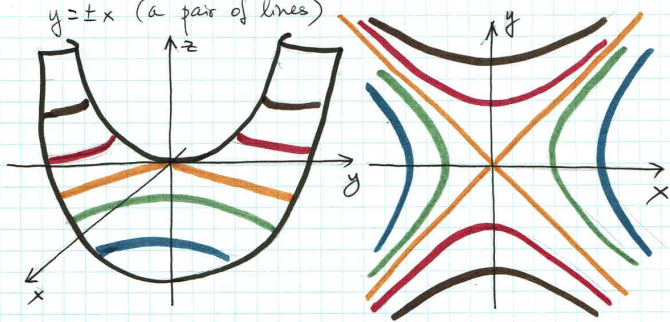


2) $f(x,y) = y^2 - x^2$. The graph $z = y^2 - x^2$ is a hyperbolic paraboloid. The level curves $y^2 - x^2 = c$ are hyperbolas in the xy -plane if $c \neq 0$. If $c = 0$, we get $y^2 - x^2 = 0$, so $y = \pm x$ (a pair of lines).

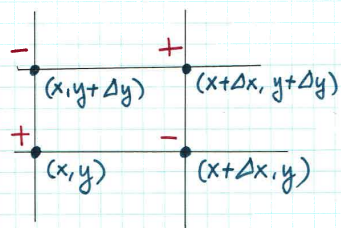


For functions $f(x,y,z)$ of 3 variables, a graph would be a hypersurface $w = f(x,y,z)$ in 4 dimensions. But we can still draw the level surfaces $f(x,y,z) = c$ for fixed c as surfaces in $\mathbb{R}^3$, and then get some idea of what the function "looks like" by varying the constant c . For example, let $f(x,y,z) = z^2 - x^2 - y^2$, then the level surface $z^2 - x^2 - y^2 = 0$ is the cone $z^2 = x^2 + y^2$, and the level surfaces $z^2 - x^2 - y^2 = c$ are hyperboloids (one sheet when $c < 0$, two sheets when $c > 0$).

Idea of proof, in the special case $f_{xy} = f_{yx}$ for a function $f(x,y)$ of 2 variables.

$$\text{Let } Q(x,y,\Delta x,\Delta y) = f(x+\Delta x, y+\Delta y) - f(x, y+\Delta y) - f(x+\Delta x, y) + f(x,y).$$

This adds up values of f at 4 vertices of a rectangle, with alternating $+/-$ signs as indicated.



Also notice that this is symmetric with respect to the order of variables.

Assume that the following double limit exists:

$$\text{(for now)} \quad \lim_{(\Delta x, \Delta y) \rightarrow (0,0)} \frac{Q(x,y,\Delta x,\Delta y)}{\Delta x \Delta y} = L. \quad (*)$$

Then both $f_{xy}(x,y)$ and $f_{yx}(x,y)$ are equal to L .

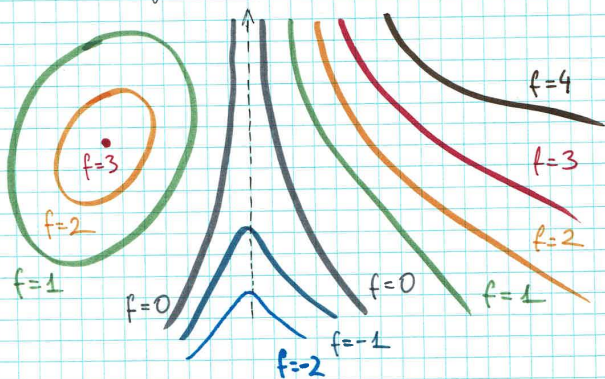
This is because

$$f_y(x,y) = \lim_{\Delta y \rightarrow 0} \frac{1}{\Delta y} (f(x, y+\Delta y) - f(x,y)),$$

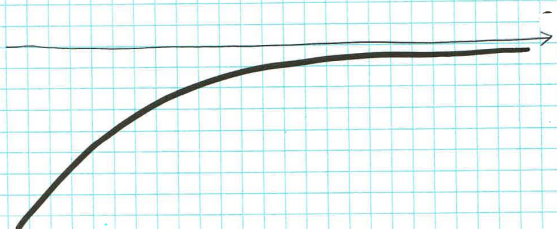
$$f_y(x+\Delta x, y) = \lim_{\Delta y \rightarrow 0} \frac{1}{\Delta y} (f(x+\Delta x, y+\Delta y) - f(x+\Delta x, y))$$

Why is the ^{test} 2nd derivative not useful here?

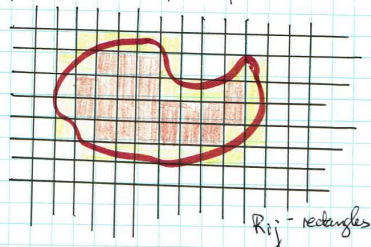
Example: $f(x,y)$ has only one local critical point, with a local but not global maximum at that point.



Cross-section along the dotted line:



Jordan area: for a planar set X , approximate it by grids ^{5/3}



$$S^* = \sum (\text{areas of } R_{ij} \text{ intersecting } X)$$

$$S_* = \sum (\text{areas of } R_{ij} \text{ contained in } X)$$

Then $S_* \leq (\text{area of } X) \leq S^*$,

X is Jordan measurable if $\lim S^* = \lim S_*$ as the diameter of the grid $\rightarrow 0$. Jordan area of X (if exists)

X has Jordan area 0 if $\lim S^* = 0$.

Example of a set that does not have Jordan area:

let $X = \{(x,y) \in [0,1]^2 : x \text{ and } y \text{ are both rational}\}$.

Then $S^* = 1$ and $S_* = 0$ for any grid. (Any rectangle contains both points with rational coordinates and points with irrational coordinates.)

