 2) = 24 \\ f_y(x, y) &= x^4 + 6xy^2 - 4y^3 \Rightarrow f_y(1, 2) = -7 \\ f_{xx}(x, y) &= 12x^2y \Rightarrow f_{xx}(1, 2) = 24 \\ f_{xy}(x, y) &= f_{yx}(x, y) = 4x^3 + 6y^2 \Rightarrow f_{xy}(1, 2) = f_{yx}(1, 2) = 28 \\ f_{yy}(x, y) &= 12xy - 12y^2 \Rightarrow f_{yy}(1, 2) = -24 \end{aligned}$$

Hence, utilizing the formula for $p_2(x, y)$ given in terms of the first- and second-order partial derivatives of f , we have

$$\begin{aligned} p_2(x, y) &= 2 + 24(x - 1) - 7(y - 2) \\ &\quad + \frac{1}{2}[24(x - 1)^2 + 2 \cdot 28(x - 1)(y - 2) - 24(y - 2)^2] \end{aligned}$$

Practice Problems

1. Find the second order Taylor polynomial of the function $f(x, y) = \sin(x + y^2)$ at $(\pi, 0)$.
2. Suppose that the function $f(x, y)$ and all its partial derivatives are continuous for all x, y , and that its second order Taylor polynomial at (a, b) is

$$p_2(x, y) = -7 + 4(x - a) - 13(x - a)^2 - 6(x - a)(y - b) + 10(y - b)^2.$$

Find the partial derivatives $f_1(a, b)$, $f_{11}(a, b)$, and $f_{12}(a, b)$.

3. Let $f(x, y) = Ax^2 + Bxy + Cy^2$ for some constants A, B, C . Prove that the second order Taylor polynomial of $f(x, y)$ at every point (a, b) in the xy -plane is equal to $f(x, y)$.
4. Let $F(x, y) = xf(x + y) + yg(x + y)$, where f, g are real-valued functions of one variable whose all derivatives are continuous.

(a) Prove that $F(x, y)$ satisfies the equation

$$F_{xx}(x, y) - 2F_{xy}(x, y) + F_{yy}(x, y) = 0$$

for all x, y .

(b) Find the second order Taylor polynomial of F at the point $(0, 0)$ in terms of the derivatives of f, g at 0.

5. In Sections 3.2 and 3.3, we have seen examples of functions that are not differentiable at $(0, 0)$ even though their first order partials exist at that point. What happens then is, we can try to define the linear approximation $L(x, y)$ using the partials we have, but this $L(x, y)$ is not going to be a good approximation to $f(x, y)$ near $(0, 0)$ (so that $z = L(x, y)$ is not a tangent plane to the graph of $z = f(x, y)$ in any reasonable sense).

In the question below, we see something similar happen for the second order Taylor approximation. The function is differentiable at $(0, 0)$ and the second order partials exist at $(0, 0)$, so that we can still set up the second order Taylor polynomial $p_2(x, y)$ at $(0, 0)$. However, at that level of approximation things start to go wrong, so that $p_2(x, y)$ is not a better approximation to $f(x, y)$ than $L(x, y)$ was.

Warning: this is the simplest example of this type with an explicit formula for $f(x, y)$, but the calculations are still rather long.

Let

$$f(x, y) = \begin{cases} \frac{x^4}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

(a) Compute $f_1, f_2, f_{11}, f_{12}, f_{21}, f_{22}$ at $(0, 0)$. Be careful and make sure to use the definition of the derivative where necessary. As an intermediate step, it may also be necessary to find the general expressions for some of these partials at points other than the origin.

(b) Based on your answer from (a), write the second order Taylor polynomial $p_2(x, y)$ for $f(x, y)$ at $(0, 0)$. Is it true that

$$\lim_{(x,y) \rightarrow (0,0)} \frac{f(x, y) - p_2(x, y)}{x^2 + y^2} = 0?$$

Prove your answer.

Chapter 4

Extreme Values

4.1 Critical Points

In this section, we assume that $f : D \rightarrow \mathbb{R}$ where $D \subset \mathbb{R}^n$ is a domain. For a point $\mathbf{a} \in \mathbb{R}^n$, we say that:

- f has a local minimum at $\mathbf{a}$ if there is a neighbourhood U of $\mathbf{a}$ such that $f(\mathbf{x}) \geq f(\mathbf{a})$ for all $\mathbf{x} \in U \cap D$.
- f has a global minimum at $\mathbf{a}$ if $f(\mathbf{x}) \geq f(\mathbf{a})$ for all $\mathbf{x} \in D$.
- a local/global maximum follows similar definitions, but with $f(\mathbf{x}) \leq f(\mathbf{a})$.

Local maxima and minima can be characterized in terms of the gradient of f .

Theorem 4.1.1 (Necessary Conditions for Maxima/Minima). *Suppose that $f : D \rightarrow \mathbb{R}$ has a local minimum or maximum at $\mathbf{a} \in D$. Then at least one of the following holds:*

1. $\nabla f(\mathbf{a}) = \mathbf{0}$; this is a critical point of f ;
2. $\nabla f(\mathbf{a})$ does not exist; this is a singular point of f ;
3. the point $\mathbf{a}$ is a boundary point of the domain D .

Proof for two-variable functions. Suppose f has a local minimum or maximum at (a, b) , that (a, b) is not on the boundary of D (so there is a neighbourhood U of (a, b) which is entirely contained in D), and further suppose that $\nabla f(\mathbf{a})$ exists.

Consider the function $f(x, b)$ of one variable. This function is defined on some interval of x values around a and must have a local minimum/maximum at a . This follows since, for all x close enough to a , we have

$$f(x, b) \geq f(a, b) \text{ if } f \text{ has a minimum}$$

$f(x, b) \leq f(a, b)$ if f has a maximum

Further, since we have assumed that $\nabla f(\mathbf{a})$ exists, we know that $\frac{\partial}{\partial x} f(x, b)|_{x=a}$ exists. Therefore, by one variable calculus, we know that $f_x(a, b) = 0$. A similar argument considering the function $f(a, y)$ shows that $f_y(a, b) = 0$. Therefore, $\nabla f(a, b) = \mathbf{0}$, as claimed. $\square$

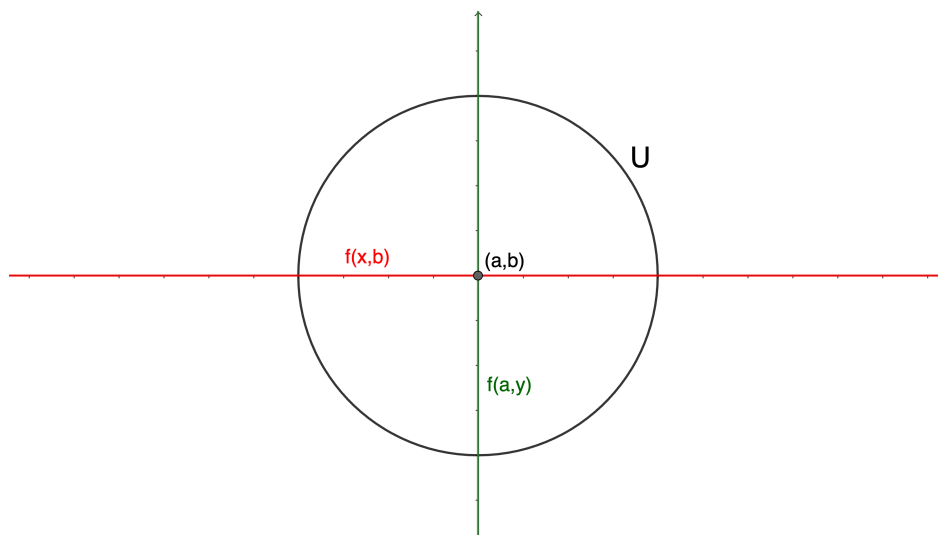


Figure 4.1: In the neighbourhood U of (a, b) , we examined the single-variable functions $f(x, b)$ and $f(a, y)$, which are defined on the coordinate axes as depicted above.

4.1.1 Classification of Critical Points

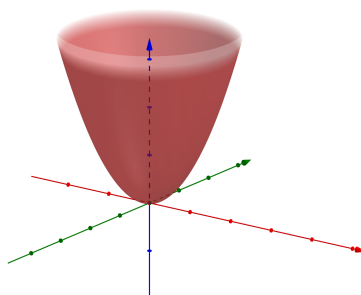
For the rest of this section, we assume that the function $f(x, y)$ has a critical point at (a, b) and is C^2 on some neighbourhood of this point. We first note that f can behave in several different ways—not just having a maximum or minimum. Three representative examples are given in Figure 4.2.

We can use Taylor's second order polynomial $p_2(x, y)$ to classify such behaviour. Suppose $f(\mathbf{x})$ —with $\mathbf{x} = (x_1, \dots, x_n)$ —has a critical point at $\mathbf{a}$. Then, near $\mathbf{a}$ we have the approximation

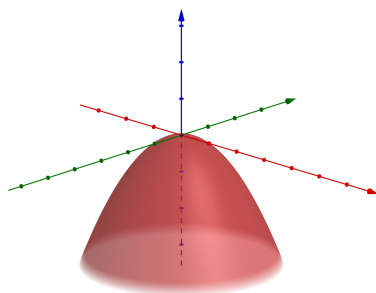
$$f(\mathbf{x}) \approx f(\mathbf{a}) + \nabla f(\mathbf{a}) \cdot (\mathbf{x} - \mathbf{a}) + \frac{1}{2}(\mathbf{x} - \mathbf{a}) \cdot \mathcal{H} f(\mathbf{x} - \mathbf{a})^t.$$

However, because $\mathbf{a}$ is a critical point, we have $\nabla f(\mathbf{a}) = \mathbf{0}$. Thus the main non-constant term in the approximation comes from the Hessian matrix. Recall that

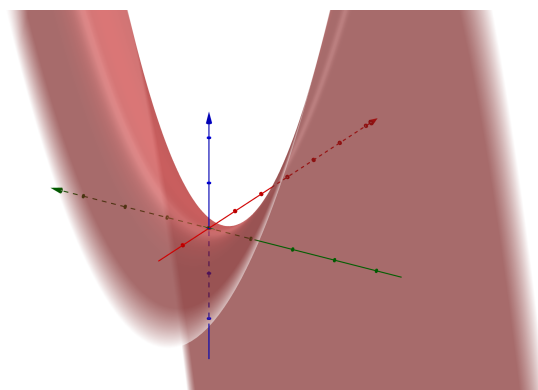
$$\mathcal{H} f = \begin{pmatrix} f_{11} & f_{12} & \cdots & f_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ f_{n1} & f_{n2} & \cdots & f_{nn} \end{pmatrix}$$



(a)



(b)



(c)

Figure 4.2: Each of the three surfaces above has a critical point at the origin $(0, 0, 0)$. At that point, they have a minimum, maximum and saddle point, respectively. The equations are given by $z = x^2 + y^2$, $z = -(x^2 + y^2)$, and $z = y^2 - x^2$.

If f is C^2 near $\mathbf{a}$, then the error terms satisfy

$$|f(\mathbf{x}) - p_2(\mathbf{x})| = o(|\mathbf{x} - \mathbf{a}|^2). \quad (4.1.1)$$

Let $Q(\mathbf{x}) = \frac{1}{2}(\mathbf{x} - \mathbf{a}) \cdot \mathcal{H}f(\mathbf{x} - \mathbf{a})^t$. Consider the following cases:

1. Suppose that $Q(\mathbf{x}) > 0$ for all $\mathbf{x} \neq \mathbf{a}$. (In the language of algebra, $\mathcal{H}f$ is a positive definite matrix; see your textbook for more about that.) Then

$$f(\mathbf{x}) \approx f(\mathbf{a}) + Q(\mathbf{x}) \geq f(\mathbf{a})$$

for all $\mathbf{x}$ sufficiently close to $\mathbf{a}$. Hence the function $f(\mathbf{a}) + Q(\mathbf{x})$ has a local minimum at $\mathbf{a}$. The estimate (4.1.1) guarantees that the same is true for the original function $f(\mathbf{x})$ (exercise: prove this!).

(Note: if $\mathbf{x}$ is not so close to $\mathbf{a}$ then the error terms $f(\mathbf{x}) - p_2(\mathbf{x})$ may be very large, so that Taylor's approximation cannot be used to determine whether f has a global maximum or minimum at $\mathbf{a}$.)

2. Suppose $Q(\mathbf{x}) < 0$ for all $\mathbf{x} \neq \mathbf{a}$. (In the language of algebra, $\mathcal{H}f$ is negative definite.) Then,

$$f(\mathbf{x}) \approx f(\mathbf{a}) + Q(\mathbf{x}) \leq f(\mathbf{a})$$

for all $\mathbf{x}$ close enough to $\mathbf{a}$. Hence, $f(\mathbf{a}) + Q(\mathbf{x})$ has a local maximum at $\mathbf{a}$. By (4.1.1), so does f .

3. It could also happen that, in every neighbourhood of $\mathbf{a}$, $Q(\mathbf{x})$ takes both positive and negative values. That is, there may be points $\mathbf{x} \neq \mathbf{y}$ near $\mathbf{a}$ such that $Q(\mathbf{x}) > 0$ and $Q(\mathbf{y}) < 0$ —which means that $\mathcal{H}f$ is indefinite. In this case, f has a saddle point at $\mathbf{a}$.

The functions $f(x, y) = x^2 + y^2$, $g(x, y) = -x^2 - y^2$, and $h(x, y) = y^2 - x^2$ in Figure 4.2 are examples of situations (1), (2), and (3), respectively, with $\mathbf{a} = (0, 0)$.

Remark 4.1.1. Notice that this does not cover all cases. For example, we could have $Q(\mathbf{x}) \geq 0$ for all $\mathbf{x} \neq \mathbf{a}$ and $Q(\mathbf{x}) = 0$ for some $\mathbf{x} \neq \mathbf{a}$. In this case, $Q(\mathbf{x})$ is positive semidefinite but not positive definite. In such cases, the behaviour of f near $\mathbf{a}$ also depends on the errors $f(\mathbf{x}) - p_2(\mathbf{x})$. As such, it cannot be determined just based on the second order derivatives of f . Take a look at the graphs of $f_{\pm}(x, y) = x^2 \pm y^4$ in Figure 4.3. Both of the functions $f_{\pm}$ have the same second order approximation $p_2(x, y) = x^2$ near $(0, 0)$.

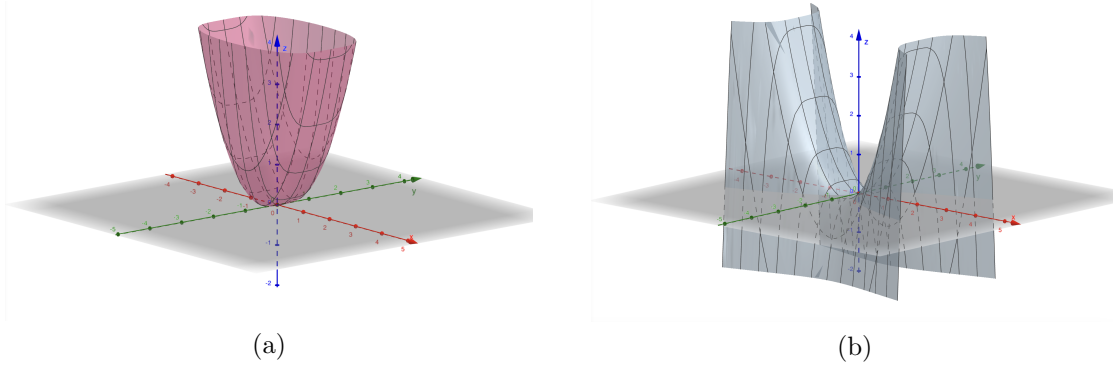


Figure 4.3: The surfaces $f_+(x, y) = x^2 + y^4$ and $f_-(x, y) = x^2 - y^4$ are shown in the above figures (A) and (B), respectively. The associated Hessian matrices are positive semi-definite (for (A)) and negative semi-definite (for (B)).

4.1.2 Determining the behaviour of $\mathcal{H}f$

In order to determine whether $\mathcal{H}f$ is positive/negative definite, indefinite or neither, we must consider the determinants of the principal minors of $\mathcal{H}f$. These are the determinants of the sub-matrices, which are given by

$$D_1 = f_{11}, \quad D_2 = \begin{vmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{vmatrix} \quad D_3 = \begin{vmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{vmatrix}, \text{ etc.}$$

We have the following conditions categorizing the behaviour of the Hessian matrix:

- If $D_1 > 0, D_2 > 0, D_3 > 0, \dots, D_n > 0$ then $\mathcal{H}f$ is positive definite, so that f has a local minimum at $\mathbf{a}$.
- If the principal minors $D_1, \dots, D_n$ satisfy $D_j < 0$ whenever j is odd and $D_k > 0$ whenever k is even, then $\mathcal{H}f$ is negative definite, which implies that f has a local maximum at $\mathbf{a}$.
- If $D_n = \det(\mathcal{H}f) \neq 0$ but the sign sequence of the principle minors is anything other than $+, +, \dots, +$ or $-, +, -, \dots$ then $\mathcal{H}f$ is indefinite, which implies that f has a saddle point at $\mathbf{a}$.
- However, if $D_n = \det(\mathcal{H}f) = 0$, then the test is inconclusive, and so we cannot decide the type of critical point at $\mathbf{a}$ just from the Hessian matrix $\mathcal{H}f$.

4.1.3 Why does the test work?

We start with an important special case. Assume that the Hessian matrix is diagonal, so that

$$\mathcal{H}f = \begin{pmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ 0 & 0 & \lambda_3 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & \lambda_n \end{pmatrix}$$

This means that $f_{11}(\mathbf{a}) = \lambda_1, f_{22}(\mathbf{a}) = \lambda_2, \dots, f_{nn}(\mathbf{a}) = \lambda_n$ and $f_{ij}(\mathbf{a}) = 0$ for $i \neq j$. Then, the second order Taylor polynomial is given by

$$p_2(\mathbf{x}) = \frac{1}{2} \underbrace{[\lambda_1(x_1 - a_1)^2 + \lambda_2(x_2 - a_2)^2 + \cdots + \lambda_n(x_n - a_n)^2]}_{Q(\mathbf{x})=Q(x_1, \dots, x_n)} + f(\mathbf{a})$$

Note also that

$$D_1 = \lambda_1, D_2 = \lambda_1\lambda_2, \dots, D_n = \lambda_1\lambda_2 \cdots \lambda_n.$$

Now, consider the sign of $Q(\mathbf{x})$:

- If $\lambda_j > 0$ for all j , then $Q(\mathbf{x}) > 0$ for all $\mathbf{x} \neq \mathbf{a}$. This gives a local minimum. We also have the $++++ \dots$ pattern for the $D_1, \dots, D_n$ sequence.
- If $\lambda_j < 0$ for all j , then $Q(\mathbf{x}) < 0$ for all $\mathbf{x} \neq \mathbf{a}$, so that we have a local maximum. In terms of principal minors, we get $D_1 = \lambda_1 < 0, D_2 = \lambda_1\lambda_2 > 0, D_3 = \lambda_1\lambda_2\lambda_3 < 0$, and so on. This corresponds to the $-+-+ \dots$ pattern.
- If $\lambda_j \neq 0$ for all j (so that $\det(\mathcal{H}f) = \lambda_1 \cdots \lambda_n \neq 0$) but the sign of the λ_j is not constant, then some of the $\lambda_j(x_j - a_j)^2$ terms are positive and others are negative for $x_j \neq a_j$, so that the function has a “directional minimum” in some directions and a “directional maximum” in other directions. This implies a saddle point.

(In particular, the $+ - + - \dots$ pattern does not work the same way as $- + - + \dots$! With a $+ - + - \dots$ pattern, we would have $\lambda_1 > 0$ and $\lambda_2 < 0$, so that $\mathcal{H}f$ is indefinite.)

- If $\lambda_j = 0$ for some j , then the error terms may be large (in some direction) relative to the size of $Q(\mathbf{x})$, so that the test is inconclusive.

When $\mathcal{H}f$ is not necessarily diagonal, we can always perform a linear change of variables which reduces to the diagonal case. Specifically, let A be a symmetric $n \times n$ matrix; then there is an orthogonal matrix Q and a diagonal matrix D , both $n \times n$, such that $A = Q^{-1}DQ$. (This should be covered in your co-requisite linear algebra course at some point.) The matrix Q provides the needed change of variables.

Worked Problems

1. Find and classify the critical points of the function $f(x, y) = x + y - x^2y - xy^2$.

We proceed in two steps:

- The first order partials are $f_x(x, y) = 1 - 2xy - y^2$ and $f_y(x, y) = 1 - 2xy - x^2$. To find critical points, we must set $f_x = f_y = 0$, which gives:

$$x^2 = y^2 \Rightarrow x = \pm y.$$

If we set $x = y$, then,

$$1 - 2x^2 - x^2 = 0 \Rightarrow 1 - 3x^2 = 0 \Rightarrow x = \pm \frac{1}{\sqrt{3}}.$$

If $x = -y$, we have

$$1 + 2x^2 - x^2 = 1 + x^2 = 0$$

which has no solutions. The critical points are $(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$ and $(-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}})$.

- To classify the critical points, we evaluate the Hessian matrix at each point. First, the second order derivatives are

$$f_{xx}(x, y) = -2y, f_{xy} = f_{yx} = -2x - 2y, f_{yy}(x, y) = -2x.$$

So, at $(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$, we have

$$\mathcal{H}f = \begin{pmatrix} -\frac{2}{\sqrt{3}} & -\frac{4}{\sqrt{3}} \\ -\frac{4}{\sqrt{3}} & -\frac{2}{\sqrt{3}} \end{pmatrix}$$

The principal minors are $D_1 = -\frac{2}{\sqrt{3}} < 0$ and $D_2 = \frac{4}{3} - \frac{16}{3} < 0$. So, at the point $(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$, f has an indefinite Hessian matrix, which implies a saddle point.

At $(-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}})$, the Hessian is given by

$$\mathcal{H}f = \begin{pmatrix} \frac{2}{\sqrt{3}} & \frac{4}{\sqrt{3}} \\ \frac{4}{\sqrt{3}} & \frac{2}{\sqrt{3}} \end{pmatrix}$$

The principal minors are given by $D_1 = \frac{2}{\sqrt{3}}$ and $D_2 = \frac{4}{3} - \frac{16}{3} < 0$. At $(-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}})$, f has an indefinite Hessian matrix, so that f again has a saddle point.

2. Find and classify the critical points of the function $f(x, y) = e^{-y}(x^2 - y^2)$.

We solve this in two steps.

- The first order partials are $f_x(x, y) = 2xe^{-y}$ and $f_y(x, y) = -e^{-y}(x^2 - y^2) - 2ye^{-y}$. To determine the critical points of f , we set $f_x = f_y = 0$. This gives,

$$\begin{aligned} f_x = 0 &\Rightarrow x = 0 \Rightarrow f_y(0, y) = e^{-y}(y^2 - 2y) \\ f_y = 0 &\Rightarrow y^2 - 2y = 0 \Rightarrow y(y - 2) = 0. \end{aligned}$$

Hence, we see that our critical points occur at $(0, 0)$ and $(0, 2)$.

- To classify the critical points, we calculate the Hessian matrix at each critical points. First, the second derivatives are given by

$$\begin{aligned} f_{xx}(x, y) &= 2e^{-y} & f_{xy}(x, y) &= f_{yx}(x, y) = 2xe^{-y}, \\ f_{yy}(x, y) &= e^{-y}(x^2 - y^2 + 4y - 2). \end{aligned}$$

At $(0, 0)$ the Hessian matrix is given by

$$\mathcal{H}f(0, 0) = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix} \Rightarrow D_1 = 2 > 0, D_2 = 4 < 0 \Rightarrow \text{indefinite, saddle point.}$$

At $(0, 2)$ the Hessian matrix is given by

$$\mathcal{H}f(0, 2) = \begin{pmatrix} 2e^{-2} & 0 \\ 0 & 2e^{-2} \end{pmatrix} \Rightarrow D_1 = 2e^{-2} > 0, D_2 = 4e^{-4} > 0,$$

so that the Hessian matrix is positive definite and f has a local minimum.

(Alternatively, since the Hessian for both points is diagonal, we could just read the classification of the critical points from the signs of the eigenvalues as in the lecture notes—without evaluating determinants. At $(0, 0)$, one eigenvalue is positive and one is negative, so we have a saddle point. At $(0, 2)$, both eigenvalues are positive, so this is a local minimum.)

The conclusion is that f has a saddle point at $(0, 0)$ and a minimum at $(0, 2)$.

Practice Problems

1. Let $f(x, y) = \frac{y}{x} + \frac{1}{y} + x$.

(a) This function has exactly one critical point. Find it.

- (b) Find the second order Taylor polynomial of $f(x, y)$ at the critical point found in (a).
- (c) Does $f(x, y)$ have a minimum, maximum, or a saddle point at this critical point?
2. Let $f(x, y) = x^3 + 3x^2y - y - y^3$. Find all critical points of f and classify them as local minima, maxima, or saddle points.
3. Let $f(x, y) = x^4 - 4xy + y^2$. Find all critical points of f and classify them as local minima, maxima, or saddle points.
4. Let $f(x, y) = x^3 - 3x^2y + xy - y^2$. Do there exist constants A, B such that the function $g(x, y) = f(x, y) - Ax - By$ has a local minimum or maximum at $(1, 2)$? If so, find all such A and B . If no, explain why.
5. Let $f(x, y, z) = x^2 - 2x + 2y^2 + 3z^2$.
- (a) This function has one critical point. Find it and classify it.
- (b) Why can we conclude that the critical point from part (a) is, in fact, a global minimum?
6. Let $f(x, y, z) = x^2y^2 \sin(z)$.
- (a) Determine all critical points of the function $f(x, y, z)$.
- (b) Calculate the Hessian matrix $\mathcal{H}(x, y, z)$ for an arbitrary point (x, y, z) .
- (c) Based on your answers to (a) and (b), explain why the second-order test must necessarily fail at all critical points of this function.
7. Let $f(x, y)$ be a function defined for all $(x, y) \in \mathbb{R}^2$. Assume that f and all its partial derivatives of all orders are continuous everywhere in the plane. Assume also that the second order Taylor polynomial of f at $(a, b) = (0, 0)$ is

$$p_2(x, y) = 2x^2 + 3y^2,$$

and that the error $f(x, y) - p_2(x, y)$ satisfies

$$\lim_{(x,y) \rightarrow (0,0)} \frac{f(x, y) - p_2(x, y)}{x^2 + y^2} = 0.$$

Prove that f has a local minimum at $(0, 0)$.

8. Let $f(x, y)$ be a function defined for all $(x, y) \in \mathbb{R}^2$. Assume that f and all its partial derivatives of all orders are continuous everywhere in the plane. Assume also that

$$f(x, y) = x^4 - 2y^4 + g(x, y),$$

where the function $g(x, y)$ satisfies $g(0, 0) = 0$ and

$$\lim_{(x,y) \rightarrow (0,0)} \frac{g(x, y)}{x^4 + y^4} = 0.$$

Prove that f does not have a local minimum or maximum at $(0, 0)$.

4.2 Global Extrema on Restricted Domains

We define a class of domains that will be important in this section.

Definition 4.2.1. A domain $X \subset \mathbb{R}^n$ is **compact** if it is bounded and closed.

For some examples, the closed ball in $\mathbb{R}^3$ and a rectangle together with its boundary in $\mathbb{R}^2$ are both compact. The following theorem, given here without proof, says that a function f continuous on a compact set X attains its global extrema on that set.

Theorem 4.2.2. If $X \subset \mathbb{R}^n$ is compact and $f : X \rightarrow \mathbb{R}$ is continuous, then f attains its global minimum and maximum values on X . That is, there are points $\mathbf{a}_{min} \in X$ and $\mathbf{a}_{max} \in X$ such that $f(\mathbf{a}_{min}) \leq f(\mathbf{a}) \leq f(\mathbf{a}_{max})$ for all $\mathbf{a} \in X$.

Note that, if f has a global minimum at $\mathbf{a}_{min}$ then it must also have a local minimum at $\mathbf{a}_{min}$. Therefore, the point $\mathbf{a}_{min} \in X$ has to be one of the following:

- a boundary point of X ;
- a singular point of f (so that ∇f does not exist at $\mathbf{a}_{min}$); or,
- a critical point of f (so that $\nabla f(\mathbf{a}_{min}) = 0$).

Of course, a similar statement holds for $\mathbf{a}_{max}$. Therefore, the procedure to minimize or maximize f on a compact domain X is as follows.

Maximizing/Minimizing Continuous Functions on Compact Sets

- (1) Find all critical and singular points $\mathbf{a}_1, \dots, \mathbf{a}_k \in X$ of the function f . Then, compute the values $f(\mathbf{a}_1), \dots, f(\mathbf{a}_k)$.
- (2) Determine the minimum or maximum values of f on the boundary of X and note the points $\mathbf{a}_-$ and $\mathbf{a}_+$ where f attains these minimum/maximum values on the boundary.
- (3) Compare all values $f(\mathbf{a}_1), \dots, f(\mathbf{a}_k)$ from Step (1) with the values $f(\mathbf{a}_-)$ and $f(\mathbf{a}_+)$ determined in Step (2). Then, from this list, choose $\mathbf{a}_{min}$ and $\mathbf{a}_{max}$ by determining where f is minimized or maximized (respectively).

Examples of using this procedure are given in the Worked Problems below. We note some additional considerations:

- If our domain $X \subset \mathbb{R}^n$ is not compact, we may still be able to find a global minimum or maximum of f on X . However, we must also consider what happens as our input $\mathbf{x} = (x_1, \dots, x_n)$ approaches infinity (that is, when one or more of the variables x_j satisfy $|x_j| \rightarrow \infty$) if X is not bounded, and/or as $\mathbf{x}$ approaches the boundary of X if X is not closed. A concrete example is considered in the Worked Problems below.
- We do not recommend using the second derivative test in (1) if your goal is only to find the largest and smallest values of f on a domain (and not to classify its critical points). While the second derivative test allowed us to determine whether the function had a local extremum at a critical point, it is entirely possible for a function $f : X \rightarrow \mathbb{R}$ to have local extrema without ever attaining a global minimum or maximum. Take, as an example, the contour plot of the function $f(x, y)$ in Figure 4.4.

You could use the second derivative test to eliminate some of the critical points (if f has a saddle point at a point $\mathbf{a}$ in the interior of X , then it does not have a minimum or maximum there), but it is easier to just evaluate f at those points.

- Min/max on the boundary: in the Worked Problems and Practice Problems below, we can determine the minimum and maximum values on the boundary of X using single-variable calculus. In general, especially for higher dimensional sets $X \subset \mathbb{R}^n$ with $n > 2$, we may need more tools. There is a separate procedure for this (the method of *Lagrange multipliers*) that we will cover in the next section.

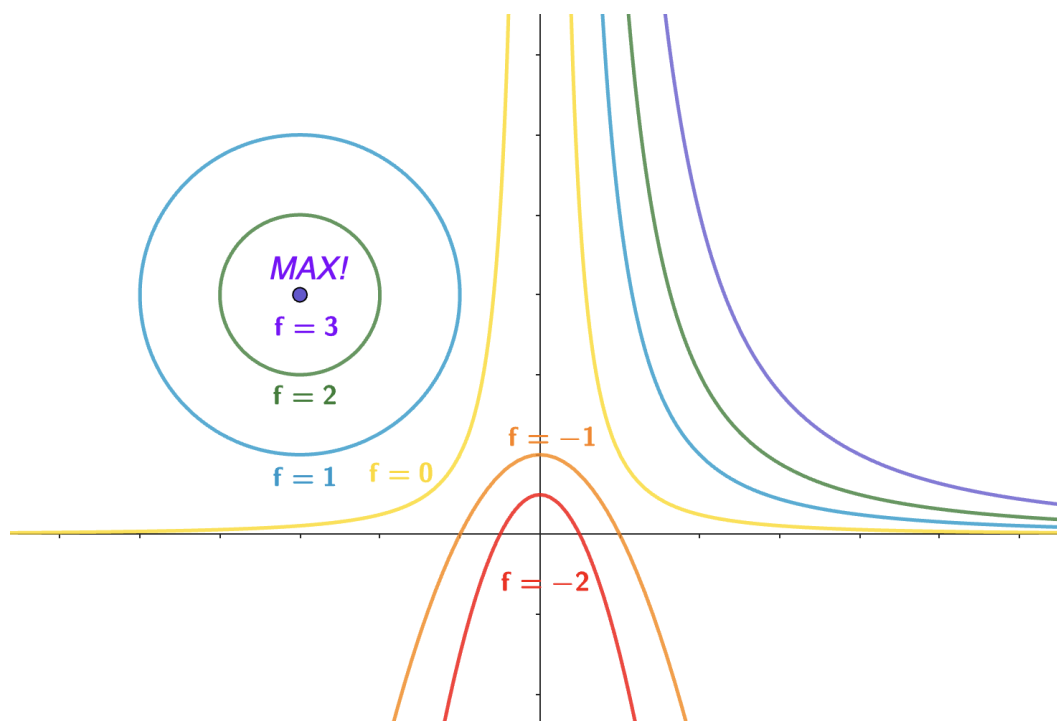
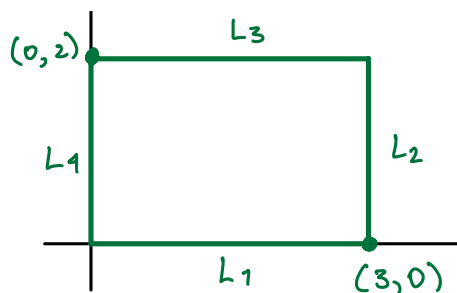


Figure 4.4: The function $f(x, y)$ has a local maximum at the marked point. However, since f grows as $x \rightarrow \infty$, we cannot conclude that f has a global maximum anywhere. Note: different colors correspond to distinct values of C in the equation $f(x, y) = C$.

Worked Problems

1. Find the largest and smallest values of the function $f(x, y) = x^2 - 2xy + 2y$ on the rectangle $0 \leq x \leq 3$, $0 \leq y \leq 2$.



We follow the procedure for determining a global maximum and minimum outlined previously.

- (a) Let us first find the critical points of $f(x, y)$. We know that $f_x(x, y) = 2x - 2y$ and $f_y(x, y) = -2x + 2$. So,

$$f_x(x, y) = f_y(x, y) = 0 \Leftrightarrow 2x - 2y = 0, -2x + 2 = 0 \Leftrightarrow x = 1, y = 1.$$

So, $f(x, y)$ has exactly one critical point interior to the rectangle: the point $(x, y) = (1, 1)$. Also, the function f has no singular points; this follows because f is C^1 everywhere, and in particular its first-order derivatives exist everywhere on the rectangle.

At the one critical point $(x, y) = (1, 1)$, we know that $f(1, 1) = 1$.

- (b) For the boundary of the rectangle, we split the boundary into four line segments as shown in the diagram above. We then find the minimum and maximum value of f on each segment. We have:

$$\text{on } L_1, f(x, 0) = x^2, 0 \leq x \leq 3 \Rightarrow \min f(0, 0) = 0, \max f(3, 0) = 9.$$

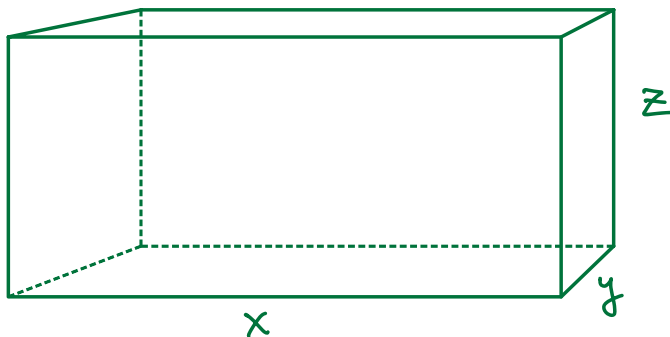
$$\text{on } L_2, f(3, y) = 9 - 4y, 0 \leq y \leq 2 \Rightarrow \min f(3, 2) = 1, \max f(3, 0) = 9.$$

$$\text{on } L_3, f(x, 2) = (x - 2)^2, 0 \leq x \leq 3 \Rightarrow \min f(2, 2) = 0, \max f(0, 2) = 4.$$

$$\text{on } L_4, f(0, y) = 2y, 0 \leq y \leq 2 \Rightarrow \min f(0, 0) = 0, \max f(0, 2) = 4.$$

- (c) Comparing the largest and smallest values found in (a) and (b), we see that $f(3, 0) = 9$ and $f(0, 0) = f(2, 2) = 0$ are the global (over the rectangle) maximum and minima (respectively) of the function $f(x, y)$.

2. We want to make a rectangular box without lid from $12m^2$ of cardboard. Find the maximal volume of such a box. **Note:** this worked problem gives an example where the domain of our function V is unbounded, hence not compact.



We seek to maximize the volume of the box, which is given by the formula $V(x, y, z) = xyz$ for $x, y, z > 0$. Using that we have $12m^2$ of cardboard (an initial constraint),

we may write $z = z(x, y)$ as a function of the length x and width y . This follows since

$$\text{Surface Area} = 2xz + 2yz + xy = 12 \Rightarrow z(x, y) = \frac{12 - xy}{2(x + y)}.$$

Hence, we seek to maximize the volume function

$$V(x, y) = xyz(x, y) = xy \left(\frac{12 - xy}{2(x + y)} \right) = \frac{12xy - x^2y^2}{2(x + y)}$$

We have $x \geq 0$ and $y \geq 0$ (the dimension of a box cannot be negative), and $xy \leq 12$ (the area xy of the bottom of the box is bounded by the total area of available cardboard). Note that the above formula for $V(x, y)$ is undefined when $(x, y) = (0, 0)$ (since then $x + y = 0$). However, as $(x, y) \rightarrow (0, 0)$, we have

$$V(x, y) = \frac{12xy - x^2y^2}{2(x + y)} \leq \frac{12xy - x^2y^2}{2x} \leq 12y - xy^2 \rightarrow 0,$$

so that we can define $V(0, 0) := 0$. With that addition, $V(x, y)$ is a continuous function of (x, y) on the domain

$$X = \{(x, y) \in \mathbb{R}^2 : x \geq 0, y \geq 0, xy \leq 12\}$$

bounded by the lines $x = 0$, $y = 0$ and the hyperbola $xy = 12$. This is not a compact set.

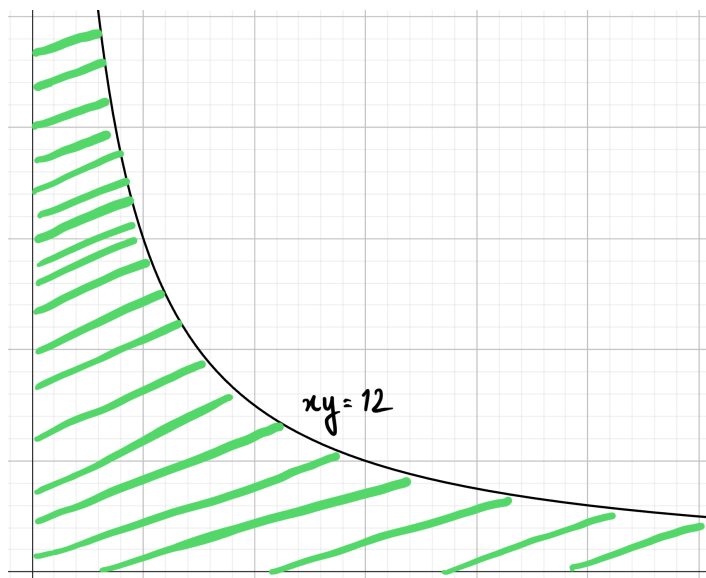


Figure 4.5: The domain of $V(x, y)$ is an unbounded set—hence, non-compact.

However, in this example, we can still find the global maximum of $V(x, y)$. Common sense says that the largest volume will be attained when the box is “reasonably”

sized, so that very long and narrow boxes should not maximize the volume. To make this argument rigorous, we will prove that the function $V(x, y)$ is small at the tail ends of the domain, so that we can discard these tail ends and maximize V on a smaller compact domain instead. We implement this in the steps (a)-(c) below.

(a) We first determine the critical points of $V(x, y)$. We have:

$$\begin{aligned}\frac{\partial V}{\partial x} &= \frac{(12y - 2xy^2)(x + y) - (12xy - x^2y^2)}{2(x + y)^2} \\ &= \frac{1}{2(x + y)^2} [12y^2 - x^2y^2 - 2xy^3]\end{aligned}$$

Since $V(x, y) = V(y, x)$ —meaning that V is symmetric under the change of variables $x \mapsto y$ —we get

$$\frac{\partial V}{\partial y} = \frac{1}{2(x + y)^2} [12x^2 - x^2y^2 - 2x^3y]$$

by interchanging the variables. To determine the critical points of V , we solve

$$\begin{aligned}\frac{\partial V}{\partial x} = \frac{\partial V}{\partial y} = 0 &\Rightarrow 12y^2 - x^2y^2 - 2xy^3 = 12x^2 - x^2y^2 - 2x^3y = 0 \\ &\Rightarrow y^2(12 - x^2 - 2xy) = x^2(12 - y^2 - 2xy) = 0.\end{aligned}$$

If $x = 0$ or $y = 0$, we are on the boundary of X ; we defer these cases to the next part of the procedure. If $x \neq 0$ and $y \neq 0$, we get

$$12 - x^2 - 2xy = 0 \text{ and } 12 - y^2 - 2xy = 0.$$

Subtracting the above equations, we have

$$y^2 - x^2 = 0 \Rightarrow x^2 = y^2 \Rightarrow x = y,$$

since both x and y are positive. Now, back-substitution gives

$$0 = 12 - x^2 - 2xy = 12 - x^2 - 2x^2 = 12 - 3x^2 \Rightarrow x = 2.$$

Hence, our only critical point occurs at $(x, y) = (x, x) = (2, 2)$. At this point,

$$V(2, 2) = \frac{12 \cdot 4 - 4 \cdot 4}{2 \cdot 4} = 4.$$

- (b) We now consider the boundary of X . When $(x, y) = (0, y)$ with $y \neq 0$ or $(x, y) = (x, 0)$ with $x \neq 0$, we have $V(x, y) = xyz(x, y) = 0$. Moreover, if $xy = 12$, then

$$V(x, y, z) = xy \frac{12 - xy}{2(x + y)} = 0,$$

so that $V(x, y, z)$ clearly does not attain its maximum value at any point of the boundary of X .

- (c) We have one additional step, since the domain of $V(x, y)$ is unbounded. Namely, we need to bound $V(x, y)$ for “large enough” values of x and y . To accomplish this, first notice that since $xy \leq 12$ on the domain of $V(x, y)$, we necessarily have

$$V(x, y) = \frac{12xy - x^2y^2}{2(x + y)} \leq \frac{12xy}{2(x + y)} = \frac{6xy}{x + y} \leq \frac{6 \cdot 12}{x + y} = \frac{72}{x + y}$$

Since $x > 0$ and $y > 0$, we know that if either $x \geq 72$ or $y \geq 72$, then

$$V(x, y) = \frac{72}{x + y} \leq \frac{72}{\max\{x, y\}} \leq \frac{72}{72} = 1.$$

Importantly, we have $1 \leq V(2, 2) = 4$! So, in the portion of the domain where $x \geq 72$ or $y \geq 72$, V cannot attain its global maximum.

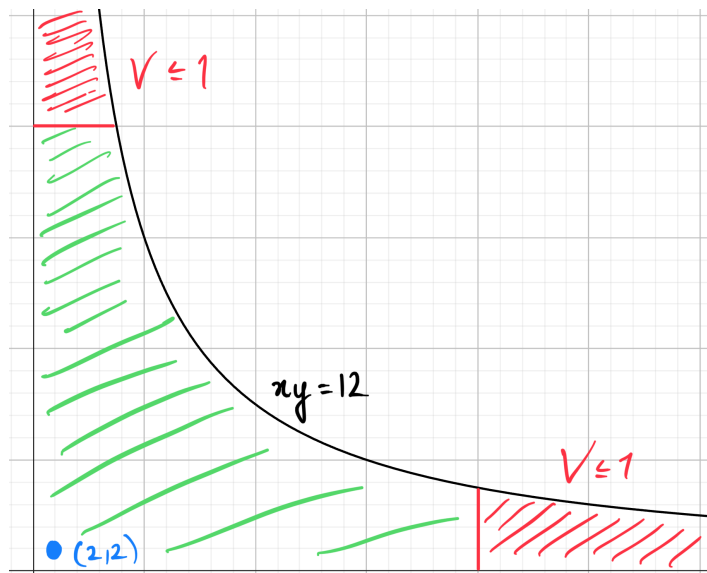


Figure 4.6: The domain of $V(x, y)$ is unbounded. However, whenever we are “far enough” away from the origin—so that either x or y is large—the function $V(x, y)$ is small.

We can now put the argument together as follows. We write our domain X as $X = X_1 \cup X_2 \cup X_3$, where

$$\begin{aligned} X_1 &= \{(x, y) \in X : x \leq 72, y \leq 72\}, \\ X_2 &= \{(x, y) \in X : x \geq 72\}, \\ X_3 &= \{(x, y) \in X : y \geq 72\}. \end{aligned}$$

The domain X_1 is compact, so we can apply the procedure for compact sets here. We found a critical point $(2, 2)$ with $V(2, 2) = 4$; we checked in (b) that on the parts of the boundary of X_1 where $x = 0$, $y = 0$, or $xy = 12$, we have $V(x, y) = 0$; and in (c), we checked that on the parts of the boundary where $x = 72$ or $y = 72$, we have $V(x, y) \leq 1$. Therefore the global maximum on X_1 is $V(2, 2) = 4$. But we also checked in (c) that $V(x, y) \leq 1$ on $X_2 \cup X_3$, so that adding X_2 and X_3 to the domain does not change the outcome.

We conclude that $V(2, 2) = 4$ is the global maximum, and choose a box of dimensions $(x, y, z) = (2, 2, 1)$ to maximize the volume.

Additional notes on the above argument:

- We needed to split X into X_1 , X_2 , and X_3 so that the largest value of V on $X_2 \cup X_3$ would be less than the largest value on X_1 . That's why we did Step (a) first, and then chose the decomposition of X after we had a guaranteed value $V(2, 2) = 4$ on what would become X_1 .
- There is nothing particular about the number 72 as the cut-off for x and y . We chose it because it was easy to use, but any other cut-off with the same property (that V should be less than 4 on $X_2 \cup X_3$) would also work. Also, we estimated $V(x, y)$ by $72(x + y)^{-1}$ at the beginning of part (c), but alternative solutions using different estimates might have also been possible.

Practice Problems

1. Find the largest and smallest values of the function $f(x, y) = x^3 - 3xy + 3y^2$ on the closed triangle with vertices $(0, 0)$, $(1, 0)$, $(1, 1)$.
2. Find the largest and smallest values of the function $f(x, y) = 4x - 2xy + y^2$ on the square $\{(x, y) : 0 \leq x \leq 2, 0 \leq y \leq 2\}$.
3. Find, with proof, the global minimum and maximum of the function $f(x, y) = \frac{xy}{x^4 + y^4 + 32}$ defined for all x, y , by following the steps listed below.
 - (a) Find all critical points of f , evaluate $f(x, y)$ at these points, and choose the largest and smallest value, M and m .

- (b) Prove that $x^4 + y^4 \geq \frac{1}{2}(x^2 + y^2)^2$.
- (c) Use (b) to prove that $|f(x, y)| \leq \frac{2}{x^2 + y^2}$. Based on this, prove that the values M and m from (a) are in fact the global minimum and maximum values.
4. Let $f(x, y) = 2x^2 - 7x - xy + y^2$. Find the global minimum of $f(x, y)$ on $\mathbb{R}^2$, by following the procedure below.
- (a) Find all critical points of f . Evaluate f at these points.
- (b) Prove that if $|x|$ and $|y|$ are large enough, then $f(x, y)$ is greater than the values you found in (a).
- (c) Conclude that the global minimum is attained at a point found in (a).
5. Find the global minimum and maximum of $f(x, y) = (2x^2 - y^2)e^{-x^2 - y^2}$ on $\mathbb{R}^2$, with proof.
6. Let $f(x, y) = x^2y - 2xy^2 + 3y$. Find the largest and smallest values of $f(x, y)$ on the following sets, or else prove that they do not exist:
- (a) $D_1 = \{(x, y) : x \geq 0, 0 \leq y \leq 2\}$,
- (b) $D_2 = \{(x, y) : x \geq 0, y \geq 0, xy \leq 16\}$.

Make sure that you account for both regions being unbounded.

4.3 The Method of Lagrange Multipliers

Our goal in this section is to determine the extreme values of a multivariate function $f(\mathbf{x})$ subject to a constraint $g(\mathbf{x}) = c$. The intended application is minimizing or maximizing f on a region D whose boundary is defined by this constraint.

For a point $\mathbf{a} \in \mathbb{R}^n$, we say that f has a local minimum at $\mathbf{a}$ subject to the constraint $g(\mathbf{x}) = c$ if:

- $g(\mathbf{a}) = c$,
- there is a neighbourhood U of $\mathbf{a}$ such that $f(\mathbf{x}) \geq f(\mathbf{a})$ for all $\mathbf{x} \in U$ such that $\mathbf{x}$ belongs to the domain of f and that $g(\mathbf{a}) = c$.

The definition of a local maximum is similar, but with $f(\mathbf{x}) \leq f(\mathbf{a})$. For a global minimum or maximum of f subject to a constraint $g(\mathbf{x}) = c$, we need the inequality in question to hold for all $\mathbf{x}$ in the domain of f such that $g(\mathbf{a}) = c$, not just in some neighbourhood of the given point.

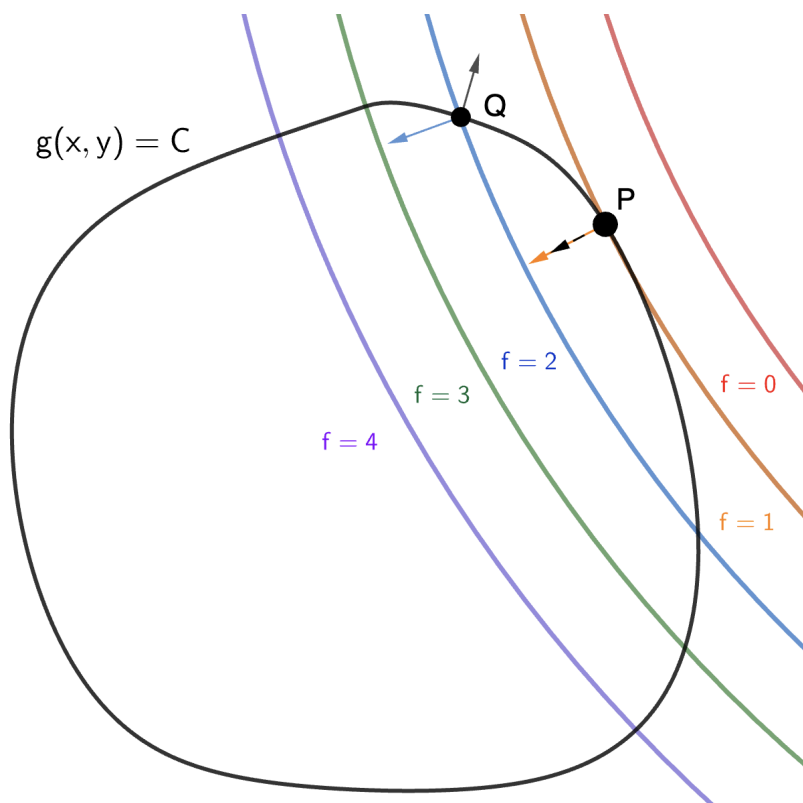


Figure 4.7: A zoomed-out picture showing the level curve $g(x, y) = C$ and its intersection with the level curves of $f(x, y)$.

4.3.1 Some diagrams

The main idea of our approach is easiest to explain when f and g are functions of two variables. Assume that $g(P) = c$ at some point P , that both $f(x, y)$ and $g(x, y)$ are C^1 on some neighbourhood of P , and that $\nabla g(P) \neq \mathbf{0}$. Then, again on some neighbourhood of P , the equation $g(x, y) = c$ defines a curve $\mathcal{C}$ that can be parametrized by a vector-valued C^1 function $(x(t), y(t))$. (This follows from the Implicit Function Theorem, see Section 3.4.) Since P is a point on $\mathcal{C}$, we have $P = (x(t_0), y(t_0))$ for some t_0 .

Suppose that $f(x, y)$ restricted to $\mathcal{C}$ has a local maximum or minimum at P . Then $f(x(t), y(t))$ has a local minimum or maximum at $t = t_0$, so that

$$0 = \left. \frac{d}{dt} f(x(t), y(t)) \right|_{t=t_0} = f_x(P)x'(t_0) + f_y(P)y'(t_0).$$

This means that $\nabla f(P)$ should be perpendicular to $\langle x'(t_0), y'(t_0) \rangle$. But $\langle x'(t_0), y'(t_0) \rangle$ is tangent to $\mathcal{C}$, therefore perpendicular to $\nabla g(P)$! Therefore the two gradient vectors should be parallel to each other.

Intuitively, we consider how $\mathcal{C}$ is situated relative to the level curve of f near P .

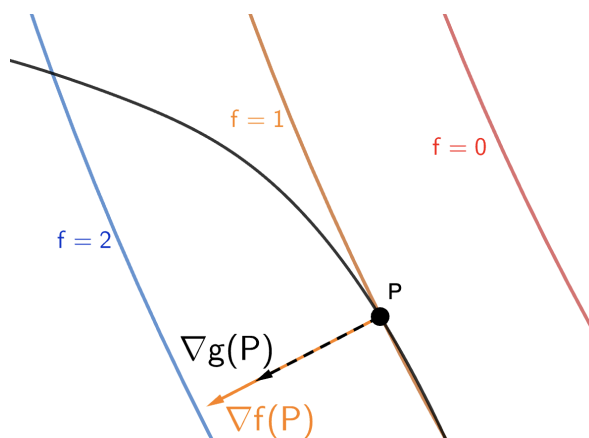


Figure 4.8: We zoom in on the critical point P from the previous figure. The key observation is that the gradient vector $\nabla f(P)$ lies in the span of the gradient vector $\nabla g(P)$. This means that $\nabla f(P) = \lambda \nabla g(P)$ for some real number $\lambda \neq 0$.

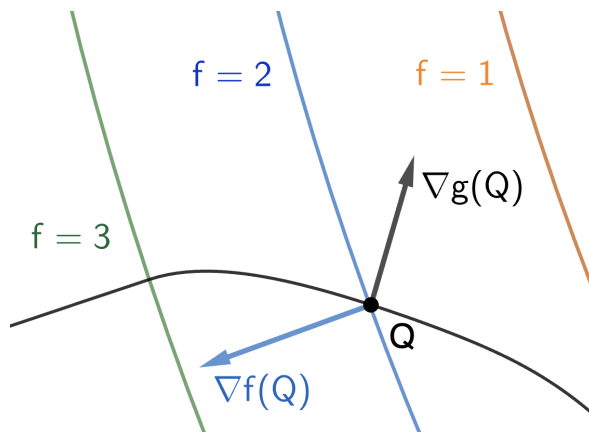


Figure 4.9: If we instead zoom in on a non-critical point Q , we see that the gradient vector $\nabla f(Q)$ no longer lies in the span of the gradient vector $\nabla g(Q)$.

At the point P on $\mathcal{C}$ where the function $f(x, y)$ is maximized or minimized, $\mathcal{C}$ should be tangent to the level curve of f . (Think about climbing part of the way up a hill, then levelling off before you reach the top, then coming down.) This means that the corresponding normal vectors ∇f and ∇g should be parallel at that point. Conversely, if Q is a point with $\nabla g(Q) \neq \mathbf{0}$ where the level curves of f and g are not tangent to each other, then they have to cross each other, so we have a situation with f taking values both larger than $f(Q)$ and greater than $f(Q)$ in any neighbourhood of Q .

4.3.2 Procedure

We assume that f and g are C^1 on an open set containing P , and that $\nabla g(P) \neq \mathbf{0}$. Let $\mathcal{C}$ be the curve $g(x, y) = g(P) = c$. Suppose that $f(x, y)$, restricted to $\mathcal{C}$, has a local

minimum or maximum at P . Then $\nabla f(P) = \lambda \nabla g(P)$ for some $\lambda \in \mathbb{R}$. Equivalently, there is a $\lambda \in \mathbb{R}$ such that $(P, \lambda) \in \mathbb{R}^2 \times \mathbb{R}$ is a critical point of the “Lagrange function”

$$L(x, y, \lambda) = f(x, y) - \lambda(g(x, y) - c), \text{ on } \mathbb{R}^3.$$

We have the following procedure for minimizing or maximizing the function $f(x, y)$ on the curve $\mathcal{C} := \{(x, y) : g(x, y) = c\}$.

Method of Lagrange Multipliers

1. Find all points (x, y) on $\mathcal{C}$ where:
 - $\nabla f(x, y) = \lambda \nabla g(x, y)$ for some $\lambda \in \mathbb{R}$;
 - $\nabla g(x, y) = 0$;
 - $\nabla f(x, y)$ or $\nabla g(x, y)$ does not exist;
 - $\mathcal{C}$ has an endpoint.
2. Evaluate $f(x, y)$ at all these “candidate points”.
3. Choose the largest and smallest values of $f(x, y)$ at each of these points.

We will implement this method in Worked Problems Section.

4.3.3 Higher dimensions

In higher dimensions (for functions of $n > 2$ variables), the procedure is still similar, but may be more complicated. For example, to minimize or maximize a function $f(x, y, z)$ on a surface with boundary given by a curve, we may need separate Lagrange multiplier procedures for both the surface and its boundary curve.

Worked Problems

1. Find the smallest and largest values of $f(x, y) = xy$ on the ellipse $4x^2 + y^2 = 16$.

We follow the Method of Lagrange Multipliers. Let $g(x, y) = 4x^2 + y^2$. Then both f and g are C^1 and

$$\nabla f(x, y) = \langle y, x \rangle \quad \nabla g(x, y) = \langle 8x, 2y \rangle.$$

First, notice that ∇f and ∇g exist everywhere, and that the curve $g(x, y) = 4x^2 + y^2 = 16$ is closed (so it has no endpoints). Further, if $\nabla g(x, y) = \mathbf{0}$, then $x = 0$ and $y = 0$. But then $4x^2 + y^2 = 0 \neq 16$. Therefore no points on our ellipse satisfy the equation $\nabla g(x, y) = 0$.

Suppose now that $\nabla f(x, y) = \lambda \nabla g(x, y)$ on for some point $(x, y) \in \mathcal{C}$. This generates the system of equations

$$y = \lambda \cdot 8x, \quad x = \lambda \cdot 2y \quad (4.3.1)$$

$$4x^2 + y^2 = 16 \quad (4.3.2)$$

From (4.3.1) we have

$$x = 2\lambda y = 2\lambda(8\lambda x) = 16\lambda^2 x.$$

So, either $x = 0$ or else $16\lambda^2 = 1$, $\lambda = \pm \frac{1}{4}$.

If $x = 0$, then $y = \pm\sqrt{16-0} = \pm 4$ from (4.3.1). At the points $(0, \pm 4)$ we have $\nabla f = \langle \pm 4, 0 \rangle$ while $\nabla g = \langle 0, \pm 8 \rangle$. These vectors are not parallel; so, in particular, we know that we cannot have a minimum or maximum at $(0, \pm 4)$. (Alternatively, it would also be fine to keep these points as *possible* minima or maxima, evaluate f at these points, and keep these values for the final comparison later on.)

If $\lambda = \pm \frac{1}{4}$, we return to (4.3.1) and find that

$$y = \pm \frac{1}{4} \cdot 8x = \pm 2x \Rightarrow y^2 = 4x^2 \Rightarrow 4x^2 + y^2 = 8x^2 = 16.$$

Then, $x = \pm\sqrt{2}$ and $y = \pm 2x = \pm 2\sqrt{2}$. Note that all four combinations of $\pm$ are possible!

To conclude, we simply evaluate f at each of these points

$$f(\sqrt{2}, 2\sqrt{2}) = f(-\sqrt{2}, -2\sqrt{2}) = 4 \leftarrow (\max)$$

$$f(-\sqrt{2}, 2\sqrt{2}) = f(\sqrt{2}, -2\sqrt{2}) = -4 \leftarrow (\min).$$

2. Find the minimum and maximum values of $f(x, y) = xy + 2y$ on the region $D := \{(x, y) : x^2 + 2y^2 \leq 4\}$.

We use our earlier procedure for minimizing/maximizing f on a compact domain D , with Lagrange multipliers used to find the min/max on the boundary of D .

The function f is C^1 and $\nabla f(x, y) = \langle y, x + 2 \rangle$. If $\nabla f = \mathbf{0}$, then $x = -2$ and $y = 0$ so that $f(-2, 0) = 0$. We record this information and move on to the boundary.

For the ellipse, we let $g(x, y) = x^2 + 2y^2$, which is C^1 with $\nabla g(x, y) = \langle 2x, 4y \rangle$. Clearly, ∇f and ∇g both exist everywhere and the ellipse has no endpoints. Further, if $\nabla g = 0$ then $x = y = 0$ and $(0, 0)$ is not on the elliptic boundary.

We set up the equations for $\nabla f = \lambda \nabla g$ and $g(x, y) = 4$:

$$y = 2\lambda x, \quad x + 2 = 4\lambda y \quad (4.3.3)$$

$$x^2 + 2y^2 = 4 \quad (4.3.4)$$

We have

$$x(x + 2) = x(4\lambda y) = 2y(2\lambda x) = 2y^2 = 4 - x^2.$$

This simplifies to

$$x^2 + 2x = 4 - x^2 \Rightarrow x^2 + x - 2 = 0,$$

so that $x = 1$ or $x = -2$.

If $x = 1$, then $2y^2 = 3$, so that $y = \pm\sqrt{3/2}$. If $x = -2$, then $2y^2 = 0$, so that $y = 0$. We have

$$\begin{aligned} f(1, \sqrt{\frac{3}{2}}) &= 3\sqrt{\frac{3}{2}}, \quad f(1, -\sqrt{\frac{3}{2}}) = -3\sqrt{\frac{3}{2}} \\ f(-2, 0) &= 0 \end{aligned}$$

The maximum value is $f(1, \sqrt{\frac{3}{2}}) = 3\sqrt{\frac{3}{2}}$, and the minimum value is $f(1, -\sqrt{\frac{3}{2}}) = -3\sqrt{\frac{3}{2}}$.

3. Find the minimum and maximum values of $f(x, y, z) = yz + xy$ on the sphere $x^2 + y^2 + z^2 = 4$.

Let $g(x, y, z) = x^2 + y^2 + z^2$, then f and g are C^1 everywhere and

$$\nabla f(x, y, z) = \langle y, x + z, y \rangle, \quad \nabla g(x, y, z) = \langle 2x, 2y, 2z \rangle.$$

If $\nabla g = \mathbf{0}$, then $x = y = z = 0$ which is not on the sphere $x^2 + y^2 + z^2 = 4$. So, we set up our Lagrange multipliers.

For convenience, we take the equation $\nabla f = \frac{1}{2}\lambda \nabla g$ —since, if $\lambda \in \mathbb{R}$ is a free parameter then $\frac{1}{2}\lambda \in \mathbb{R}$ is also free. This produces the equations

$$y = \lambda x, \quad x + z = \lambda y, \quad y = \lambda z \quad (4.3.5)$$

$$x^2 + y^2 + z^2 = 4 \quad (4.3.6)$$

From (4.3.5), we see that $\lambda x = y = \lambda z$. So, either $x = z$ or $\lambda = 0$.

If $\lambda = 0$, then $y = 0$ and $x + z = 0$ so that $x = -z$. From the boundary condition (4.3.6), we have $x^2 + 0^2 + (-x)^2 = 2x^2 = 4$ so that $x = \pm\sqrt{2} = -z$. We evaluate

$$f(\sqrt{2}, 0, -\sqrt{2}) = f(-\sqrt{2}, 0, \sqrt{2}) = 0.$$

If $x = z$, then from (4.3.5)

$$x + z = 2x = \lambda y = \lambda^2 x \Rightarrow \lambda^2 x = 2x.$$

So either $x = 0$ or $\lambda = \pm\sqrt{2}$. If $x = 0$ then $z = 0$ and $y = \pm 2$. Evaluating,

$$f(0, \pm 2, 0) = 0.$$

If instead $\lambda = \pm\sqrt{2}$ then $y = \pm\sqrt{x}$ and

$$x^2 + 2x^2 + x^2 = 4x^2 = 4 \Rightarrow x = z = \pm 1 \Rightarrow y = \pm\sqrt{2}.$$

Evaluating,

$$f(1, \sqrt{2}, 1) = f(-1, -\sqrt{2}, -1) = 2\sqrt{2} \longleftarrow (\max)$$

$$f(-1, \sqrt{2}, -1) = f(1, -\sqrt{2}, 1) = -2\sqrt{2} \longleftarrow (\min).$$

Practice Problems

1. Find the largest and smallest values of the function $f(x, y, z) = x + 2y$ in the closed disc $x^2 + y^2 \leq 25$.
2. Find the largest and smallest values of the function $f(x, y) = x^2 - 3x - \frac{y^2}{2}$ in the closed disc $x^2 + y^2 \leq 4$.
3. Find the largest and smallest values of the function $f(x, y) = \frac{x}{y}$ in the closed region $x^2 + (y - 2)^2 \leq 1$.
4. Let $A > 0$ be some real number. Find the point(s) on the surface $xyz = A^3$ that is closest to the origin, and calculate the minimum distance of this surface to the origin. Your answer will depend upon the constant A .
5. Find the largest and smallest values of $xy^2 + yz^2$ in the ball $x^2 + y^2 + z^2 \leq 1$.

Chapter 5

Integration

5.1 Introduction to the Riemann Integral

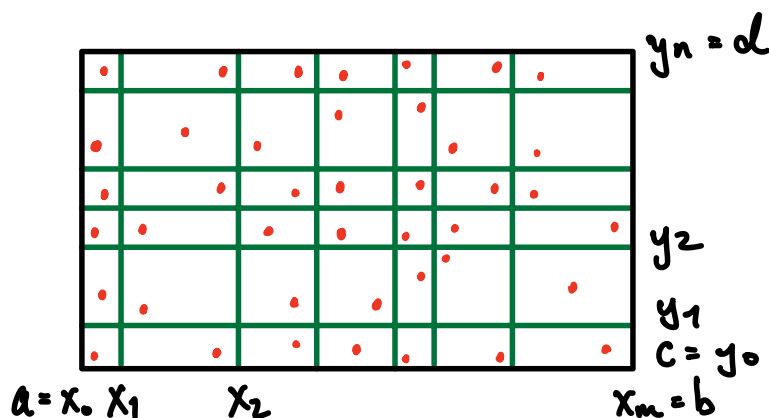
We introduce integration in many variables, starting with functions $f(x, y)$ of two real variables.

5.1.1 The Riemann integral on rectangles

Let

$$D := \{(x, y) \in \mathbb{R}^2 : a \leq x \leq b, c \leq y \leq d\},$$

a closed and bounded rectangle in the plane. The purpose of this subsection is to define the integral of $f(x, y)$ over D , denoted by $\iint_D f(x, y) dA$ (where A stands for area). We will refer to the following figure extensively in our construction.



To begin, we partition the rectangle using the grid as shown above. Here, each of

the variables x and y is partitioned as

$$a = x_0 < x_1 < \cdots < x_m = b$$

$$c = y_0 < y_1 < \cdots < y_n = d$$

The intervals in our partitions need not have equal length. Let $\Delta x_i = x_i - x_{i-1}$, $\Delta y_j = y_j - y_{j-1}$ and $\Delta A_{ij} = \Delta x_i \Delta y_j$. With this notation, ΔA_{ij} is the area of the small rectangle with left bottom corner at the point (x_{i-1}, y_{j-1}) . We then let P_{ij}^* be any point of this small rectangle (these are the dots appearing in the previous figure). Note that P_{ij}^* may lie on the boundary of its small rectangle, for example it could be one of its vertices. We use $\mathcal{P}$ to denote this collection of sub-rectangles together with the choice of points P_{ij}^* .

The **Riemann sum** associated with $f(x, y)$ and the partition $\mathcal{P}$ is defined as

$$\mathcal{R}(f, \mathcal{P}) := \sum_{i=0}^m \sum_{j=0}^n f(P_{ij}^*) \Delta A_{ij}.$$

We say that the function $f(x, y)$ is **Riemann integrable** on D if $\mathcal{R}(f, \mathcal{P})$ converges to a limit as the **norm of the partition**

$$\|\mathcal{P}\| := \max_{i,j} \sqrt{(\Delta x_i)^2 + (\Delta y_j)^2}.$$

approaches 0. This limit (if it exists) is the **Riemann integral** of f on D , and we use $\iint_D f(x, y) dA$ to denote it. To make sense of the limit “as $\|\mathcal{P}\|$ approaches 0”, we use the epsilon-delta machinery as follows.

Definition 5.1.1. *Let $f(x, y)$ be a function defined on a closed and bounded rectangle D . We say that*

$$\iint_D f(x, y) dA = L$$

if the following holds: for every $\epsilon > 0$ there is a $\delta > 0$ such that if $\mathcal{P}$ is any partition of D with $\|\mathcal{P}\| < \delta$, then

$$|\mathcal{R}(f, \mathcal{P}) - L| < \epsilon.$$

In this course, we will only use Riemann integrals (defined above), so from now on we will refer to them simply as integrals. Similarly, when we say that a function is integrable, we will mean it to say that it is Riemann integrable. However, you should be aware that there are also other types of integrals, notably Lebesgue integrals in $\mathbb{R}^n$ and more general integrals associated with abstract measure spaces. These integrals are studied in much more depth in higher level analysis courses.

5.1.2 Sufficient conditions for integrability, and integration on more general domains

Many functions encountered in calculus are Riemann integrable. This includes the following important special cases.

- Let D be a bounded and closed rectangle in $\mathbb{R}^2$. If $f(x, y)$ is continuous on D (including its boundary!), it is integrable on D .
- If $f(x, y)$ is continuous on a compact domain $X \subset \mathbb{R}^2$, we let D be a bounded and closed rectangle containing X . We then *extend* f to a function $\tilde{f}$ on D by setting

$$\tilde{f}(x, y) = \begin{cases} f(x, y) & \text{if } (x, y) \in X \\ 0 & \text{if } (x, y) \in D \setminus X \end{cases}$$

If we then assume that the boundary of X is a union of finitely many curves of finite length, then

$$\iint_X f(x, y) dA := \iint_D \tilde{f}(x, y) dA$$

exists.

The second bullet point above allows us to integrate functions on domains X that are not rectangles, provided that X satisfies the conditions above and f is continuous on X . A further extension (improper integrals, in cases when X is not compact and/or f is not continuous on X) will be considered later on. We do not provide proofs of the above statements (see your textbook for proofs and/or more details), but see Section 5.1.3 for a taste of what is involved.

What is “a curve of finite length”? We will actually study this in detail in MATH 227 (among other things, we will define the length of a curve and derive a formula to compute it). For now, the following will suffice. Suppose that the curve $\mathcal{C}$ is given by an equation of the form $g(x, y) = c$, where g is a C^1 function such that $\nabla g(x, y) \neq \mathbf{0}$ for all $(x, y) \in \mathcal{C}$. (This is the same condition that allows us to apply the Implicit Function Theorem and solve, locally, either for y as a C^1 function of x or the other way around. See Section 3.4.) Then any bounded piece of $\mathcal{C}$ has finite length. For example:

- The circle $x^2 + y^2 = 4$ has finite length (and you should already know the formula for it!). Note that it can be represented by the equation $g(x, y) = 4$ with $g(x, y) = x^2 + y^2$, and $\nabla g(x, y) = 2x\mathbf{i} + 2y\mathbf{j}$ is never zero on the circle.
- The parabola $y = x^2$ has infinite length, but if we only consider a bounded segment of it (for example, from $(0, 0)$ to $(1, 1)$), then this part has finite length.
- Many fractal curves are bounded but have infinite length, for instance the Koch snowflake curve. (We encourage you to look it up.)

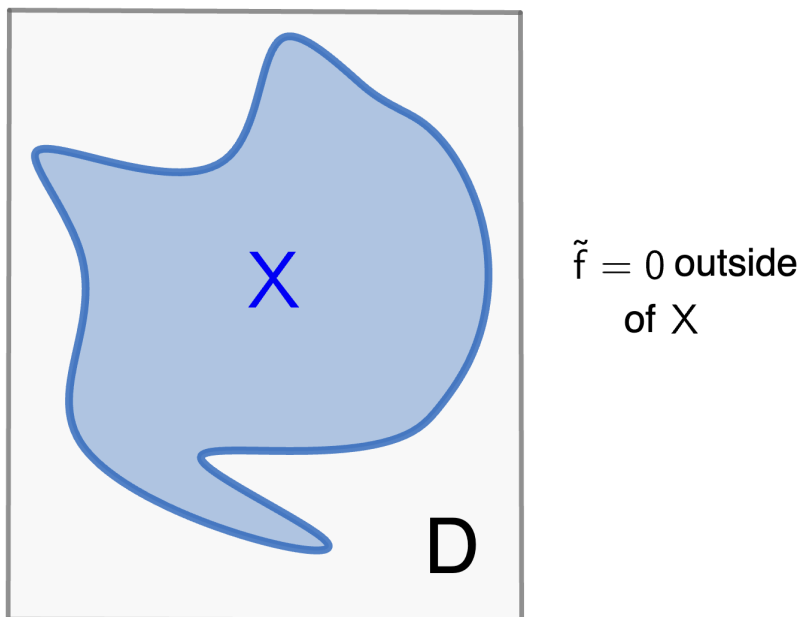


Figure 5.1: The function $f(x, y)$, originally defined on the compact domain X , can be extended to a function $\tilde{f}(x, y)$ defined on the whole rectangular region D .

5.1.3 A More General Theorem on Integrability

This section is, perhaps, more thorough than necessary for this course. However, we include it to make more sense of the two “special cases” listed above.

Suppose that $D \subset \mathbb{R}^2$ is, again, a closed and bounded rectangle and assume that $f(x, y)$ is bounded on D . That is, there is a constant $M > 0$ such that $|f(x, y)| \leq M$ for all $(x, y) \in D$. Then we have the following theorem.

Theorem 5.1.2. *Let $f(x, y)$ be a bounded function on a closed and bounded rectangle D . Then f is integrable on D if and only if the set*

$$\{(x, y) \in D : f \text{ is discontinuous at } (x, y)\}$$

*has **Jordan area** zero.*

We will not prove this here, but we will define the Jordan area of a planar set X and explain how the special cases above fit in this picture. Our starting point is to try to approximate X by unions of squares on a grid. We divide the plane into congruent squares R_{ij} (with the obvious notation) as shown below. Let

$$S^* := \text{sum of areas of } R_{ij} \text{ intersecting } X$$

$$S_* := \text{sum of areas of } R_{ij} \text{ contained within } X$$

We will use S^* and S_* as the upper and lower approximations for the area of X . By definition, we know that

$$S_* \leq (\text{Area of } X) \leq S^*$$

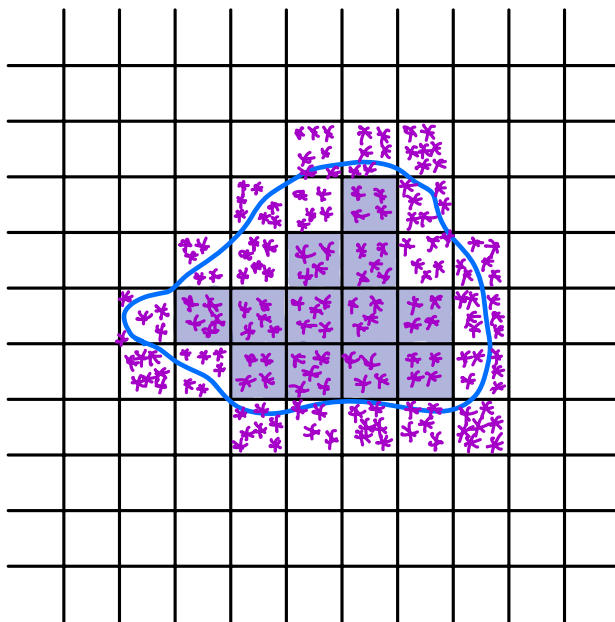


Figure 5.2: An approximation of X by grids. The solid colouring indicates squares contained entirely in X ; the starred squares are those that intersect X (but are not necessarily contained entirely in it).

We say that X is **Jordan measurable** if $\lim S^* = \lim S_*$ as the diameter of the grid goes to zero. We then define the area of X to be equal to this common limit. In other words, for a Jordan-measurable X , both the upper and lower approximations converge to the actual area of X as the grid diameter goes to 0. Further, we say that X has **Jordan area zero** if $\lim S^* = 0$.

We now revisit the special cases mentioned above:

- If a function $f(x, y)$ is continuous on a bounded rectangle D including its boundary, then $f(x, y)$ is bounded and its set of discontinuities is empty (therefore has Jordan area 0).
- Suppose that $f(x, y)$ is continuous on a compact domain X contained in a bounded rectangle D . Define the extension $\tilde{f}$ of f to D as in the second special case. Then the set of discontinuities of $\tilde{f}$ is contained in the boundary of X .

(why?). If the boundary of X is a union of finitely many curves of finite length, it must have Jordan area zero¹, so that Theorem 5.1.2 again applies.

- Theorem 5.1.2 also covers many cases that are not included in our special cases. For example, X could be bounded by fractal curves that have infinite length but still have Jordan area zero.

As an example of a set that does not have a well-defined Jordan area, consider

$$X := \{(x, y) \in [0, 1]^2 : x \text{ and } y \text{ are both rational}\}.$$

Then $S^* = 1$ and $S_* = 0$ for any grid. Why? Because any rectangle contains both points with rational coordinates and also points with irrational coordinates!

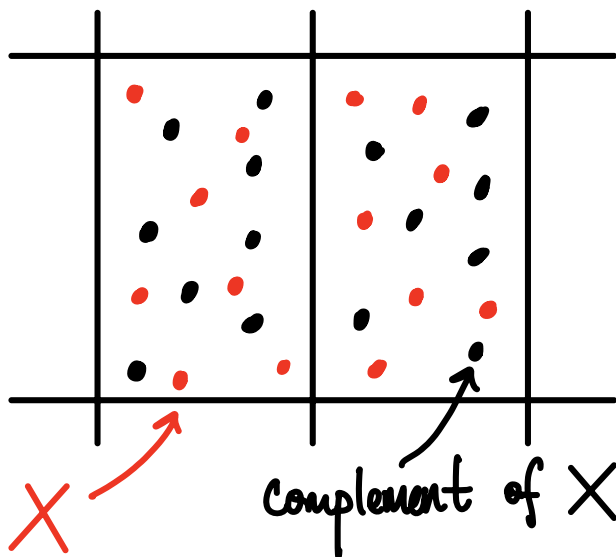


Figure 5.3: A set X that cannot be well approximated by grid squares.

Worked Problems

1. Let $f(x, y)$ be a function on the domain $D = \{(x, y) : 0 \leq x \leq 1, 0 \leq y \leq 1\}$, defined by

$$f(x, y) = \begin{cases} 1 & \text{if } x < y, \\ 0 & \text{if } x \geq y. \end{cases}$$

Prove **directly from the definition** (using Riemann sums) that $\iint_D f(x, y) = 1/2$, by following the procedure below.

¹This does require a proof. As an exercise, can you prove that the circle $x^2 + y^2 = 4$ has Jordan area zero?

(a) Let $\mathcal{P}$ be a partition of D , with partition rectangles $D_{ij} = \{(x, y) : x_{i-1} \leq x \leq x_i, y_{j-1} \leq y \leq y_j\}$. Let $\mathcal{R}$ be the Riemann sum associated with this partition. Let $d_{ij} = \sqrt{(x_i - x_{i-1})^2 + (y_j - y_{j-1})^2}$ be the length of the diagonal of D_{ij} . Assume that $0 < d_{ij} < \epsilon$ for all i, j , and for some $\epsilon < 0.001$. Prove that

$$\frac{1}{2} - 100\epsilon < \mathcal{R} < \frac{1}{2} + 100\epsilon.$$

(The constant 100 is not optimal and you may be able to get a better one.)

(b) What happens when $\epsilon \rightarrow 0$?

We start with (a). Let X be the union of those rectangles of $\mathcal{P}$ that intersect the diagonal $\{(x, x) : 0 \leq x \leq 1\}$. We claim that $X \subset Y$, where

$$Y := \{(x, y) \in D : x - 2\epsilon \leq y \leq x + 2\epsilon\}.$$

To prove this, let D_{ij} be a rectangle of $\mathcal{P}$ that intersects the diagonal. Then there is an x such that $0 \leq x \leq 1$ and $(x, x) \in D_{ij}$. The diagonal of D_{ij} has length at most ϵ , so that for any other point $(x', y') \in D_{ij}$ we have $|x' - x| \leq \epsilon$ and $|y' - x| \leq \epsilon$. Therefore

$$|x' - y'| = |(x' - x) + (x - y')| \leq |x - x'| + |y - y'| \leq 2\epsilon.$$

This means that D_{ij} is contained in Y .

Now we bound the Riemann sum $\mathcal{R}$. For an upper bound, we choose sampling points P_{ij} with $f(P_{ij}) = 1$ where possible. That choice can only be made for all rectangles D_{ij} either contained in the triangle below the line $y = x$ or intersecting that line. In both cases, D_{ij} lies below the line $y = x + 2\epsilon$. Therefore $\mathcal{R}$ is bounded from above by the area of $Y_+ := \{(x, y) \in D : y \leq x + 2\epsilon\}$. The complement of Y_+ in D is a right triangle of area $\frac{1}{2}(1 - 2\epsilon)^2$. Therefore

$$\begin{aligned} \mathcal{R} &\leq (\text{Area of } Y_+) \\ &= 1 - \frac{1}{2}(1 - 2\epsilon)^2 = 1 - \frac{1}{2}(1 - 4\epsilon + 4\epsilon^2) = \frac{1}{2} + 2\epsilon - 2\epsilon^2 < \frac{1}{2} + 2\epsilon. \end{aligned}$$

For a lower bound, we choose sampling points P_{ij} with $f(P_{ij}) = 0$ where possible. That choice can only be made for all rectangles D_{ij} either contained in the triangle above the line $y = x$ or intersecting that line – in both cases, above the line $y = x - 2\epsilon$. For every rectangle D_{ij} intersecting the triangle $Y_- := \{(x, y) \in D : y \geq x - 2\epsilon\}$, we have to choose $f(P_{ij}) = 1$. Therefore $\mathcal{R}$ is bounded from below by the area of Y_- . This is a right triangle of area $\frac{1}{2}(1 - 2\epsilon)^2$. Therefore

$$\begin{aligned} \mathcal{R} &\geq (\text{Area of } Y_-) \\ &= \frac{1}{2}(1 - 2\epsilon)^2 = \frac{1}{2}(1 - 4\epsilon + 4\epsilon^2) = \frac{1}{2} - 2\epsilon + 2\epsilon^2 > \frac{1}{2} - 2\epsilon. \end{aligned}$$

Now we can answer the question in (b). As $\epsilon \rightarrow 0$, both $\frac{1}{2} - 2\epsilon$ and $\frac{1}{2} + 2\epsilon$ approach $\frac{1}{2}$, so that $\mathcal{R} \rightarrow \frac{1}{2}$.

Practice Problems

1. Let $f(x, y) = x + y$, and let $D = \{(x, y) : 0 \leq x \leq 1, 0 \leq y \leq 1\}$. Prove that for any $\epsilon > 0$, there is a $\delta > 0$ such that the following holds:

Let $\mathcal{P}_1, \mathcal{P}_2$ be two partitions of D together with choices of sampling points. If $\|\mathcal{P}_1\| < \delta$ and $\|\mathcal{P}_2\| < \delta$, then

$$|\mathcal{R}(f, \mathcal{P}_1) - \mathcal{R}(f, \mathcal{P}_2)| < \epsilon.$$

(This property is used in the proof that f is integrable on D .)

Hint: let $\mathcal{P}$ be a common refinement of $\mathcal{P}_1$ and $\mathcal{P}_2$ – that is, a partition that includes all the partition points of both $\mathcal{P}_1$ and $\mathcal{P}_2$. What can you say about $\mathcal{R}(f, \mathcal{P}_1) - \mathcal{R}(f, \mathcal{P})$ and $\mathcal{R}(f, \mathcal{P}_2) - \mathcal{R}(f, \mathcal{P})$?

2. Let $D = \{(x, y) : 0 \leq x \leq 1, 0 \leq y \leq 1\}$, and let $S = \{P_1, \dots, P_n\} \subset D$ be a set of finitely many points. Let $f(x, y)$ be a function such that $f(P_1) = c_1, \dots, f(P_n) = c_n$ for some arbitrary (but finite) values $c_1, \dots, c_n \in \mathbb{R}$, and $f(x, y) = 0$ for all $(x, y) \notin S$.

Let $\mathcal{P}$ be a partition of D , with partition rectangles $D_{ij} = \{(x, y) : x_{i-1} \leq x \leq x_i, y_{j-1} \leq y \leq y_j\}$. Let $\mathcal{R}$ be the Riemann sum for f associated with this partition. Let $d_{ij} = \sqrt{(x_i - x_{i-1})^2 + (y_j - y_{j-1})^2}$ be the length of the diagonal of D_{ij} . Assume that $0 < d_{ij} < \epsilon$ for all i, j . Prove that

$$|\mathcal{R}| < C\epsilon^2$$

for some constant C that may depend on $n, c_1, \dots, c_n$, but is independent of ϵ . What does this say about the integral $\iint_D f(x, y) dA$?

5.2 Integration in Two Real Variables

5.2.1 Iterated Integrals in Two Variables

In order to actually compute integrals in two variables, we will write them as iterated integrals and then use antiderivatives. The easiest case is when the region of integration is a rectangle. Here is another theorem that we will not prove (a special case of Fubini's Theorem).

Theorem 5.2.1. *Let $f(x, y)$ be a bounded function on a closed and bounded rectangle $D = \{(x, y) \in \mathbb{R}^2 : a \leq x \leq b, c \leq y \leq d\}$. Assume that f is integrable on D . Then*

$$\iint_D f(x, y) dA = \int_a^b \left(\int_c^d f(x, y) dy \right) dx = \int_c^d \left(\int_a^b f(x, y) dx \right) dy.$$

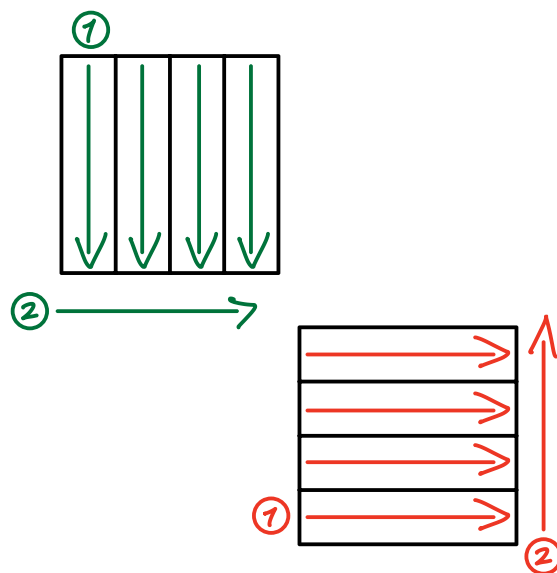


Figure 5.4: If the region of integration is a rectangle, we can arrange the order of summation in Riemann sums in either one of the two ways shown here.

Intuitively, if we think about the double integral as being approximated by Riemann sums, then the two iterated integrals correspond to the two orders of summation shown in Figure 5.4.

When the region of integration is not a rectangle, it may still be possible to write the region of integration in one of the two ways shown in Figure 5.5.

Suppose D is y -simple as defined in Figure 5.5, f is continuous on D , and $c(x)$ and $d(x)$ are continuous on the interval $[a, b]$. Then, we write the iterated (double) integral of $f(x, y)$ over D as

$$\iint_D f(x, y) dA = \int_a^b \left[\int_{c(x)}^{d(x)} f(x, y) dy \right] dx.$$

Of course, we have a similar expression when D is x -simple, with the x and y variables interchanged. We always perform summation in the simple direction first. If D is both x -simple and y -simple, we can choose whichever order we prefer.

5.2.2 Applications of Double Integrals

Let $D \subset \mathbb{R}^2$ be a bounded region, and let $f(x, y)$ be a function defined and bounded on D . We assume that both f and D are nice enough so that the integral $\iint_D f dA$ exists (see Section 5.1 for sufficient conditions).

One basic application is as follows:

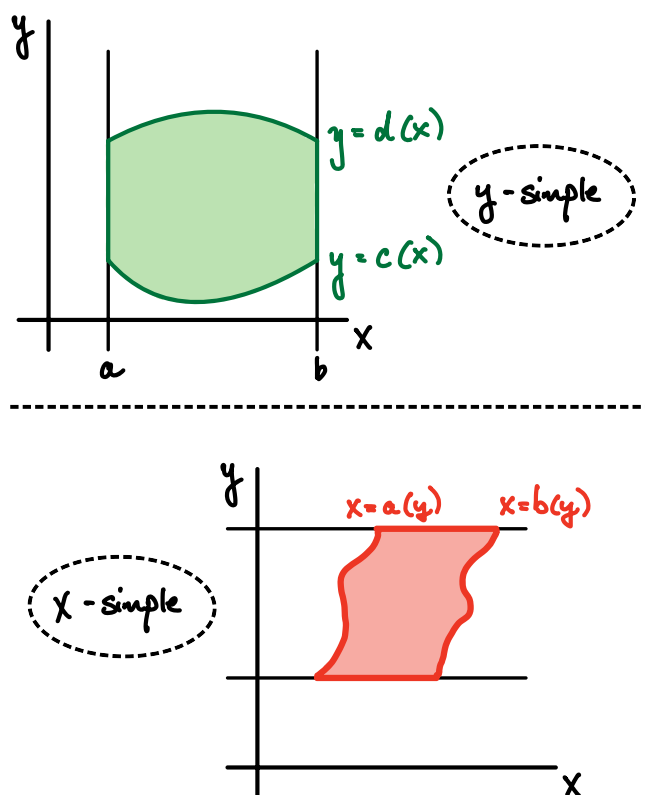


Figure 5.5: Two cases of sets D where the range of the x and/or y coordinates can be written as functions of the other variable.

$$\iint_D 1 dA = (\text{area of } D). \quad (5.2.1)$$

For an explanation, you may want to revisit the notes on Jordan area in Section 5.1. The upper and lower approximations, S^* and S_* , to the area of a bounded planar region X are actually Riemann sums for the function $f(x, y)$ that is equal to 1 on X and 0 on its complement. (This is often called the *characteristic function* or *indicator function* of X .) Both the Jordan area of X and the integral $\iint_X f dA$ are defined as the limit of such sums as the partition diameter goes to 0.

We can use double integrals to evaluate volumes under graphs of functions of two variables. If $f(x, y) \geq 0$ for all $(x, y) \in D$, then

$$\iint_D f(x, y) dA = (\text{volume under the graph of } z = f(x, y) \text{ above } D). \quad (5.2.2)$$

Here, by “above D ”, we mean above D when D is viewed as a subset of the plane $z = 0$ in $\mathbb{R}^3$. This is similar to the interpretation of an integral $\int_a^b f(x) dx$ of a nonnegative function $f(x)$ as the area under its graph. See Worked Problem 5 for an example.

We also use integrals to calculate averages of functions. If the region $D \subset \mathbb{R}^2$ has finite area, then

$$(\text{Average value of } f(x, y) \text{ on } D) = \frac{\iint_D f(x, y) dA}{\text{Area of } D}. \quad (5.2.3)$$

As a special case of the average value formula, if $D \subset \mathbb{R}^2$ is a region, then the **centroid** of D is the point $(\bar{x}, \bar{y}) \in \mathbb{R}^2$ where

$$\bar{x} := \frac{\iint_D x dA}{\text{Area of } D} \quad \bar{y} := \frac{\iint_D y dA}{\text{Area of } D}.$$

In words, the centroid encodes the average values of the x and y coordinates on D . (Note that the centroid is not always a point of D .)

Worked Problems

1. Let $D := \{(x, y) : 0 \leq x \leq 2, 0 \leq y \leq 3\}$. Evaluate the integral of $f(x, y) = x^2 y$ over D .

We evaluate the integral of $f(x, y)$ over D as an iterated integral. Here, D is a rectangle, therefore both x -simple and y -simple. In this example, the order of integration does not make much difference.

$$\begin{aligned} \iint_D x^2 y dA &= \int_0^2 \int_0^3 x^2 y dy dx = \int_0^2 \frac{x^2 y^2}{2} \Big|_{y=0}^{y=3} dx \\ &= \int_0^2 \frac{9}{2} x^2 dx = \frac{3}{2} x^3 \Big|_0^2 \\ &= 12 \end{aligned}$$

2. Find the integral of $f(x, y) = x^2 y$ on $D := \{(x, y) : 0 \leq x \leq 1, 0 \leq y \leq x\}$.

The region $D \subset \mathbb{R}^2$ is a right triangle with vertices $(0, 0)$, $(1, 0)$ and $(1, 1)$.

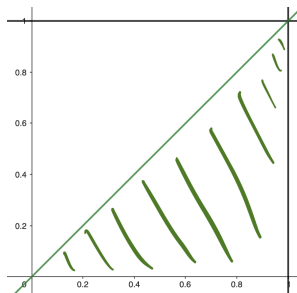


Figure 5.6: The triangle with vertices $(0, 0)$, $(1, 0)$ and $(1, 1)$.

This is, again, both an x -simple and a y -simple region. If we follow the y -simple order of integration, we see that $y = y(x) = x$ at the upper boundary of D and $y = y(x) = 0$ at the lower boundary. Evaluating as an iterated integral then gives

$$\begin{aligned}\iint_D x^2 y dA &= \int_0^1 \int_0^x x^2 y dy dx = \int_0^1 \frac{x^2 y^2}{2} \Big|_0^x dx \\ &= \int_0^1 \frac{x^4}{2} dx = \frac{x^5}{10} \Big|_0^1 \\ &= 10\end{aligned}$$

3. Find the integral of $f(x, y) = y$ on $D = \{(x, y) : 0 \leq x \leq 1, 0 \leq y \leq x^2\}$.

We again evaluate this integral on D as a y -simple domain.

$$\begin{aligned}\iint_D y dA &= \int_0^1 \left[\int_0^{x^2} y dy \right] dx = \int_0^1 \frac{y^2}{2} \Big|_0^{x^2} dx = \int_0^1 \frac{x^4}{2} dx = \frac{x^5}{10} \Big|_0^1 \\ &= \frac{1}{10}\end{aligned}$$

4. Find the integral of $f(x, y) = 3y$ over the region D , where D is the region bounded by the curves $xy^2 = 1$, $y = x$, $x = 0$ and $y = 3$.

The region D is slightly more complicated than those in the previous examples. We give a picture in Figure 5.7.

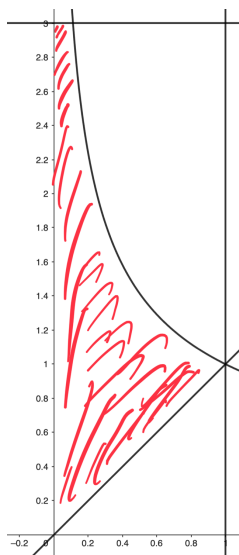


Figure 5.7: The region D in Worked Problem 4.

Notice that the curves $y = x$ and $xy^2 = 1$ intersect at the point $(x, y) = (1, 1)$. Moreover, if we split the region D as $D = D_1 \cup D_2$ where D_1 contains the part of D below the line $y = 1$ and D_2 contains the part above, we see that D_1 and D_2 are both x -simple. Importantly, they are also disjoint except for boundary, so that we can just add up the integrals. This allows us to evaluate the area integral as

$$\iint_D 3y dA = \iint_{D_1} 3y dA + \iint_{D_2} 3y dA.$$

Evaluating each integral separately, we have

$$\iint_{D_1} 3y dA = \int_0^1 \int_0^y 3y dx dy = \int_0^1 3y^2 dy = y^3 \Big|_0^1 = 1$$

and then

$$\iint_{D_2} 3y dA = \int_1^3 \int_0^{y^{-2}} 3y dx dy = \int_1^3 \frac{3}{y} dy = 3 \ln y \Big|_1^3 = 3 \ln 3$$

Summing together, we have

$$\iint_D 3y dA = 1 + 3 \ln 3.$$

5. Find the volume above the square $D = [-1, 1]^2$ and below the paraboloid $z = 4 - x^2 - y^2$.

Figure 5.8 shows a rough diagram illustrating the region $D \subset \mathbb{R}^2$ (which we view as a subset of the plane $z = 0$ in $\mathbb{R}^3$) and the portion of the paraboloid above D . The volume of this region can be written as the iterated integral

$$\iint_D (4 - x^2 - y^2) dA = \int_{-1}^1 \int_{-1}^1 (4 - x^2 - y^2) dx dy,$$

which, by using the symmetry of the surface, we evaluate as

$$\begin{aligned} 4 \int_0^1 \int_0^1 (4 - x^2 - y^2) dx dy &= 4 \int_0^1 \left[4x - \frac{x^3}{3} - y^2 x \right]_{x=0}^1 dy \\ &= 4 \int_0^1 \left(\frac{11}{3} - y^2 \right) dy = 4 \left[\frac{11}{3} y - \frac{y^3}{3} \right]_0^1 \\ &= \frac{40}{3}. \end{aligned}$$

Therefore, we have:

$$\text{Volume} = \iint_D (4 - x^2 - y^2) dx dy = \frac{40}{3}.$$

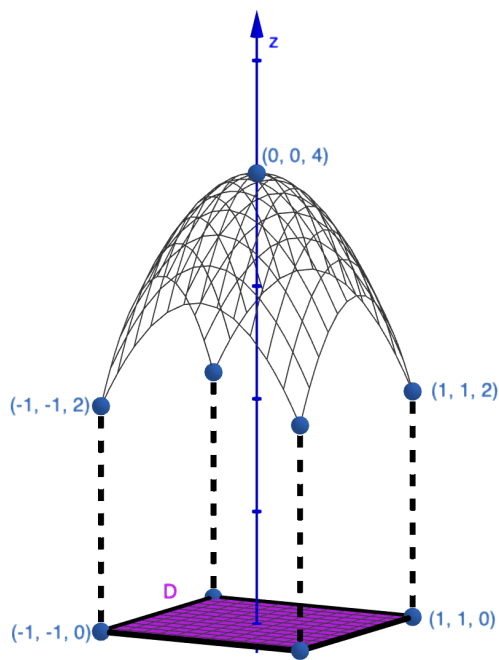


Figure 5.8: The region D in the xy -plane, and the 3-dimensional region above D and below the paraboloid.

6. Find the volume above the triangle with vertices $(0, 0)$, $(0, 1)$ and $(1, 0)$ and under the plane $z = 2x + y$.

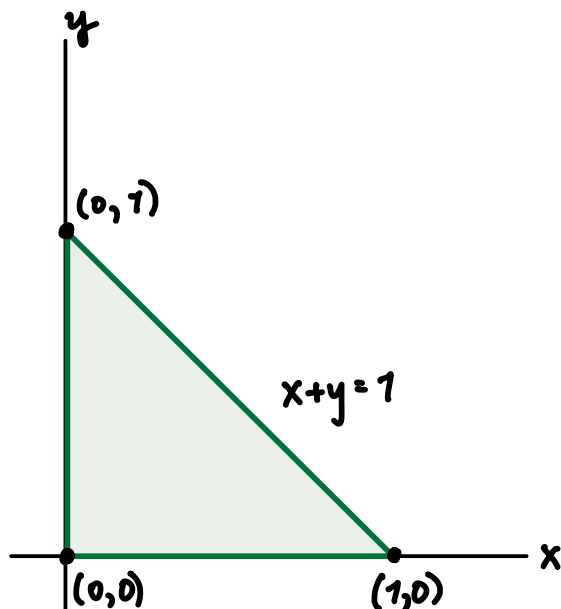
The triangular region D is shown in Figure 5.9. We evaluate the volume as the following iterated integral

$$\iint_D (2x + y) dA = \int_0^1 dx \int_0^{1-x} (2x + y) dy.$$

Note that we wrote the dx variable in front of the dy integral—this is just another way of writing double integrals you will likely encounter. We then calculate

$$\begin{aligned} \int_0^1 dx \int_0^{1-x} (2x + y) dy &= \int_0^1 \left[2xy - \frac{y^2}{2} \right]_{y=0}^{1-x} dx = \int_0^1 \left[2x(1-x) + \frac{(1-x)^2}{2} \right] dx \\ &= \int_0^1 \left[-\frac{3x^2}{2} + x + \frac{1}{2} \right] dx = \left[-\frac{x^3}{2} + \frac{x^2}{2} + \frac{x}{2} \right]_{x=0}^1 \\ &= \frac{1}{2}. \end{aligned}$$

So, the volume enclosed by this triangular region and the given plane is $\frac{1}{2}$.

Figure 5.9: The triangle with vertices $(0,0)$, $(0,1)$ and $(1,0)$

7. Find the average value $f(x,y) = x \sin(x+y)$ on the rectangle $D := \{(x,y) : 0 \leq x \leq 1, 0 \leq y \leq \pi\}$.

We use the averaging formula in (5.2.3). We first find the area of the region, which is $1 \cdot \pi = \pi$. We then evaluate the integral of $f(x,y)$ over D as the iterated integral

$$\iint_D f(x,y) dA = \int_0^1 \int_0^\pi x \sin(x+y) dy dx$$

The calculation is

$$\begin{aligned} \int_0^1 \int_0^\pi x \sin(x+y) dy dx &= \int_0^1 x \left(-\cos(x+y) \right) \Big|_{y=0}^\pi dx \\ &= \int_0^1 x (\cos x - \cos(\pi+x)) dx \\ &= \int_0^1 2x \cos x dx \end{aligned}$$

At the last step of the above calculation, we used that $\cos(\pi+x) = -\cos(x)$ for any real-number x . Therefore

$$\iint_D f(x,y) dA = \int_0^1 2x \cos(x) dx = 2 \sin(1) + 2 \cos(1) - 2,$$

where the last equality is obtained via integration by parts. (Exercise: complete the calculations!) Hence, from the average formula (5.2.3), we have

$$\text{Average of } f(x, y) \text{ on } D = \frac{2}{\pi}(\sin(1) + \cos(1) - 1)$$

Practice Problems

1. Evaluate the integral $\iint_D 3dA$, if D is the region bounded by the parabola $y^2 - x - 5 = 0$ and the line $x + 2y = 3$.
2. Reverse the order of integration to evaluate the following integrals. (In this question, the functions to be integrated (such as $\sin(y^3)$) do not have antiderivatives that could be written in terms of elementary functions. The only way to evaluate the integrals is to interchange the order of integration and then take advantage of the additional factors from integrating in x .)

(a) $\int_0^1 \int_{\sqrt{x}}^1 \cos(y^3 + 1) dy dx.$

(b) $\int_0^1 \int_{x^2}^1 x^3 \sin(y^3) dy dx.$

3. Change the order of integration in the integral

$$\int_0^4 \int_{x/2}^x \cos(y^2) dy dx + \int_4^8 \int_{x/2}^4 \cos(y^2) dy dx.$$

Then evaluate the integral.

4. Evaluate the double integrals. Use any method you like.

(a) $\iint_D y dA$, D is the triangle in the xy -plane with vertices $(0, 0)$, $(1, 2)$, $(3, 0)$.

(b) $\int_1^2 \int_{1/x}^1 ye^{xy} dy dx$

5. Find the volume under the plane $z = x$ and above the region in the xy -plane bounded by $y = x$ and $y = x^2 - x$.

6. Evaluate the double integral $\iint_D x \, dA$, where D is the triangle with vertices $(0, 0)$, $(2, 1)$, $(4, 0)$.
7. Let T be the triangle with vertices $(0, 0)$, $(s, 0)$ and (s, t) for some $s, t > 0$. Determine the centroid of T .

5.3 Improper Integrals

Until this time, we have only considered the integral of $f(x, y)$ over D in the special case where D is bounded and also $f(x, y)$ is bounded (i.e. there is some $M > 0$ so that $|f(x, y)| \leq M$ for all $(x, y) \in D$).

If one (or both) of these assumptions fail, then the integral of $f(x, y)$ over D might not be well defined even if we allow integrals to have infinite values (see below for an example). However, if we assume that $f \geq 0$ on D , then:

- either $\iint_D f(x, y) \, dA$ is finite (or **convergent**) and can be evaluated as a limit of real numbers;
- or, the integral takes on infinite value (that is, it is **divergent**).

For non-negative functions, we can still evaluate improper double integrals as iterated (and possibly improper) one-dimensional integrals in the variables x and y . See examples in the Worked Problems section.

5.3.1 Determining Convergence/Divergence of Improper Integrals

In many cases it is possible to use inequalities and estimates to determine whether an improper integral is convergent or divergent, even if we cannot (or would rather not) evaluate the actual value of the integral.

The general principle is as follows. Suppose that $D \subset \mathbb{R}^2$ is some region and that $f(x, y)$ and $g(x, y)$ satisfy:

$$0 \leq g(x, y) \leq f(x, y), \text{ for all } (x, y) \in D.$$

Then, we necessarily have

$$\iint_D f(x, y) \, dA \geq \iint_D g(x, y) \, dA$$

Therefore:

$$\iint_D g(x, y) \, dA = \infty \Rightarrow \iint_D f(x, y) \, dA = \infty,$$

$$\iint_D f(x, y) dA < \infty \Rightarrow \iint_D g(x, y) < \infty.$$

5.3.2 Without positivity, improper integrals may fail to exist.

All of the above methods (estimates, iterated integrals) assume that $f(x, y)$ is non-negative on the domain of integration $D \subset \mathbb{R}^2$. This guarantees that its integral either converges to a number or else is infinite. If $f(x, y)$ takes on both positive and negative values on its domain D , the area integral $\iint_D f dA$ may be undefined for other reasons. To show how this can happen, we construct a function $f(x, y)$ on the domain $D := \{(x, y) : x \geq 0, y \geq 0\}$ such that

$$\int_0^\infty \int_0^\infty f(x, y) dx dy$$

converges, but

$$\int_0^\infty \int_0^\infty f(x, y) dy dx$$

diverges.

The function we choose is defined using the following rule. For $x, y \geq 0$, and for $n = 1, 2, \dots$, let

$$f(x, y) = \begin{cases} n, & \text{if } n-1 \leq x < n \text{ and } n-1 < y \leq n \\ -n, & \text{if } n \leq x < n+1 \text{ and } n-1 < y \leq n. \end{cases}$$

We also let $f(x, y) = 0$ for all other pairs (x, y) . The values of $f(x, y)$ are shown below.

Now, for any $c > 0$, the horizontal line $y = c$ intersects a green square with label n and a red square with label $-n$; moreover, the intersection of this line with the green square has the same length as the intersection with the red square. Consequently,

$$\int_0^\infty f(x, y) dx = \lim_{N \rightarrow \infty} \int_0^N f(x, y) dx = 0 \text{ for each } y > 0.$$

However, let us now see what happens if we use vertical lines $x = c$ instead. For $0 \leq c < 1$, the line intersects only the first green square with label 1 in an interval of length 1. For $c \geq 1$, the vertical line $x = c$ intersects a green square with label $n+1$ and a red square with label $-n$, with both intersections having length 1. As a consequence, we see that

$$\int_0^\infty f(x, y) dy = \lim_{N \rightarrow \infty} \int_0^N f(x, y) dy = 1 \text{ for each } x > 0.$$

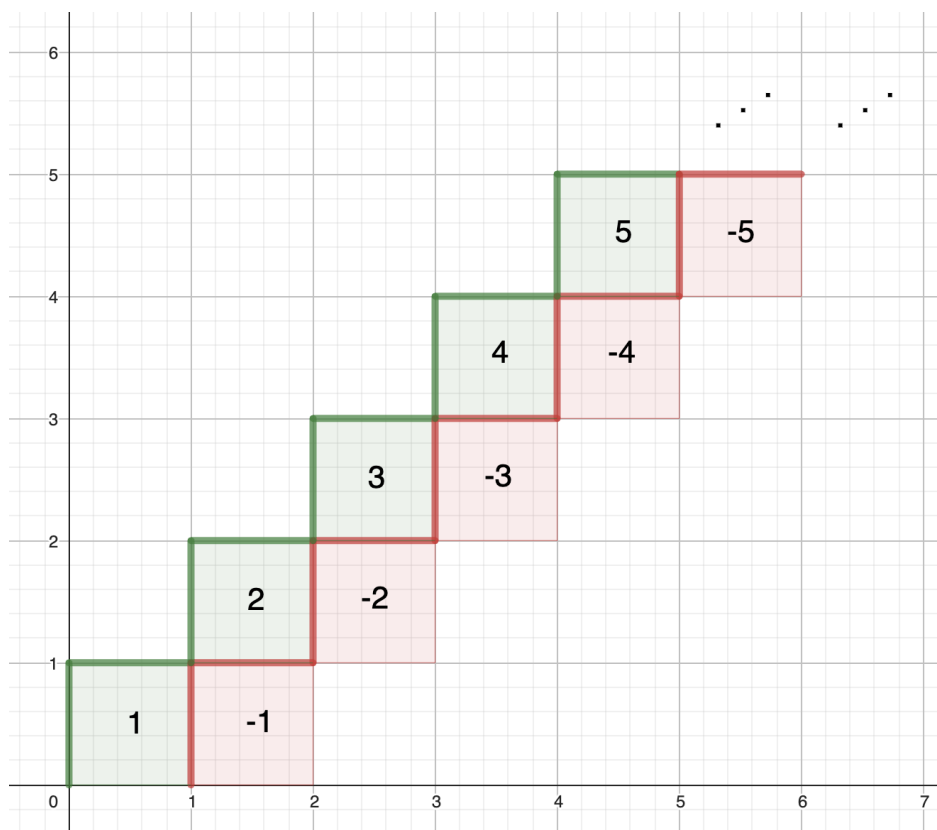


Figure 5.10: A function whose double integral is not well defined. The vertical dots at the upper right-hand corner are to remind you that this sequence of squares continues infinitely.

So, evaluating the iterated integrals of f gives

$$\int_0^\infty \int_0^\infty f(x, y) dx dy = \int_0^\infty 0 dy = 0$$

$$\int_0^\infty \int_0^\infty f(x, y) dy dx = \int_0^\infty 1 dx = \infty.$$

Hence the iterated integrals attain different values depending upon the order of integration. Consequently, the integral $\iint_D f(x, y) dA$ fails to exist.

Worked Problems

1. Determine, for each of the following functions f and domains D , whether the domain is bounded and whether $f(x, y)$ is bounded on D :

(a) $f(x, y) = \frac{1}{x}$ and $D = [0, 1]^2$;

(b) $f(x, y) = \frac{1}{x}$ and $D = \{(x, y) : 0 \leq x \leq 1, 0 \leq y \leq x^2\}$;

(c) $f(x, y) = \frac{1}{x^3}$ and $D = \{(x, y) : 0 \leq x \leq 1, 0 \leq y \leq x^2\}$;

(d) $f(x, y) = \frac{1}{x^2 y^2}$ and $D := \{(x, y) : x \geq 2, y \geq 2\}$.

Then, evaluate the associated improper integrals to determine if they converge or diverge.

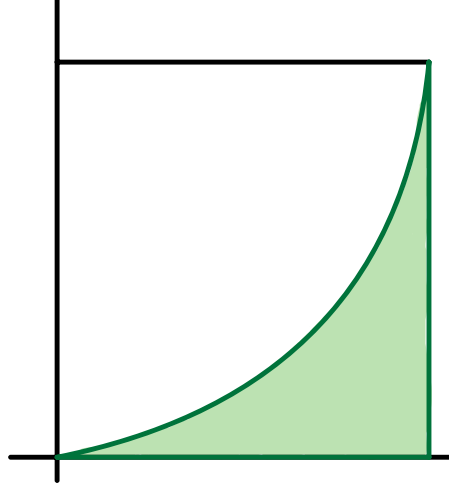


Figure 5.11: The region D in (b) and (c)

The solutions are as follows:

- (a) The domain is a closed square of side-length 1, therefore is bounded. However, since $\frac{1}{x} \rightarrow \infty$ as $(x, y) \rightarrow (0, 0)$, we see that $f(x, y)$ is not bounded on D . The iterated integral is

$$\iint_D \frac{1}{x} dA = \int_0^1 \int_0^1 \frac{1}{x} dy dx = \int_0^1 \frac{dx}{x} = +\infty.$$

- (b) The domain is, again, bounded, since $|x| \leq 1$ and $|y| \leq x^2 \leq 1^2 = 1$. Again, $f(x, y)$ is unbounded since $\frac{1}{x} \rightarrow 0$ as $(x, y) \rightarrow (0, 0)$. Nevertheless, we now have that

$$\iint_D f(x, y) dA = \int_0^1 \int_0^{x^2} \frac{1}{x} dy dx = \int_0^1 x dx = \frac{1}{2},$$

and so the integral of $f(x, y)$ converges over this choice of domain.

- (c) The domain D is the same as in (b) and is thus bounded. We still have $f(x, y) = \frac{1}{x^3} \rightarrow \infty$ as $(x, y) \rightarrow (0, 0)$, therefore $f(x, y)$ is again unbounded on D . We have

$$\iint_D f(x, y) dA = \int_0^1 \int_0^{x^2} \frac{1}{x^3} dy dx = \int_0^1 \frac{1}{x} dx = \infty,$$

so that the integral now diverges! Comparing (a), (b), (c) shows that both the domain and the function $f(x, y)$ matter when determining convergence and divergence of the improper integral.

- (d) In this example, the domain is unbounded, since both x and y may be as large as we wish. However, in this domain, we have:

$$f(x, y) = \frac{1}{x^2 y^2} \leq \frac{1}{2^2 \cdot 2^2} = \frac{1}{16},$$

so that f is indeed bounded (even though the domain is unbounded!) Evaluating the associated improper integral gives

$$\begin{aligned} \iint_D f(x, y) dA &= \int_2^\infty \int_2^\infty \frac{1}{x^2 y^2} dy dx \\ &= 2 \int_2^\infty \left[-\frac{1}{x^2 y} \right]_{y=2}^\infty dx = \int_2^\infty \frac{1}{2x^2} dx = -\frac{1}{2x} \Big|_2^\infty \\ &= \frac{1}{4}. \end{aligned}$$

And so, the integral of $f(x, y)$ over the unbounded domain D converges.

2. For the following functions f and domains:

(a) $f(x, y) = \frac{1}{\sqrt{x^2 + y^2}}$ and $D = \{(x, y) : 0 \leq x \leq 1, 0 \leq y \leq x^2\}$;

(b) $f(x, y) = \frac{1}{10 + x + \sin y}$ and $D = \{(x, y) : x \geq 0, 0 \leq y \leq 1\}$,

use the convergence/divergence test to determine the behaviour of the associated improper integral.

The solutions are as follows:

- (a) We estimate

$$f(x, y) = \frac{1}{\sqrt{x^2 + y^2}} \leq \frac{1}{\sqrt{x^2}} = \frac{1}{x}$$

where we used that $x \geq 0$ on D in the last equality. However, from Worked Problem 1(b), we know that

$$\iint_D \frac{1}{x} dA < \infty \Rightarrow \iint_D \frac{1}{\sqrt{x^2 + y^2}} dA < \infty.$$

We are done.

- (b) Notice that, on our domain D , we have $0 \leq y \leq 1$ so that $0 \leq \sin y \leq \sin 1 < \sin \frac{\pi}{2} = 1$. This is a consequence of the region having bounded y values, even though the x values are unbounded, as shown in Figure 5.12.

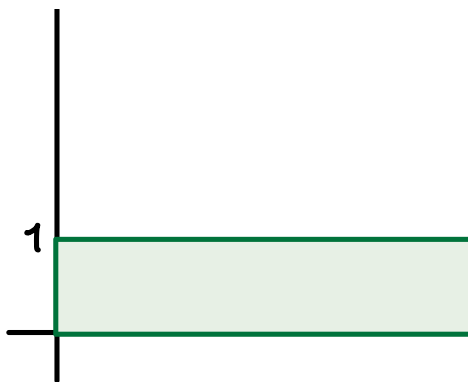


Figure 5.12: The unbounded domain D in (b)

Hence

$$f(x, y) = \frac{1}{10 + x + \sin y} \geq \frac{1}{10 + x + 1} = \frac{1}{x + 11}, \text{ for all } (x, y) \in D.$$

Let $g(x, y) = \frac{1}{11+x}$. Then,

$$\iint_D g(x, y) dA = \int_0^\infty \int_0^1 \frac{1}{x + 11} dy dx = \int_0^\infty \frac{1}{x + 11} dx = \log(x + 11) \Big|_{x=0}^\infty = \infty$$

Now, since $f(x, y) \geq g(x, y)$ for all (x, y) in D , we see that

$$\iint_D f(x, y) dA \geq \iint_D g(x, y) dA = \infty,$$

and so the integral of f over D diverges.

Practice Problems

1. Determine whether the integral

$$\iint_Q \frac{x}{(1+x)(1+y)} dA,$$

where Q is the first quadrant of the xy -plane, converges or diverges. If it converges, evaluate it.

2. Determine whether the integral

$$\iint_Q e^{y-x} dA$$

converges or diverges, if

$$Q := \{(x, y) : x \geq 1, 0 \leq y \leq \ln(x^2 + x + 1)\}$$

If it converges, evaluate it.

3. Determine whether the integral

$$\iint_D \frac{e^y}{y^{3/2}} dA$$

is convergent, if D is the triangle in the xy -plane with vertices $(0, 0)$, $(1, 0)$, $(1, 1)$. (You do not have to evaluate the integral.)

4. Let D be the region $\{(x, y) : x \geq 2, 0 \leq y \leq \pi\}$. Prove that the integral

$$\iint_D \frac{1 + e^{-x}}{x^2 - \sin^2 y} dA$$

is convergent, by comparing it to a different integral that is easier to evaluate.

5.4 Change of Variables

5.4.1 Polar coordinates

We now consider change of variables in double integration. One of the more common changes of variables is the following.

Polar Coordinates in $\mathbb{R}^2$

For (x, y) in $\mathbb{R}^2$, let

$$x = x(r, \theta) = r \cos \theta \quad y = y(r, \theta) = r \sin \theta, \quad (5.4.1)$$

where

$$\begin{aligned} r = r(x, y) &= \sqrt{x^2 + y^2} && \text{(Distance to origin)} \\ \theta = \theta(x, y) &&& \text{(Angle with horizontal axis)} \end{aligned}$$

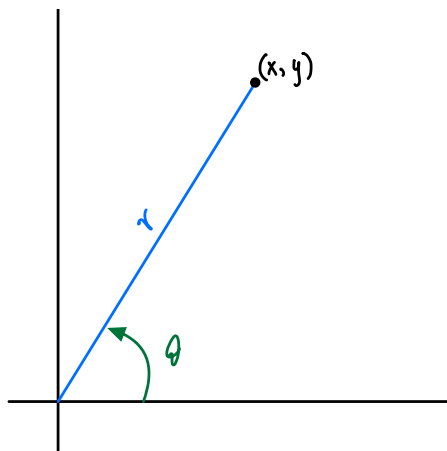


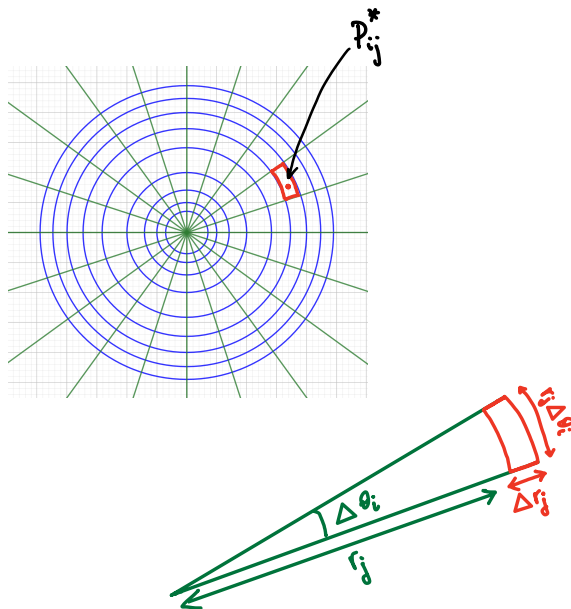
Figure 5.13: A point (x, y) given in Cartesian coordinates can be rewritten in terms of its distance to the origin $r \geq 0$ and its angular orientation relative to the horizontal axis $0 \leq \theta < 2\pi$.

We will always assume that $r = \sqrt{x^2 + y^2} \geq 0$; the usual range of θ is $[0, 2\pi]$, but it can be more convenient sometimes to use other intervals, for example $[-\pi, \pi]$. Expressing θ in terms of x and y is a little bit tricky. We have $\tan \theta = y/x$, but this does not determine θ uniquely since $\tan \theta = \tan(\theta + \pi)$. We just have to work back from the $x = r \cos \theta$ and $y = r \sin \theta$ formulas.

Many domains (e.g. discs) and functions (e.g. radially symmetric functions) have simple representations in polar coordinates, meaning that our calculations can be much simpler if we make the change of variables $(x, y) \mapsto (r \cos \theta, r \sin \theta)$. As such, we want to develop a formula for integration in polar coordinates.

5.4.2 Riemann Sums and Integrals in Polar Coordinates

Assume that f is continuous on a compact domain D , so that integrability is guaranteed.



From the diagram, we see that

$$(\text{Area element}) \quad \Delta A_{ij} \approx \Delta r_j \cdot (r_j \Delta \theta_i) = r_j \Delta \theta_i \Delta r_j.$$

Hence, if P_{ij}^* is any element taken from the associated region, then we have

$$(\text{Riemann Sum}) \quad \sum_{i,j} f(P_{ij}^*) \Delta A_{ij} \approx \sum_{i,j} f(P_{ij}^*) \underbrace{r_j \Delta \theta_i \Delta r_j}_{\text{Area element}}$$

Taking an associated limit over partitions (as in Section 5.1 in Cartesian coordinates) we get

$$(\text{Integral in Polar Coordinates}) = \iint_D f(r, \theta) r d\theta dr.$$

The term $r d\theta dr$ in the integral comes from the term $r_j \Delta \theta_i \Delta r_j$ in the Riemann sum.

The Worked Problems below show how to use the above conversion formula in practice. In addition to evaluating proper integrals, we often use polar coordinates to evaluate improper integrals and/or to determine their convergence. Examples of that are also given below.

In the calculation of the conversion formula, we used $r_j \Delta\theta_i \Delta r_j$ as an approximate value of the area element. Actually, the indicated area can be calculated exactly. If $\Delta r_j = r_j - r_{j-1}$, then the area of the angular segment of the annulus is

$$\frac{\Delta\theta_i}{2\pi}(\pi r_j^2 - \pi r_{j-1}^2) = \frac{1}{2}\Delta\theta_i(r_j + r_{j-1})(r_j - r_{j-1}) = \frac{r_j + r_{j-1}}{2}\Delta\theta_i\Delta r_j.$$

This is the same as our approximate value, but with a factor of $(r_j + r_{j-1})/2$ instead of just r_j . If we made this change in the Riemann sum above, it would still approximate the same integral.

5.4.3 More General Changes of Variables

We will also use more general changes of variables $x = x(u, v)$, $y = y(u, v)$. We will assume that both x and y are C^1 in the u and v variables. We will need the **Jacobian** $\left|\frac{\partial(x, y)}{\partial(u, v)}\right|$ defined in Section 3.4.

More General Variable Changes

If we assume that

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} \neq 0,$$

then the conversion formula for area elements is

$$dA = dx dy = \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du dv \quad (5.4.2)$$

The conversion formula for polar coordinates follows this scheme with $u = r$ and $v = \theta$. Both $x(r, \theta) = r \cos \theta$ and $y(r, \theta) = r \sin \theta$ are C^1 , with Jacobian

$$\left| \frac{\partial(x, y)}{\partial(r, \theta)} \right| = \begin{vmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{vmatrix} = r \cos^2 \theta + r \sin^2 \theta = r \underbrace{(\cos^2 \theta + \sin^2 \theta)}_{=1} = r.$$

so that the area element is $dA = r dr d\theta$ as discussed above. Question 4 in the Worked Problems Section features an integral involving a more general change of variables.

The assumption that $\frac{\partial(x, y)}{\partial(u, v)} \neq 0$ can sometimes be relaxed a little bit. In particular, we will sometimes allow the Jacobian to be 0 at isolated points or on the boundary of the domain of integration. Polar coordinates provide a good example of this: if we integrate in polar coordinates on any region containing the origin, then the Jacobian at $(x, y) = (0, 0)$ is $r = 0$. This is fine. However, we do not want “degenerate” changes of variables. For example, if we let $x = u + v$ and also $y = u + v$ (with Jacobian equal to 0 everywhere), then that does not even define u and v uniquely in the first place.

5.4.4 Geometric interpretation of the Jacobian

The Jacobian term appearing in the area element formula (5.4.2) has a geometric interpretation, which we now explore. For simplicity, let us assume that the change of variables $(x, y) \mapsto (u, v)$ is linear, so that

$$\begin{pmatrix} x \\ y \end{pmatrix} = A \begin{pmatrix} u \\ v \end{pmatrix}, \text{ where } A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

Differentiating in u and v gets us $\begin{pmatrix} x_u & x_v \\ y_u & y_v \end{pmatrix} = A$ (you should check this!), so that the conversion formula above gives $dx dy = |\det(A)| du dv$.

Now, let us see what this means geometrically. In Cartesian coordinates, the area element corresponds to a rectangle, with sides parallel to the $\mathbf{i}$ and $\mathbf{j}$ directions, and with dimensions $\Delta u \times \Delta v$. Applying the linear transformation gives

$$A(\Delta u \mathbf{i}) = \Delta u (A\mathbf{i}) = \Delta u (a_{11}\mathbf{i} + a_{21}\mathbf{j}), \quad A(\Delta v \mathbf{j}) = \Delta v (A\mathbf{j}) = \Delta v (a_{12}\mathbf{i} + a_{22}\mathbf{j})$$

so that the image of our rectangle under the linear transformation A is a parallelogram spanned by the two vectors above.

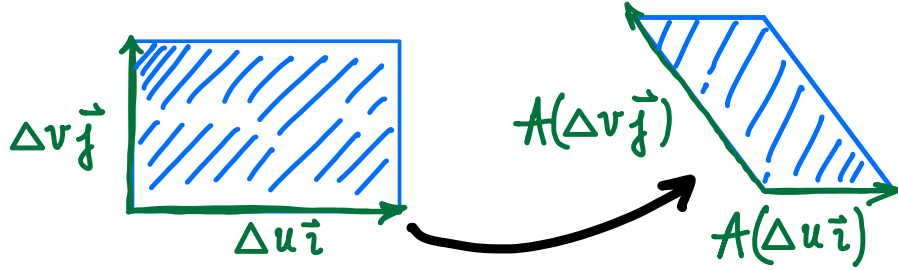


Figure 5.14: In a linear change of variables, rectangles become parallelograms

We have a formula for the area of a parallelogram in terms of the cross product:

$$\begin{aligned} (\text{Area of Parallelogram}) &= |A(\Delta u \mathbf{i}) \times A(\Delta v \mathbf{j})| = \left| \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_{11} & a_{21} & 0 \\ a_{12} & a_{22} & 0 \end{vmatrix} \right| \Delta u \Delta v \\ &= |\det(A)| \Delta u \Delta v \end{aligned}$$

So, we see that the area element satisfies (5.4.2) in the special case when the change of variables $(x, y) \mapsto (u, v)$ is given by the linear transformation A .

For general C^1 changes of variables, we use the linear approximation of $x = x(u, v)$ and $y = y(u, v)$ near some point $P_{i,j}^* = (x_i, y_j)$. Let $(x, y) = \mathbf{F}(u, v)$, then

$$\begin{pmatrix} x \\ y \end{pmatrix} \approx \begin{pmatrix} x_i \\ y_j \end{pmatrix} + D\mathbf{f} \begin{pmatrix} \Delta u \\ \Delta v \end{pmatrix}.$$

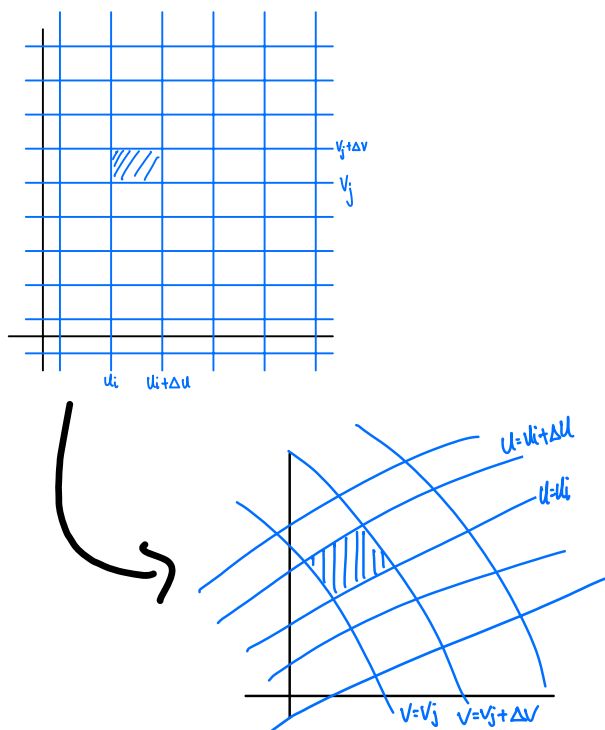


Figure 5.15: A more general change of variables

where $D\mathbf{f} = \begin{pmatrix} x_u & x_v \\ y_u & y_v \end{pmatrix}$ is the derivative matrix evaluated at (x_i, y_j) (see Section 3.4). Then the argument is as above, with $D\mathbf{f}$ playing the role of A , and the C^1 assumption guarantees that the error terms do not spoil the picture.

5.4.5 A remark on computing the Jacobian

It will often be to our advantage to define the new variables u, v in terms of x, y (as in Worked Problem 4 below). Instead of solving for x, y in terms of u, v and then computing the derivatives to find the Jacobian, we can proceed more directly by using the Implicit Function Theorem! Going back to Section 3.4, we can write formulas (3.4.4) and (3.4.5) together as

$$\begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = I,$$

the identity matrix. Taking the determinants, we get

$$\frac{\partial(x, y)}{\partial(u, v)} = \left(\frac{\partial(u, v)}{\partial(x, y)} \right)^{-1}.$$

Worked Problems

1. Find the volume under the paraboloid $z = 4 - x^2 - y^2$ and above the xy -plane.

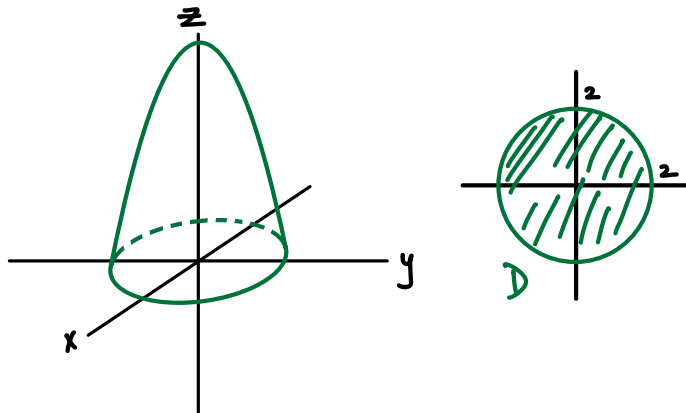


Figure 5.16: The region and its intersection with the xy -axis.

The volume is given by the integral

$$\iint_D (4 - x^2 - y^2) dA, \quad D := \{(x, y) : x^2 + y^2 \leq 4\}.$$

The key observation is that both the function $f(x, y) = 4 - x^2 - y^2$ and the region D are *radially symmetric*. This means that they only depend upon the distance $r = \sqrt{x^2 + y^2}$ from (x, y) to the origin. So, we convert the integral to polar coordinates:

$$f(x(r, \theta), y(r, \theta)) = f(r, \theta) = 4 - r^2$$

$$D := \{(r, \theta) : r^2 \cos^2 \theta + r^2 \sin^2 \theta \leq 4\} = \{(r, \theta) : 0 \leq r \leq 2, 0 \leq \theta < 2\pi\}$$

The associated area element is given by $dA = r dr d\theta$. Evaluating the integral, we have

$$\begin{aligned} \iint_D (4 - x^2 - y^2) dA &= \int_0^2 dr \int_0^{2\pi} d\theta \underbrace{(4 - r^2)}_{f(r, \theta)} \underbrace{r}_{\text{Jacobian}} = \int_0^2 \int_0^{2\pi} (4r - r^3) d\theta dr \\ &= 2\pi \int_0^2 (4r - r^3) dr = 2\pi \left[2r^2 - \frac{r^4}{4} \right]_{r=0}^2 \\ &= 8\pi \end{aligned}$$

2. Evaluate the improper integral

$$\iint_{\mathbb{R}^2} \frac{dA}{(1 + x^2 + y^2)^3}$$

We convert to polar coordinates, since

$$\mathbb{R}^2 = \{(r, \theta) : 0 \leq r < \infty, 0 \leq \theta < 2\pi\}$$

$$f(x(r, \theta), y(r, \theta)) = f(r, \theta) = \frac{1}{(1 + r^2)^3}$$

so that both the integrand $f(x, y)$ and region $D = \mathbb{R}^2$ are radially symmetric. We have

$$\begin{aligned} \iint_{\mathbb{R}^2} \frac{dA}{(1 + x^2 + y^2)^3} &= \int_0^{2\pi} d\theta \int_0^\infty \frac{r dr}{(1 + r^2)^3} = \int_0^{2\pi} \left[\frac{-1}{4(1 + r^2)^2} \right]_{r=0}^\infty d\theta \\ &= \frac{1}{4} \int_0^{2\pi} d\theta \\ &= \frac{\pi}{2}. \end{aligned}$$

3. For which value(s) of $a \in \mathbb{R}$ is the integral $\iint_D (x^2 + y^2)^{-a} dA$ convergent, assuming that $D := \{(x, y) : x^2 + y^2 \leq 1\}$?

Again, since our integrand and domain are both radially symmetric, we evaluate the associated (improper) integral in polar coordinates. We have

$$\begin{aligned} \iint_D (x^2 + y^2)^{-a} dA &= \int_0^1 \int_0^{2\pi} r^{-2a} \cdot r d\theta dr \\ &= 2\pi \int_0^1 r^{1-2a} dr. \end{aligned}$$

Now, we have three cases depending upon the value of a

$$2\pi \int_0^1 r^{1-2a} dr = \begin{cases} 2\pi \frac{r^{2-2a}}{2-2a} \Big|_0^1 = \frac{\pi}{1-a}, & \text{if } a < 1 \\ 2\pi \ln r \Big|_0^1 = \infty, & \text{if } a = 1 \\ 2\pi \frac{r^{2-2a}}{2-2a} \Big|_0^1 = \infty, & \text{if } a > 1 \end{cases}$$

In the first and third cases above, we used that if $a < 1$, then $2 - 2a > 0$, so that $r^{2-2a}|_{r=0} = 0$, whereas if $a > 1$, then $2 - 2a < 0$ and $r^{2-2a} \rightarrow \infty$ as $r \rightarrow 0$. Hence, we see that the integral converges if and only if $a < 1$.

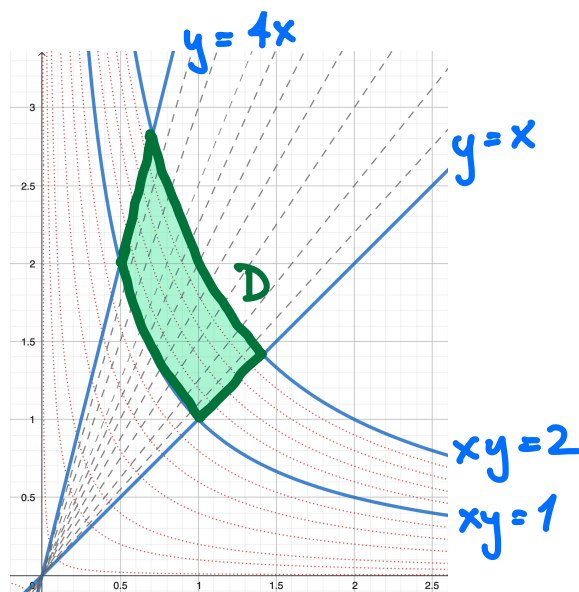


Figure 5.17: The region bounded by the four curves. The dotted/dashed lines correspond to the non-linear coordinates introduced below.

4. Find the area of the region D bounded by the curves $y = x$, $y = 4x$, $xy = 1$ and $xy = 2$.

The area of D is

$$\text{Area of } D = \iint_D 1 dA.$$

To evaluate the area integral, we will introduce “non-linear” coordinates adjusted to the shape of D . Specifically, let $u = xy$ and $v = \frac{y}{x}$. Then we can rewrite D in the new coordinates:

$$\text{Bounded by } xy = 1, xy = 2 \Rightarrow 1 \leq u \leq 2,$$

$$\text{Bounded by } y = x, y = 4x \Rightarrow 1 \leq v \leq 4,$$

so that $D = \{(u, v) : 1 \leq u \leq 2, 1 \leq v \leq 4\}$.

Let us compute the Jacobian of the change of variables $(x, y) \mapsto (u, v)$.

$$\begin{aligned} \frac{\partial(u, v)}{\partial(x, y)} &= \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = \begin{vmatrix} y & x \\ -\frac{y}{x^2} & \frac{1}{x} \end{vmatrix} = \frac{2y}{x} = 2v \\ \Rightarrow \frac{\partial(x, y)}{\partial(u, v)} &= \left(\frac{\partial(u, v)}{\partial(x, y)} \right)^{-1} = \frac{1}{2v} \end{aligned}$$

Now, by (5.4.2), we have

$$\begin{aligned}\iint_D 1 dA &= \int_1^2 \int_1^4 \frac{dv}{2v} du = \frac{1}{2} \int_1^2 \ln v \Big|_{v=1}^{v=4} du = \frac{1}{2} \ln 4 \int_1^2 du \\ &= \ln 2\end{aligned}$$

Practice Problems

1. Evaluate the double integrals:

(a) $\iint_D y dA$, where $D = \{(x, y) : y \geq 0, 1 \leq x^2 + y^2 \leq 9\}$.

(b) $\iint_{\mathbb{R}^2} \frac{dA}{(4 + x^2 + y^2)^5}$

2. Evaluate the integral $\iint_D (\frac{x}{y} - 1) dA$, if D is the region in the first quadrant bounded by the hyperbolas $x^2 - y^2 = 1$, $x^2 - y^2 = 4$, and the lines $x = 2y$, $x = 3y$.

3. Use a change of variables to evaluate the integral $\iint_D (x - y) dx dy$, where D is the region in the first quadrant bounded by the hyperbolas $x^2 - y^2 = 1$, $x^2 - y^2 = 4$, and the lines $x + y = 3$, $x + y = 6$.

4. Evaluate the area of the region in the first quadrant bounded by the hyperbolas $xy = 1$, $xy = 5$, and the parabolas $y = x^2$, $y = 4x^2$. (Hint: introduce new coordinates in which the region of integration is a rectangle.)

5. Determine whether the integral $\iint_{\mathbb{R}^2} \frac{|y|}{(1 + x^2 + y^2)^3} dA$ is convergent.

5.5 Integration in Three Variables

This section focuses on integration of functions $f(x, y, z)$ over regions $R \subset \mathbb{R}^3$:

$$\iiint_R f(x, y, z) dV,$$

where dV indicates the 3-dimensional volume element.

The general theory is essentially the same as in $\mathbb{R}^2$. The integral above is again defined as a limit of Riemann sums, except that this time we approximate a 3-dimensional

region by unions of small 3-dimensional rectangular boxes. Integrability conditions, iterated integrals, and improper integrals are also similar conceptually to the 2D case. (Please make sure to check your textbook for details.)

We will also use change of variables in 3D integrals. As in the 2-dimensional case, this multiplies the volume element by a Jacobian factor. The following is essentially a 3D version of polar coordinates.

Triple Integrals in Cylindrical Coordinates

Cylindrical coordinates in $\mathbb{R}^3$ are related to Cartesian coordinates (x, y, z) by making the change of variables $(x, y) \mapsto (r, \theta)$ to polar coordinates and preserving the z -variable. That is,

$$(x, y, z) \mapsto (r, \theta, z) \text{ where } x = r \cos \theta, y = r \sin \theta, z = z. \quad (5.5.1)$$

In this coordinate system, we have

$$dV = dx dy dz = r dr d\theta dz$$

where the extra r factor comes from the change of variables $(x, y) \mapsto (r, \theta)$.

The name *cylindrical coordinates* comes from the fact that, for fixed $z = z_0$, the variables $(x, y, z) = (r \cos \theta, r \sin \theta, z_0)$ parametrize a (two dimensional) circle of radius r at “height” z_0 in $\mathbb{R}^3$. Allowing z to vary from z_0 to z_1 (so that $z_0 \leq z \leq z_1$) we obtain a circular cylinder of diameter $2r$ and height $z_1 - z_0 = \Delta z$,

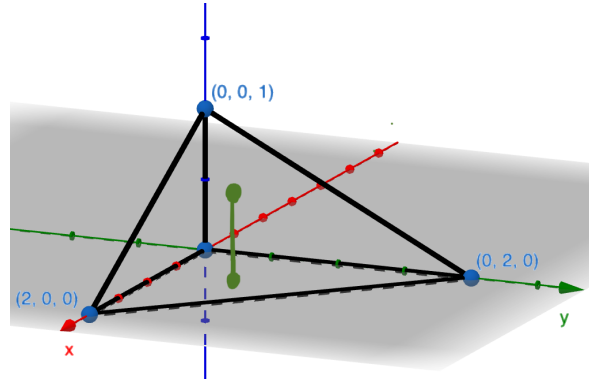
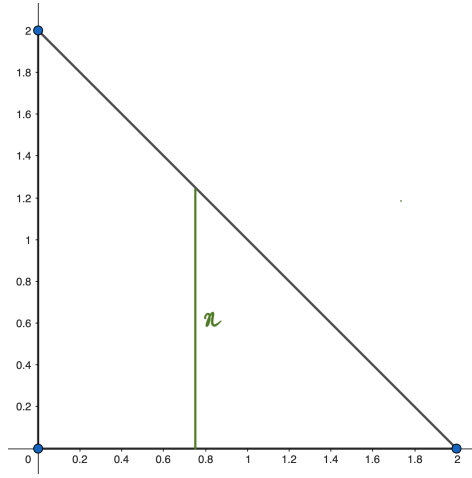
What becomes significantly more difficult in 3 dimensions is setting up the limits of integration in iterated integrals. The Worked Problems section includes several common types of regions. Whenever you need to write a 3D integral as an iterated integral, we recommend very strongly that you draw a picture of the region (or find some other way to visualize it). It is very difficult to guess the iterated limits of integration without such visualization. It can also be helpful to draw more than one picture. For example, if you start by writing the volume integral as

$$\iiint_D f(x, y, z) dV = \iint_X \left(\int_{g(x, y)}^{h(x, y)} f(x, y, z) dz \right) dy dx$$

for some planar region X , you might want to draw pictures of both D and X . See Worked Problems for examples of this.

Worked Problems

1. Find $\iiint_D e^{-z} dV$ if D is the tetrahedron bounded by the coordinate planes and the plane $x + y + 2z = 2$.

Figure 5.18: The region D in Worked Problem 1.Figure 5.19: D intersected with the xy -plane.

From Figures 5.18 and 5.19, we see that for (x, y, z) in D we must have

$$z \geq 0 \text{ and also } 2z \leq 2 - x - y \Rightarrow 0 \leq z \leq 1 - \frac{x}{2} - \frac{y}{2};$$

the second condition follows since our segment begins at the xy plane and terminates at the plane with equation $x + y + 2z = 2$. For bounds on y in terms of x , we use Figure 5.19 to determine that

$$0 \leq y \leq 2 - x,$$

since our segment begins at the line $y = 0$ and terminates at the line $y = 2 - x$. This also implies that $0 \leq x \leq 2$. So, we write the volume integral as an iterated integral

$$\iiint_D e^{-z} dV = \int_0^2 \int_0^{2-x} \int_0^{1-\frac{x}{2}-\frac{y}{2}} e^{-z} dz dy dx.$$

The calculation is as follows:

$$\begin{aligned}
 \int_0^2 \int_0^{2-x} \int_0^{1-\frac{x}{2}-\frac{y}{2}} e^{-z} dz dy dx &= \int_0^2 \int_0^{2-x} (-e^{-z}) \Big|_{z=0}^{1-\frac{x}{2}-\frac{y}{2}} dy dx \\
 &= \int_0^2 \int_0^{2-x} (1 - e^{\frac{x}{2}+\frac{y}{2}-1}) dy dx \\
 &= \int_0^2 \left[y - 2e^{\frac{x}{2}+\frac{y}{2}-1} \right]_{y=0}^{2-x} dx \\
 &= \int_0^2 (2-x - 2[e^{\frac{x}{2}+\frac{2-x}{2}-1}] + 2e^{\frac{x}{2}-1}) dx \\
 &= \int_0^2 (2-x-2+2e^{\frac{x}{2}-1}) dx = \int_0^2 (2e^{\frac{x}{2}-1} - x) dx \\
 &= \left[4e^{\frac{x}{2}-1} - \frac{x^2}{2} \right]_{x=0}^2 \\
 &= 2 - 4e^{-1}
 \end{aligned}$$

2. Find $\iiint_D y dV$ if D is the spatial region bounded by the parabolic cylinder $y = x^2$, the plane $y + z = 4$ and the xy -plane (defined by the equation $z = 0$).

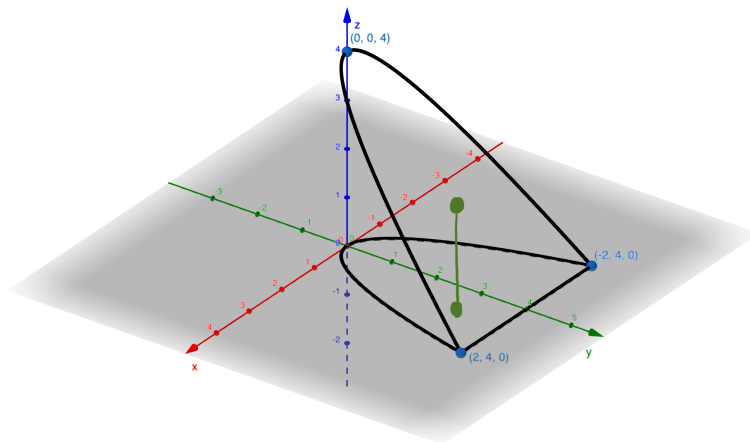
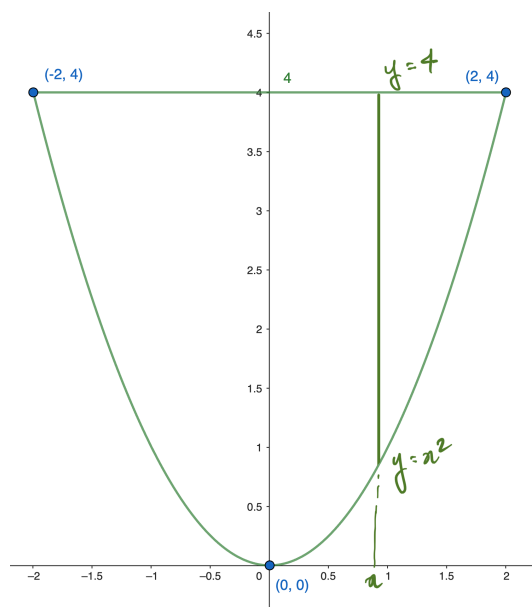


Figure 5.20: The region D in Worked Problem 2.

For fixed (x, y) , we examine Figure 5.20 above, determining that $0 \leq z \leq 4 - y$. Then, fixing x and examining Figure 5.21, we have $x^2 \leq y \leq 4$ and $-2 \leq x \leq 2$. This leads us to evaluate the volume integral as the iterated integral

$$\iiint_D y dV = \int_{-2}^2 \int_{x^2}^4 \int_0^{4-y} y dz dy dx$$

Figure 5.21: D intersected with the xy -plane.

The calculation follows:

$$\begin{aligned}
 \int_{-2}^2 \int_{x^2}^4 \int_0^{4-y} y dz dy dx &= \int_{-2}^2 \int_{x^2}^4 [yz]_{z=0}^{z=4-y} dy dx = \int_{-2}^2 \int_{x^2}^4 (4y - y^2) dy dx \\
 &= \int_{-2}^2 \left[2y^2 - \frac{y^3}{3} \right]_{y=x^2}^{y=4} dx = \int_{-2}^2 \left[32 - \frac{64}{3} - 2x^4 + \frac{x^6}{3} \right] dx \\
 &= \left[32x - \frac{64x}{3} - \frac{2x^5}{5} + \frac{x^7}{21} \right]_{-2}^2
 \end{aligned}$$

You can use your calculator to find the precise answer; up to three decimal places, this gives

$$\int_{-2}^2 \int_{x^2}^4 \int_0^{4-y} y dz dy dx \approx 29.257$$

3. Evaluate $\iiint_D z dV$ if D is the solid region bounded by the xy -plane and the paraboloid $z = 4 - x^2 - y^2$.

In cylindrical coordinates, our region D can be written as

$$D = \{(r, \theta, z) : 0 \leq r \leq 2, 0 \leq \theta \leq 2\pi, 0 \leq z \leq 4 - r^2\}$$

Notice that this gives the region D with two parameters r and θ independent of any variables—a fact which makes the iterated triple integral much easier to calculate.

We have

$$\iiint_D z dv = \int_0^{2\pi} d\theta \int_0^2 dr \int_0^{4-r^2} z r dz$$

where the r term comes from the Jacobian induced by our change of variables $(x, y, z) \mapsto (r, \theta, z)$. The calculation is:

$$\begin{aligned} & \int_0^{2\pi} d\theta \int_0^2 dr \int_0^{4-r^2} z r dz \\ &= \int_0^{2\pi} d\theta \int_0^2 dr \left[\frac{z^2 r}{2} \right]_{z=0}^{z=4-r^2} = \int_0^{2\pi} d\theta \int_0^2 \frac{1}{2} (4-r^2)^2 r dr \\ &= \int_0^{2\pi} d\theta \int_0^2 \frac{1}{2} (16r - 8r^3 + r^5) dr = \frac{1}{2} \int_0^{2\pi} \left[8r^2 - 2r^4 + \frac{r^6}{6} \right]_0^2 d\theta \\ &= \frac{1}{2} \int_0^{2\pi} \left[32 - 32 + \frac{64}{6} \right] d\theta = \frac{1}{2} \int_0^{2\pi} \frac{32}{3} d\theta \\ &= \frac{32\pi}{3} \end{aligned}$$

4. Find $\iiint_D xz dV$ if D is bounded by the planes $z = 0$, $z = 2$, $x = y$, $y = 0$ and also the cylinder $x^2 + y^2 \leq 1$ with $x, y > 0$.

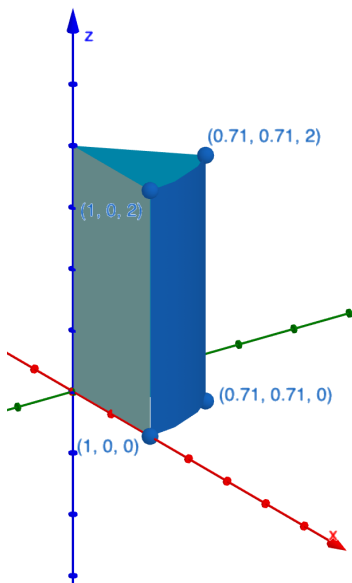
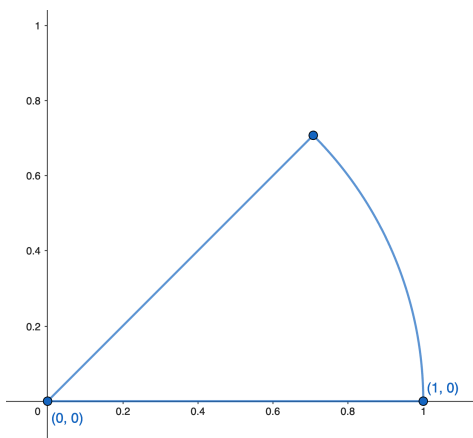


Figure 5.22: The region D in Worked Problem 4.

In cylindrical coordinates, our region has a simple description:

$$D := \{(r, \theta, z) : 0 \leq \theta \leq \frac{\pi}{4}, 0 \leq r \leq 1, 0 \leq z \leq 2\}$$

Figure 5.23: D intersected with the xy -plane.

so that the bounds on r , θ and z in our region D do not involve any of the other variables. Using $x = r \cos \theta$, and multiplying by the Jacobian factor r , we have

$$\iiint_D xz dV = \int_0^{\pi/4} \int_0^1 \int_0^2 (r \cos \theta) z r dz dr d\theta = \int_0^{\pi/4} d\theta \int_0^1 dr \int_0^2 z r^2 \cos \theta dz$$

Notice that, since $f(r, \theta, z) = f_1(r)f_2(\theta)f_3(z)$, where $f_1(r) = r^2$, $f_2(\theta) = \cos \theta$, and $f_3(z) = z$, we obtain

$$\int_0^{\pi/4} d\theta \int_0^1 dr \int_0^2 z r^2 \cos \theta dz = \left(\int_0^{\pi/4} \cos \theta d\theta \right) \left(\int_0^1 r^2 dr \right) \left(\int_0^2 z dz \right).$$

This equality *only makes sense* because both the function $f(r, \theta, z)$ and also the region D can be written in such a way that our variables r , θ and z are independent of one another. Evaluating each of these three integrals separately gives

$$\iiint_D xz dV = \left(\sin \theta \Big|_0^{\pi/4} \right) \cdot \left(\frac{r^3}{3} \Big|_0^1 \right) \cdot \left(\frac{z^2}{2} \Big|_0^2 \right) = \frac{\sqrt{2}}{3}$$

Practice Problems

1. Let R be the solid region in $\mathbb{R}^3$ bounded by the planes $x = 0$, $y = 0$, $z = 0$, and $x + 2y + z = 6$. Write

$$\iiint_R f(x, y, z) dV$$

as iterated integrals where the order of integration is (a) $dz dy dx$, (b) $dy dx dz$.

2. Evaluate the integral $\iiint_D y \, dV$, if D is the tetrahedron with vertices $(0,0,0)$, $(4,2,0)$, $(0,2,0)$, and $(0,2,4)$.
3. Let D be the spatial region above the xy -plane, inside the cylinder $x^2 + y^2 = 1$, and inside the sphere $x^2 + y^2 + z^2 = 4$. Evaluate the integral $\iiint_D z \, dV$.
4. Evaluate the integral $\iiint_R x \, dV$, if $R = \{(x, y, z) : 0 \leq z \leq y \leq x, x^2 + y^2 \leq 1\}$.

5.6 Spherical Coordinates

5.6.1 A Geometric Perspective

Let P be some point in $\mathbb{R}^3$, and let S be the (unique) sphere in $\mathbb{R}^3$ containing P and centered at the origin. Let $\mathbf{P}$ denote the vector from the origin to P . A bit of geometry shows that the location of P in three dimensional space is uniquely determined by three parameters:

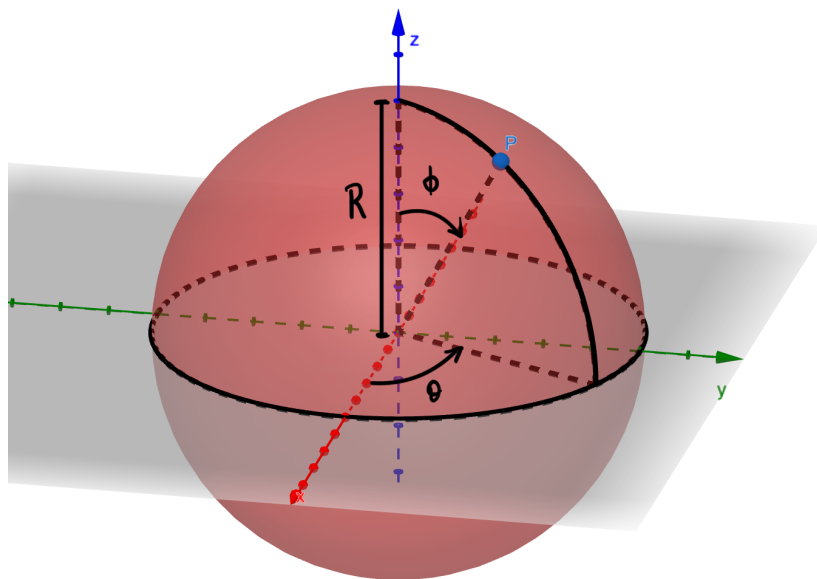


Figure 5.24: The point P and sphere S .

- The radius $R > 0$ of the sphere S ;
- The angle $0 \leq \theta < 2\pi$ made by the projection of $\mathbf{P}$ onto the xy -plane and the unit vector $\mathbf{i}$;

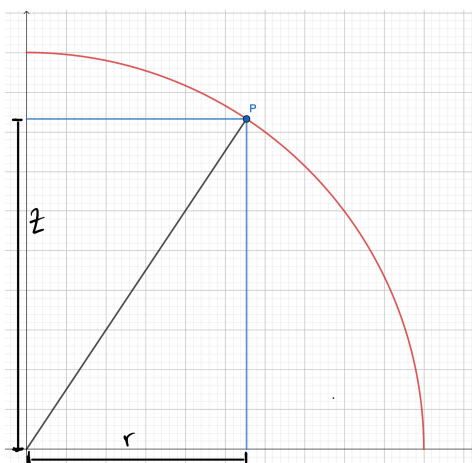


Figure 5.25: A cross-section.

- The angle $0 \leq \phi \leq \pi$ made by $\mathbf{P}$ and the unit vector $\mathbf{k}$.

This is illustrated above. To help elucidate the role of the variable ϕ , we have also included a cross-section of the sphere, which is taken in the plane passing through the z -axis and containing the point P . As an example, if $\phi = 0$, then P lies on the positive z axis; for $\phi = \pi/2$, P lies in the xy -plane; and if $\phi = \pi$, then P lies on the negative z -axis. We do not consider values of ϕ that are greater than π . For example, if P corresponds to some values of (R, θ, π) , then its antipodal point $-P$ corresponds to $(R, \theta + \pi, \pi - \phi)$ (rather than $(R, \theta, \phi + \pi)$).

In particular, we see that the parameters (R, θ, ϕ) give us a set of coordinates in $\mathbb{R}^3$. We call them **spherical coordinates**.

Change of Variables: Euclidean to Spherical Coordinates

We have $R = \sqrt{x^2 + y^2 + z^2}$, and

$$\begin{cases} x = R \cos \theta \sin \phi & R > 0 \\ y = R \sin \theta \sin \phi & 0 \leq \theta \leq 2\pi \\ z = R \cos \phi & 0 \leq \phi \leq \pi. \end{cases}$$

The easiest way to derive the conversion formulae above is to use cylindrical coordinates (r, θ, z) as an intermediate step. In the right triangle adjacent to the z -axis in the figure above, the hypotenuse has length R , the side adjacent to the angle ϕ has length z , and the side opposite to it has length r . Therefore

$$r = R \sin \phi, \quad z = R \cos \phi, \quad R = \sqrt{r^2 + z^2}.$$

Combining this with $x = r \cos \theta$, $y = r \sin \theta$, we get our conversion formulae.

5.6.2 Jacobian and Volume Elements

For a general change of variables $(x, y, z) \rightarrow (u, v, w)$ in 3D, the volume element is

$$dV = dx dy dz = \left| \frac{\partial(x, y, z)}{\partial(u, v, w)} \right| du dv dw$$

where

$$\frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{vmatrix}.$$

This can be proved the same way as for changes of variables in 2 dimensions, but using the formula for the volume of a parallelepiped (triple product) instead of the area of a parallelogram. In the special case of spherical coordinates, we have

$$\frac{\partial(x, y, z)}{\partial(R, \theta, \phi)} = -R^2 \sin \phi;$$

this is proved in most multivariable calculus textbooks, or you can do it yourself by evaluating the determinant and using that $\sin^2 t + \cos^2 t = 1$. Taking the absolute value to get the positive volume element, we get the formula

$$(\text{Jacobian}) \quad \left| \frac{\partial(x, y, z)}{\partial(R, \theta, \phi)} \right| = R^2 \sin \phi.$$

In the Worked Problems Section, we use our conversion formulas to evaluate three-dimensional volume integrals as iterated integrals in the coordinate system (R, θ, ϕ) .

Worked Problems

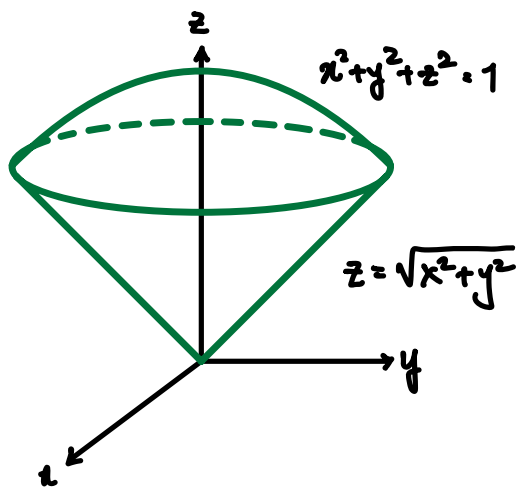
1. Find the volume of the region that lies above the cone $z = \sqrt{x^2 + y^2}$ and below the sphere $x^2 + y^2 + z^2 = 1$.

In spherical coordinates, the region D is given by

$$D := \{(R, \theta, \phi) : 0 \leq R \leq 1, 0 \leq \theta < 2\pi, 0 \leq \phi \leq \frac{\pi}{4}\},$$

which is a much simpler representation than in Euclidean coordinates. Hence, we evaluate the volume integral as the iterated integral

$$\iiint_D dV = \int_0^{\frac{\pi}{4}} d\phi \int_0^1 dR \int_0^{2\pi} d\theta \underbrace{(R^2 \sin \phi)}_{\text{Jacobian}}$$

Figure 5.26: The region D described above.

We calculate this iterated integral as the product of one-dimensional integrals:

$$\begin{aligned}
 \int_0^{\frac{\pi}{4}} d\phi \int_0^1 dR \int_0^{2\pi} d\theta (R^2 \sin \phi) &= \left(\int_0^{\frac{\pi}{4}} \sin \phi d\phi \right) \cdot \left(\int_0^1 R^2 dR \right) \cdot \left(\int_0^{2\pi} d\theta \right) \\
 &= \left(-\cos \phi \Big|_{\phi=0}^{\frac{\pi}{4}} \right) \cdot \left(\frac{R^3}{3} \Big|_{R=0}^1 \right) \cdot \left(\theta \Big|_{\theta=0}^{2\pi} \right) \\
 &= \left(1 - \frac{\sqrt{2}}{2} \right) \cdot \left(\frac{1}{3} \right) \cdot (2\pi) \\
 &= \frac{\pi}{3} (2 - \sqrt{2})
 \end{aligned}$$

2. Find the average value of the function $f(x, y, z) = z$ over the region D from Worked Problem 1.

The average value of $f(x, y, z) = z$ over the region D is defined as

$$\bar{z} := \frac{\iiint_D z dV}{\text{Volume of } D} = \frac{\iiint_D z dV}{\frac{\pi}{3}(2 - \sqrt{2})}$$

Because the region D has a simpler representation in spherical coordinates, we will evaluate the integral of f over D in spherical coordinates. We have:

$$\iiint_D z dV = \int_0^{\frac{\pi}{4}} d\phi \int_0^1 dR \int_0^{2\pi} d\theta \underbrace{(R \cos \phi)}_{f(R, \theta, \phi)} \underbrace{(R^2 \sin \phi)}_{\text{Jacobian}}$$

Again, calculating this as a product of one-dimensional integrals gives

$$\begin{aligned}\iiint_D z \, dV &= \left(\int_0^{\pi/4} \sin \phi \cos \phi \, d\phi \right) \cdot \left(\int_0^1 R^2 \, dR \right) \cdot \left(\int_0^{2\pi} d\theta \right) \\ &= \frac{2\pi}{4} \int_0^{\pi/4} \sin \phi \cos \phi \, d\phi\end{aligned}$$

Now, using the identity $\sin \phi \cos \phi = \frac{1}{2} \sin(2\phi)$, we have

$$\iiint_D z \, dV = \frac{\pi}{4} \int_0^{\pi/4} \sin(2\phi) \, d\phi = -\frac{\pi}{8} \cos(2\phi) \Big|_{\phi=0}^{\pi/4} = \frac{\pi}{8}$$

Thus

$$\bar{z} := \frac{\iiint_D z \, dV}{\text{Volume of } D} = \frac{\frac{\pi}{8}}{\frac{\pi}{3}(2 - \sqrt{2})} = \frac{3}{8(2 - \sqrt{2})} \approx 0.64$$

3. Evaluate the integral of $f(x, y, z) = x^2 + y^2$ over the region B if B is the top half of the ball of radius a centered at the origin.

The region B has a very simple representation in spherical coordinates:

$$B := \{(R, \theta, \phi) : 0 \leq R \leq a, 0 \leq \theta < 2\pi, 0 \leq \phi \leq \pi/2\}.$$

In spherical coordinates, we have

$$f(x, y, z) = f(x(R, \theta, \phi), y(R, \theta, \phi)) = R^2 \sin^2 \phi.$$

One way of deriving this with less calculation is noticing that $x^2 + y^2 = r^2$ in cylindrical coordinates, and then $r = R \sin \phi$ in spherical coordinates. Either way, our volume integral is given by the iterated integral

$$\iiint_B (x^2 + y^2) \, dV = \int_0^{2\pi} d\theta \int_0^{\pi/2} d\phi \int_0^a dR \underbrace{(R^2 \sin^2 \phi)}_{f(R, \theta, \phi)} \underbrace{(R^2 \sin \phi)}_{\text{Jacobian}}$$

Again, this is a product of one-dimensional integrals:

$$\begin{aligned}\iiint_B (x^2 + y^2) \, dV &= \left(\int_0^{2\pi} d\theta \right) \cdot \left(\int_0^a R^4 \, dR \right) \cdot \left(\int_0^{\pi/2} \sin^3 \phi \, d\phi \right) \\ &= \frac{2\pi a^5}{5} \int_0^{\pi/2} \sin^3 \phi \, d\phi\end{aligned}$$

To evaluate the final integral in the variable ϕ , recall that $\sin^3(\phi) = (1 - \cos^2 \phi) \sin \phi$, so that

$$\begin{aligned} \int_0^{\pi/2} \sin^3 \phi \, d\phi &= \int_0^{\pi/2} (1 - \cos^2 \phi) \sin \phi \, d\phi + \int_0^{\pi/2} \sin \phi \, d\phi \\ &= \left(\frac{\cos^3 \phi}{3} \Big|_{\phi=0}^{\pi/2} \right) - \left(\cos \phi \Big|_{\phi=0}^{\pi/2} \right) \\ &= -\frac{1}{3} - (-1) \\ &= \frac{2}{3} \end{aligned}$$

So, combining our previous calculations shows that

$$\iiint_B (x^2 + y^2) \, dV = \frac{4\pi a^5}{15}$$

4. *Convert the integral*

$$I = \int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_0^{\sqrt{x^2+y^2}} f(x, y, z) \, dz \, dy \, dx$$

to (a) cylindrical and (b) spherical coordinates.

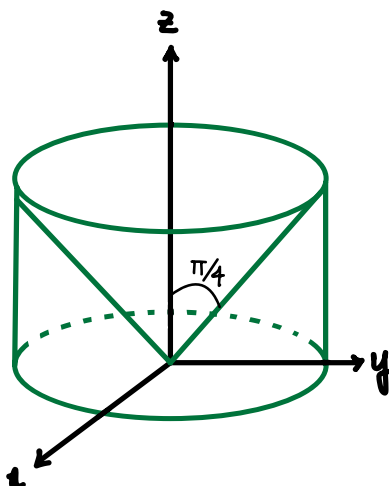


Figure 5.27: The region D of integration in Worked Problem 4, first given in Euclidean coordinates, is the region below the cone and above the xy -plane.

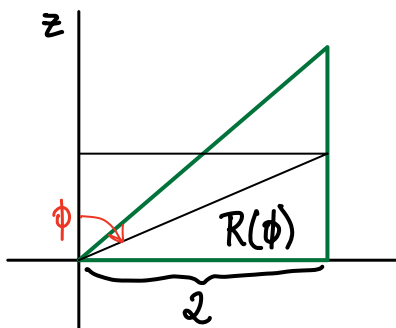


Figure 5.28: A cross section of the region in Worked Problem 4 shows the dependence of R upon the angular parameter ϕ .

The first step is to identify the region of integration D . For the z variable, we see that $0 \leq z \leq \sqrt{x^2 + y^2}$, which means that z is bounded by the xy -plane (below) and the cone $z = \sqrt{x^2 + y^2}$ (above). For the y variable, we know that $-\sqrt{4 - x^2} \leq y \leq \sqrt{4 - x^2}$ means that the (x, y) variables together form a circular disc. Putting this information together, we see that D is the region inside a circular cylinder and outside of a cone.

(a) In cylindrical coordinates, we have

$$I = \int_0^2 dr \int_0^{2\pi} d\theta \int_0^r dz f(r \cos \theta, r \sin \theta, z) \cdot r$$

(b) In spherical coordinates, we have the restrictions $0 \leq \theta < 2\pi$ and $\frac{\pi}{4} \leq \phi \leq \frac{\pi}{2}$. Now, if ϕ is given, then our R variable is bounded below by zero. The upper bound satisfies the equation $R(\phi) = \frac{2}{\sin \phi}$. This can be inferred from Figure 4 below. As such, we have

$$I = \int_0^{2\pi} d\theta \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} d\phi \int_0^{\frac{2}{\sin \phi}} dR f(R \cos \theta \sin \phi, R \sin \theta \sin \phi, R \cos \phi) R^2 \sin \phi.$$

Practice Problems

1. A surface in $\mathbb{R}^3$ has the equation $x^2 + y^2 = z^2 - 4$. Convert the equation of this surface to cylindrical and spherical coordinates.
2. A surface in $\mathbb{R}^3$ has the equation $R^2 - 4R \sin \phi + 3 = 0$ in spherical coordinates. Find its equation in cylindrical coordinates.

3. Evaluate the integral $\iiint_D \frac{\cos(x^2 + y^2 + z^2)}{\sqrt{x^2 + y^2 + z^2}} dV$, where D is the region that lies above the cone $z = \sqrt{x^2 + y^2}$ and inside the sphere $x^2 + y^2 + z^2 = 4$.
4. Write the following integrals as iterated integrals in your choice of coordinates (rectangular (x, y, z) , cylindrical (r, θ, z) , or spherical (R, θ, ϕ)). You do not need to evaluate the integrals.
- (a) $\iiint_D (x + y) dV$, where D is the region bounded by the planes $y = x$, $y = 4x$, $x = 2$, $z = 0$, and $z = x$.
- (b) $\iiint_D z^2 dV$, where D is the spatial region outside the cone $z = \sqrt{x^2 + y^2}$, inside the cylinder $x^2 + y^2 = 4$, and above the xy -plane.
5. Write out the integral $\iiint_W x^2 dV$ as an iterated integral in cylindrical coordinates, if W is the three-dimensional solid inside the cylinder $x^2 + y^2 = 1$, bounded from below by the cone $z = \sqrt{x^2 + y^2}$ and from above by the sphere $x^2 + y^2 + z^2 = 4$. (You do not need to evaluate the integral.)
6. Find the mass of the solid bounded from below by the cone $\phi = \pi/3$ and from above by the sphere $R = 2$, if the density at a point with spherical coordinates (R, θ, ϕ) is R . (Evaluate the integral.)
7. Use the change of variables $x = u^2$, $y = v^2$, $z = w^2$ to evaluate the volume of the region bounded by the coordinate planes and the surface $\sqrt{x} + \sqrt{y} + \sqrt{z} = 1$.