Notes on Multivariable Calculus I

SOLUTIONS MANUAL

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Solutions Manual

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Chapter 1

An Introduction to Euclidean Space

1.1 Sets and Coordinates

1. *Decide whether each of the sets below is open, closed, or neither.*

   (a) $\{(x, y) \in \mathbb{R}^2 : \ 0 \leq x < 1, \ 0 \leq y < 2\}$ – neither

   (b) $\{(x, y) \in \mathbb{R}^2 : \ x + y < 2\}$ – open

   (c) $\{(x, y) \in \mathbb{R}^2 : \ 0 < \sqrt{x^2 + y^2} \leq 3\}$ – neither open nor closed

   (d) $\{(x, y, z) \in \mathbb{R}^3 : \ |x| \geq 6\}$ – closed

2. *The following conditions describe sets in* $\mathbb{R}^3$*. Are these sets open, closed, or neither? Specify the boundary of each set.*

   (a) $x^2 + y^2 + z^2 \geq 16$: The boundary is the sphere $x^2 + y^2 + z^2 = 16$. Since this sphere is included in S, the set is closed.

   (b) $x^2 + y^2 \geq 4$, $z \geq 0$: This is the set above the plane $z = 0$ and on the outside of the cylinder $x^2 + y^2 = 4$, with boundary included. The boundary is the union of the cylindrical surface $\{(x, y, z) : \ x^2 + y^2 = 4, \ z \geq 0\}$, and the set $x^2 + y^2 \geq 4$ in the plane $z = 0$. This set is closed.

   (c) $0 < x < 1$, $0 < y < 1$, $z = 1$. This is the unit square in the plane $z = 1$, without the edges. The boundary is the closed square $\{(x, y, z) : \ 0 \leq x \leq 1, \ 0 \leq y \leq 1, \ z = 1\}$. This set is neither closed nor open.

   (d) $0 < \sqrt{x^2 + y^2} \leq 3$: The set is neither open nor closed. The boundary consists of the cylinder $\sqrt{x^2 + y^2} = 3$ (included in the set) and the line $x = y = 0$ in $\mathbb{R}^3$ (not included in it).

(e) $x \geq 0,\ x^2 + y^2 + z^2 \geq 4$: The set is closed. The boundary consists of the half-sphere $x \geq 0,\ x^2 + y^2 + z^2 = 4$, and the set $x = 0,\ y^2 + z^2 \geq 4$.

(f) $x > 0,\ 0 \leq y \leq 4$: The set is neither open nor closed. The boundary is

$$\begin{aligned}\{(x, y, z) \in \mathbb{R}^3 :\ x > 0,\ y = 0\} \cup \{(x, y, z) \in \mathbb{R}^3 :\ x > 0.\ y = 4\}\\ \cup\{(x, y, z) \in \mathbb{R}^3 :\ x = 0,\ 0 \leq y \leq 4\}.\end{aligned}$$

3. *Prove directly from the definition that the set $S = \{(x, y, z) \in \mathbb{R}^3 :\ x^2 + y^2 < 4\}$ is an open set in $\mathbb{R}^3$. (Given a point $P = (x, y, z) \in S$, find a neigbourhood of P which is contained in S.)*

Let $P = (x, y, z) \in S$. By the definition of S, we have $x^2 + y^2 < 4$, so that if we choose $\varepsilon = 2 - \sqrt{x^2 + y^2}$ then $\varepsilon > 0$. We claim that $B_\varepsilon(P) \subset S$. To see this, let $Q = (a, b, c) \in B_\varepsilon(P)$. Let P', Q' be the projections of P, Q onto the xy-plane, respectively. That is, let $P' = (x, y, 0)$ and $Q' = (a, b, 0)$. Then $|PQ| < \varepsilon$ by construction and thus

$$\begin{aligned}|P'Q'| &= \sqrt{(x-a)^2 + (y-b)^2}\\ &\leq \sqrt{(x-a)^2 + (y-b)^2 + (z-c)^2}\\ &= |PQ|\\ &< \varepsilon.\end{aligned}$$

By the triangle inequality, $|OQ'| \leq |OP'| + |P'Q'|$, where $O = (0, 0, 0)$ denotes the origin. This yields

$$\begin{aligned}\sqrt{a^2 + b^2} &\leq \sqrt{x^2 + y^2} + |P'Q'|\\ &< \sqrt{x^2 + y^2} + \varepsilon\\ &= 2.\end{aligned}$$

This is equivalent to $a^2 + b^2 < 4$, which shows that $Q \in S$, as required.

4. *Let $S' = \{(x, y, z) \in \mathbb{R}^3 :\ x^2 + y^2 < 4,\ z \geq 0\}$. Is S' an open set in $\mathbb{R}^3$? Is it closed? Prove your answers (again, using points and neighbourhoods).*

S' is not an open set. For example, $P = (0, 0, 0)$ belongs to S', but has no neighbourhood contained in S'. To check that last part, let $B_r(P)$ be any neighbourhood of P in $\mathbb{R}^3$. Then $B_r(P)$ contains the point $Q = (0, 0, -r/2)$, but $Q \notin S'$ since the z-coordinate of Q is negative.

S' is not a closed set, either. For this we must show that its complement $(S')^c$ is not open. Let $\tilde{P} = (2, 0, 0)$. Then $\tilde{P} \in (S')^c$, but there is no neighbourhood of $\tilde{P}$ contained in $(S')^c$. To see this, let $B_r(\tilde{P})$ be any neighbourhood of $\tilde{P}$ in $\mathbb{R}^3$. Then $B_r(\tilde{P})$ contains the point $\tilde{Q} = (2 - \frac{\min(r,4)}{2}, 0, 0)$, but $\tilde{Q} \notin (S')^c$ since $\tilde{Q} \in S'$.

5. *Prove directly from the definition that the set $S = \{(x, y) \in \mathbb{R}^2 : \ x + y > 0\}$ is an open set in $\mathbb{R}^2$. (Given a point $P = (x, y) \in S$, find an $r > 0$ such that the neigbourhood $B_r(P)$ is contained in S.)*

   Let $P = (x, y) \in S$, and choose $r = (x + y)/2$. By the definition of S, we have $x + y > 0$, so that $r > 0$. We claim that $B_r(P) \subset S$. To see this, let $Q = (x', y') \in B_r(P)$. Then
   $$|PQ| = \sqrt{(x - x')^2 + (y - y')^2} < r,$$
   so that
   $$|x' - x| \le |PQ| < r = \frac{x + y}{2}, \quad |y' - y| \le |PQ| < r = \frac{x + y}{2}.$$
   In particular, $x' - x > -\frac{x+y}{2}$ and $y' - y > -\frac{x+y}{2}$, so that
   $$\begin{aligned} x' + y' &= (x + y) + (x' - x) + (y' - y) \\ &> (x + y) - \frac{x + y}{2} - \frac{x + y}{2} \\ &= 0. \end{aligned}$$
   Therefore $Q \in S$. Since $Q \in B_r(P)$ was arbitrary, we have $B_r(P) \subset S$, as claimed.

6. *Let $S' = \{(x, y) \in \mathbb{R}^2 : \ x + y > 0, \ x - y \le 0\}$. Is S' an open set in $\mathbb{R}^2$? Prove your answer (again, using points and neighbourhoods).*

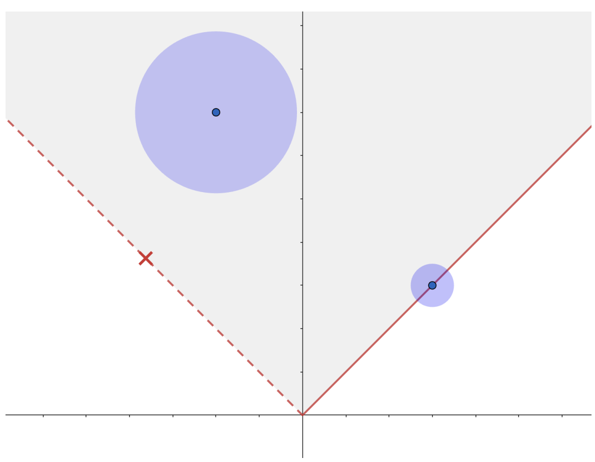

Figure 1.1: The region S' as well as a few points of S' (in blue) with associated neighbourhoods. The red X is reminding you that the boundary is *not* included in the set!

   S' is not an open set. For example, $P = (1, 1)$ belongs to S' since $1 + 1 = 2 > 0$ and $1 - 1 = 0$. However, P has no neighbourhood contained in S'. To check that last part, let $B_r(P)$ be any neighbourhood of P in $\mathbb{R}^2$. Then $B_r(P)$ contains the point $Q = (1, 1 - r/2)$, but $Q \notin S'$ since $1 - (1 - r/2) = r/2 > 0$.

7. *Let $S = \{(x, y) \in \mathbb{R}^2 : \ |x| \le 1, \ |y| < 1\}$. Prove directly from the definition (again, using points and neighbourhoods) that S is* **not** *a closed set in $\mathbb{R}^2$.*

To prove that S is not closed, we need to prove that its complement S^c is not open. We have

$$S^c = \{(x, y) \in \mathbb{R}^2 : \ |x| > 1 \text{ or } \ |y| \ge 1\}.$$

Let $P = (0, 1)$. Then $P \in S^c$. We need to prove that the open set condition for S^c fails at P; in other words, that there is no neighbourhood of P contained in $(S')^c$. To see this, let $B_r(P)$ be any neighbourhood of P in $\mathbb{R}^2$. Then $B_r(P)$ contains the point $Q = (0, 1 - \frac{r}{2})$, but $Q \notin S^c$ since $Q \in S$.

8. *Prove from the definition that the set $S := \{(x, y) : x^2 + 4y^2 < 1\}$ is open in $\mathbb{R}^2$. As above, this means: for each point $P \in S$, find an $r > 0$ such that $B_r(P) \subset S$.*

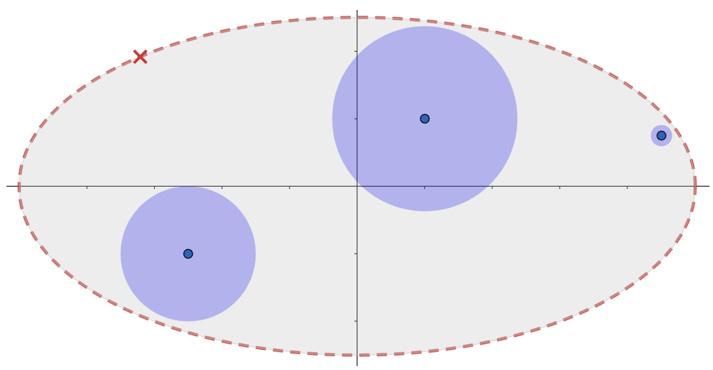

Figure 1.2: The region S is an ellipse. Included in the picture are a few points of S (in blue) with associated neighbourhoods. The red X is reminding you that the boundary is *not* included in the set!

The set S is the region enclosed by an ellipse with centre at the origin. The ellipse S has major axis in the horizontal direction with length 1; its minor axis is vertical with length 1/2. Notice for any $P = (x, y) \in S$, we necessarily have $1 - x^2 - 4y^2 > 0$.

Now, suppose $P = (x, y) \in S$ is given to us. Let $r = c(1 - x^2 - 4y^2)$ where $0 < c < 1$ is some small constant. Then $r > 0$. Moreover, if $Q = (u, v) \in B_r(P)$, we know that $|PQ| < r$. In particular, $|u - x| \le |PQ| < r$ and $|v - y| \le |PQ| < r$. By the triangle inequality, this gives,

$$|u| \le |x| + |u - x| < |x| + r \qquad \text{and} \qquad |v| \le |y| + |v - y| < |y| + r. \tag{1.1.1}$$

We now need to show that $Q \in B_r(P)$ also satisfies $Q \in S$. Since Q was an arbitrary element of $B_r(P)$, this would imply that $B_r(P) \subset S$. Since $P \in S$ was also arbitrary, this would show that S is open.

Now, $Q = (u, v)$ is an element of S if and only if $u^2 + 4v^2 < 1$. We will need some intermediate inequalities to show that this is, in fact, true. Notice that by (1.1.1)

$$\begin{aligned} u^2 + 4v^2 &\le (|x| + r)^2 + 4(|y| + r)^2 \\ &= |x|^2 + 2|x|r + r^2 + 4|y|^2 + 8|y|r + 4r^2 \\ \text{(using that } |x|, |y| \le 1 \text{ for } (x, y) \in S) &\le |x|^2 + 2r + r^2 + 4|y|^2 + 8r + 4r^2 \\ &= |x|^2 + 4|y|^2 + 10r + 5r^2 \end{aligned}$$

Now, we examine $r = c(1 - x^2 - 4y^2)$. Notice, since $c < 1$, we must have $r < 1$. So, in particular: $r^2 = r(r) < 1 \cdot r = 1$. Let's continue our inequalities

$$\begin{aligned} u^2 + 4v^2 &\le |x|^2 + 4|y|^2 + 10r + 5r^2 \\ &\le x^2 + 4y^2 + 15r. \end{aligned}$$

Now substitute the definition of r, to obtain

$$\begin{aligned} u^2 + 4v^2 &\le x^2 + 4y^2 + 15c(1 - x^2 - 4y^2) \\ &= 1 - (1 - x^2 - 4y^2) + 5c(1 - x^2 - 4y^2) \\ &= 1 - (1 - 15c)\underbrace{(1 - x^2 - 4y^2)}_{>0}. \end{aligned}$$

In particular, if we choose $c < 1/15$, then

$$1 - (1 - 15c)\underbrace{(1 - x^2 - 4y^2)}_{>0} < 1 - (1 - 15 \cdot \frac{1}{15})(1 - x^2 - 4y^2) = 1.$$

So, with this choice of $r = c(1 - x^2 - 4y^2)$, we see that $Q = (x, y) \in S$. This concludes the proof that S is open.

9. *Let $S \subset \mathbb{R}^2$ be an open set. Prove directly from the definition that the set*

$$S' = \{(x, y) \in \mathbb{R}^2 : \ (2x, 2y) \in S\}$$

is also an open set. (We know that S is open. This means that, given a point $P = (x, y) \in S$, there is an $r > 0$ such that the neigbourhood $B_r(P)$ is contained in S. You have to prove that the same condition is satisfied for all $P' \in S'$, possibly with a different value of r.)

Let $P' = (x, y) \in S'$. Then $P = (2x, 2y) \in S$. Since S is open, there is $r > 0$ such that $B_r(P) \subset S$. Let $r' = r/2 > 0$. We claim that $B_{r'}(P') \subset S'$. Since $P' \in S'$ was arbitrary, this will prove that S' is open.

We now need to prove that $B_{r'}(P') \subset S'$. Let $Q' = (x', y') \in B_{r'}(P')$, and let $Q = (2x', 2y')$. Then

$$|P'Q'| = \sqrt{(x-x')^2 + (y-y')^2} < r' = r/2,$$

so that

$$|PQ| = \sqrt{(2x-2x')^2 + (2y-2y')^2} = 2|P'Q'| < r.$$

This means that $Q \in B_r(P)$, so that $Q \in S$. But then $Q' \in S'$. This shows that any point $Q' \in B_{r'}(P')$ is in S', proving the claim.

1.2 Vectors, Dot Products, and Projections

1. *What is the distance from the point* $(1, 2, 3)$ *in* $\mathbb{R}^3$ *to the nearest point on the* x*-axis?*

   The nearest point on the x-axis is $(1, 0, 0)$. (This can be found using vector projections, but here the projection is easy enough to find by inspection.) The distance is $\sqrt{4+9} = \sqrt{13}$.

2. *Express the vector* $\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ *as a sum of vectors* $\mathbf{u}$ *and* $\mathbf{v}$*, where* $\mathbf{u}$ *is parallel to the vector* $\mathbf{i} - \mathbf{j}$ *and* $\mathbf{v}$ *is perpendicular to* $\mathbf{u}$*.*

   The vector $\mathbf{u}$ is the vector projection of $\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ on $\mathbf{i} - \mathbf{j}$. We find it from the formula

   $$\mathbf{u} = \frac{(\mathbf{i} + 2\mathbf{j} - \mathbf{k}) \cdot (\mathbf{i} - \mathbf{j})}{|\mathbf{i} - \mathbf{j}|^2}(\mathbf{i} - \mathbf{j}) = \frac{1-2}{2}(\mathbf{i} - \mathbf{j}) = -\frac{1}{2}\mathbf{i} + \frac{1}{2}\mathbf{j}.$$

   Then $\mathbf{v} = (\mathbf{i} + 2\mathbf{j} - \mathbf{k}) - (-\frac{1}{2}\mathbf{i} + \frac{1}{2}\mathbf{j}) = \frac{3}{2}\mathbf{i} + \frac{3}{2}\mathbf{j} - \mathbf{k}$.

3. *Let* $\mathbf{u} = 5\mathbf{i} + 7\mathbf{k}$*. Find vectors* $\mathbf{v}$ *and* $\mathbf{w}$ *such that:* $\mathbf{u} = \mathbf{v} + \mathbf{w}$*,* $\mathbf{v}$ *is parallel to the vector* $\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$*, and* $\mathbf{w}$ *is perpendicular to the vector* $\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$*.*

   We need $\mathbf{v}$ to be the vector projection of $\mathbf{u}$ in the direction of the vector $\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$. Applying the formula from class, we get

   $$\begin{aligned} \mathbf{v} &= \frac{(5\mathbf{i} + 7\mathbf{k}) \cdot (\mathbf{i} + 2\mathbf{j} - 2\mathbf{k})}{|\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}|^2}(\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}) \\ &= \frac{5-14}{1+4+4}(\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}) = \frac{-9}{9}(\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}) = -\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}. \end{aligned}$$

   Then

   $$\mathbf{w} = \mathbf{u} - \mathbf{v} = (5\mathbf{i} + 7\mathbf{k}) - (-\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}) = 6\mathbf{i} + 2\mathbf{j} + 5\mathbf{k}.$$

   Optional: we can check that $\mathbf{w}$ is perpendicular to $\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$ by checking the dot product,

   $$(6\mathbf{i} + 2\mathbf{j} + 5\mathbf{k}) \cdot (\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}) = 6 + 4 - 10 = 0.$$

4. *Find all values of t for which the vector $\mathbf{v} = (t-2)\mathbf{i}+2\mathbf{j}+(2t+1)\mathbf{k}$ is perpendicular to the vector $\mathbf{w} = 2t\mathbf{i} + (3-4t)\mathbf{j} + 2\mathbf{k}$.*

   We need $\mathbf{v} \cdot \mathbf{w} = 0$. This means

$$\begin{aligned} 0 &= (t-2)(2t) + 2(3-4t) + 2(2t+1) = 2t^2 - 4t + 6 - 8t + 4t + 2 \\ &= 2t^2 - 8t + 8 = 2(t^2 - 4t + 4) = 2(t-2)^2 \end{aligned}$$

   Hence the only solution is $t = 2$.

5. *Let $Q = (1, 0, -1)$. The set of points $P = (x, y, z)$ such that the vector $\vec{PQ}$ makes equal angles with the vectors $\mathbf{u} = 2\mathbf{i} - \mathbf{j}$ and $\mathbf{v} = \mathbf{i} + \mathbf{j} + 3\mathbf{k}$ can be described by an equation in x, y, z. Find that equation.*

   For vectors $\mathbf{a}$ and $\mathbf{b}$ we have $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta$, where θ is the angle between $\mathbf{a}$ and $\mathbf{b}$. Thus a vector $\vec{PQ}$ makes equal angles with the vectors $\mathbf{u}$ and $\mathbf{v}$ if and only if

$$\frac{\vec{PQ} \cdot \mathbf{u}}{|\vec{PQ}||\mathbf{u}|} = \frac{\vec{PQ} \cdot \mathbf{v}}{|\vec{PQ}||\mathbf{v}|} \tag{1.2.1}$$

   We will simplify this equation to give our equation in x, y, z. First, we calculate

$$\begin{aligned} \vec{PQ} &= (1-x, -y, -1-z)\ , \\ \vec{PQ} \cdot \mathbf{u} &= 2(1-x) + y \\ &= -2x + y + 2\ , \\ \vec{PQ} \cdot \mathbf{v} &= (1-x) - y + 3(-1-z) \\ &= -x - y - 3z - 2\ , \\ |\mathbf{u}| &= \sqrt{5}\ , \\ |\mathbf{v}| &= \sqrt{11}\ . \end{aligned}$$

   Substituting these into equation (1.2.1) gives us

$$\begin{aligned} &(|\vec{PQ}||\mathbf{v}|)(\vec{PQ} \cdot \mathbf{u}) = (|\vec{PQ}||\mathbf{u}|)(\vec{PQ} \cdot \mathbf{v}) \\ \iff\ & |\mathbf{v}|(\vec{PQ} \cdot \mathbf{u}) = |\mathbf{u}|(\vec{PQ} \cdot \mathbf{v}) \\ \iff\ & \sqrt{11}(-2x + y + 2) = \sqrt{5}(-x - y - 3z - 2) \\ \iff\ & (-2\sqrt{11} + \sqrt{5})x + (\sqrt{11} + \sqrt{5})y + 3\sqrt{5}z = -2\sqrt{5} - 2\sqrt{11}\ . \end{aligned}$$

6. *Let $\mathbf{u}$ and $\mathbf{v}$ be two non-zero vectors in $\mathbb{R}^2$ that are not parallel and not perpendicular to each other. Consider the following sequence of vectors $\mathbf{w}_1, \mathbf{w}_2, \ldots$ in $\mathbb{R}^2$:*

- $\mathbf{w}_1$ *is the vector projection of* $\mathbf{u}$ *in the direction of* $\mathbf{v}$*,*
- $\mathbf{w}_2$ *is the vector projection of* $\mathbf{w_1}$ *in the direction of* $\mathbf{u}$*,*
- $\mathbf{w}_3$ *is the vector projection of* $\mathbf{w_2}$ *in the direction of* $\mathbf{v}$*,*
- $\mathbf{w}_4$ *is the vector projection of* $\mathbf{w_3}$ *in the direction of* $\mathbf{u}$*),*
- *continue in this manner, alternating projections in the directions of* $\mathbf{u}$ *and* $\mathbf{v}$*.*

Find the general formula for the length of $\mathbf{w}_n$ *for* $n = 1, 2, \ldots,$ *in terms of* $|\mathbf{u}|$*,* $|\mathbf{v}|$*, and the angle* θ *between* $\mathbf{u}$ *and* $\mathbf{v}$*. What happens as* $n \to \infty$*? (Optional: you may want to draw a picture.)*

We have

$$|\mathbf{w}_1| = \left|\frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|}\right| = \frac{|\mathbf{u}|\,|\mathbf{v}|\,|\cos\theta|}{|\mathbf{v}|} = |\mathbf{u}|\,|\cos\theta|.$$

Since $\mathbf{w}_1$ is parallel to $\mathbf{v}$, the angle between $\mathbf{w}_1$ and $\mathbf{u}$ is either θ or $\pi - \theta$. Since $|\cos(\pi-\theta)| = |\cos\theta|$, in either case we get

$$|\mathbf{w}_2| = \left|\frac{\mathbf{w}_1\cdot\mathbf{u}}{|\mathbf{u}|}\right| = \frac{|\mathbf{w}_1|\,|\mathbf{u}|\,|\cos\theta|}{|\mathbf{u}|} = |\mathbf{w}_1|\,|\cos\theta|.$$

Taking into account the formula we already have for $|\mathbf{w}_1|$, we get that

$$|\mathbf{w}_2| = (|\mathbf{u}|\,|\cos\theta|)\,|\cos\theta| = |\mathbf{u}|\,||\cos\theta|^2.$$

Repeating the argument, we get that for general n,

$$|\mathbf{w}_n| = |\mathbf{u}|\,|\cos\theta|^n.$$

Finally, since $|\cos\theta| < 1$ by assumption, we have that $|\mathbf{w}_n| \to 0$ as $n \to \infty$.

1.3 The Cross Product

1. *Give an example of three non-zero vectors* $\mathbf{u}$*,* $\mathbf{v}$*,* $\mathbf{w}$ *such that* $\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w}) = 4$*.*

   One example is $\mathbf{u} = 4\mathbf{k}$, $\mathbf{v} = \mathbf{i}$, $\mathbf{w} = \mathbf{j}$. Then $\mathbf{v}\times\mathbf{w} = \mathbf{i}\times\mathbf{j} = \mathbf{k}$ and $\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w}) = 4$.

2. *Give an example of three non-zero vectors* $\mathbf{u}$*,* $\mathbf{v}$*,* $\mathbf{w}$*, such that* $\mathbf{v}\times\mathbf{w} \neq 0$ *but* $\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w}) = 0$*.*

   One example is $\mathbf{u} = \mathbf{v} = \mathbf{i}$, $\mathbf{w} = \mathbf{j}$. Then $\mathbf{v}\times\mathbf{w} = \mathbf{i}\times\mathbf{j} = \mathbf{k} \neq 0$ and $\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w}) = \mathbf{i}\cdot\mathbf{k} = 0$. Any other non-zero vectors such that $\mathbf{v}$ and $\mathbf{w}$ are not colinear but $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are coplanar would also work.

3. *Find the area of the triangle in $\mathbb{R}^3$ with vertices $(0,1,1)$, $(2,2,1)$, $(1,0,3)$.*

   The triangle is spanned by the vectors $\mathbf{v} = \langle 2,1,0\rangle$ and $\mathbf{w} = \langle 1,-1,2\rangle$. We have

$$\mathbf{v}\times\mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & 0 \\ 1 & -1 & 2 \end{vmatrix} = (2-0)\mathbf{i} - (4-0)\mathbf{j} + (-2-1)\mathbf{k} = 2\mathbf{i} - 4\mathbf{j} - 3\mathbf{k}.$$

   The area of the triangle is $\dfrac{1}{2}|\mathbf{v}\times\mathbf{w}| = \dfrac{1}{2}\sqrt{4+16+9} = \dfrac{\sqrt{29}}{2}$.

4. *Find the volume of the parallelepiped in $\mathbb{R}^3$ spanned by the vectors $\mathbf{i}-\mathbf{j}+4\mathbf{k}$, $3\mathbf{j}-\mathbf{k}$, $2\mathbf{i}+\mathbf{j}$.*

   The volume is

$$V = \left\|\begin{matrix} 1 & -1 & 4 \\ 0 & 3 & -1 \\ 2 & 1 & 0 \end{matrix}\right\| = |1\cdot 1 + 1\cdot 2 + 4\cdot(-6)| = |-21| = 21$$

1.4 Lines and Planes

1. *Give an example of a line parallel to the plane $x+y+z=0$ but not contained in that plane. Give the equation of the line in the vector parametric form $\mathbf{r} = \mathbf{a} + t\mathbf{v}$.*

   The plane is perpendicular to the vector $\mathbf{n} = \mathbf{i}+\mathbf{j}+\mathbf{k}$, so $\mathbf{v}$ should also be perpendicular to it. Additionally, $\mathbf{a}$ should not satisfy the equation of the plane. Thus we need $v_1+v_2+v_3 = 0$ and $a_1+a_2+a_3 \neq 0$. For example, we can take $\mathbf{v} = \mathbf{i}-\mathbf{j}$, $\mathbf{a} = \mathbf{i}$, so that the line equation is $\mathbf{r} = \mathbf{i} + t(\mathbf{i}-\mathbf{j})$.

2. *The points P, Q, R in $\mathbb{R}^3$ have coordinates $P = (1,1,-1)$, $Q = (2,-1,0)$, $R = (2,3,4)$.*

   (a) Find the area of the triangle $\triangle PQR$.

   We have $\vec{PQ} = \langle 1,-2,1\rangle$ and $\vec{PR} = \langle 1,2,5\rangle$, so that

$$\vec{PQ}\times\vec{PR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -2 & 1 \\ 1 & 2 & 5 \end{vmatrix} = -12\mathbf{i} - 4\mathbf{j} + 4\mathbf{k} = -4(3\mathbf{i}+\mathbf{j}-\mathbf{k})$$

   The area of the triangle is

$$\frac{1}{2}|\vec{PQ}\times\vec{PR}| = \frac{1}{2}\cdot 4\sqrt{9+1+1} = 2\sqrt{11}.$$

(b) Find the angle at P in the triangle $\triangle PQR$. (An answer in the form $\cos^{-1}(\cdot)$ or $\sin^{-1}(\cdot)$ is sufficient.)

We have $|\vec{PQ}| = \sqrt{1+4+1} = \sqrt{6}$ and $|\vec{PR}| = \sqrt{1+4+25} = \sqrt{30}$. We also have $\vec{PQ}\cdot\vec{PR} = 1-4+5 = 2$. Therefore

$$\cos\theta = \frac{2}{\sqrt{6}\sqrt{30}} = \frac{1}{\sqrt{45}}, \quad \theta = \cos^{-1}\left(\frac{1}{\sqrt{45}}\right).$$

Alternative solution: it is also possible to find $\sin\theta$ from

$$\sin\theta = \frac{|\vec{PQ}\times\vec{PR}|}{|\vec{PQ}||\vec{PR}|} = \sqrt{\frac{44}{45}}.$$

However, this does not distinguish between the angles θ (in this case, acute) and $\pi-\theta$ (obtuse). The first method is preferred because it does provide that information.

(c) Find the equation (in the form $Ax+By+Cz=D$) of the plane through the points P, Q, R.

The plane is perpendicular to the vector $3\mathbf{i}+\mathbf{j}-\mathbf{k}$ from (a), and passes through $P=(1,1,-1)$. Therefore the equation of the plane is

$$3(x-1)+(y-1)-(z+1)=0,$$

which simplifies to $3x-3+y-1-z-1=0$, or $3x+y-z=5$.

3. *Prove that the line $x=t+1, y=3t, z=t+3$ is parallel to the plane $4x-y-z=2$.*

   The line is parallel to the vector $\mathbf{u}=\mathbf{i}+3\mathbf{j}+\mathbf{k}$, and the plane is perpendicular to the vector $\mathbf{v}=4\mathbf{i}-\mathbf{j}-\mathbf{k}$. SInce $\mathbf{u}\cdot\mathbf{v}=4-3-1=0$, $\mathbf{u}$ is perpendicular to $\mathbf{v}$, therefore perpendicular to the plane.

4. *Find the equation (in the form $Ax+By+Cz=D$) of the plane that contains the line $x=t+1, y=3t, z=t+3$ and is perpendicular to the plane $4x-y-z=2$.*

   The plane should be parallel to both $\mathbf{u}$ and $\mathbf{v}$ from (a), therefore perpendicular to their cross product:

$$\mathbf{u}\times\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 3 & 1 \\ 4 & -1 & -1 \end{vmatrix} = -2\mathbf{i}+5\mathbf{j}-13\mathbf{k}.$$

   For a point that lies in the plane, we can take any point from the given line, for example with $t=0$ we get $(1,0,3)$. Hence the equation of the plane is

$$-2(x-1)+5(y-0)-13(z-3)=0,$$

   which simplifies to $-2x+2+5y-13z+39=0$, or $2x-5y+13z=41$.

5. *Find the equation of the plane that contains the line $x = t, y = 2t, z = 0$, and the point $(3, -1, -1)$.*

The plane has to be parallel to the vector $\mathbf{i} + 2\mathbf{j}$ (the direction vector of the line). We are given that the point $(3, -1, -1)$ lies in the plane, and so does $(0, 0, 0)$ since it lies on the given line, so another vector parallel to the plane is $3\mathbf{i} - \mathbf{j} - \mathbf{k}$. We now find the normal vector

$$\mathbf{n} = (3\mathbf{i} - \mathbf{j} - \mathbf{k}) \times \mathbf{i} + 2\mathbf{j} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -1 & -1 \\ 1 & 2 & 0 \end{vmatrix} = 2\mathbf{i} - \mathbf{j} + 7\mathbf{k}.$$

The plane perpendicular to $\mathbf{n}$ and through $(0, 0, 0)$ is $2x - y + 7z = 0$.

6. *Write the vector $\mathbf{w} = \mathbf{i} - \mathbf{j} + 4\mathbf{k}$ as a sum of two vectors $\mathbf{w} = \mathbf{u} + \mathbf{v}$ such that $\mathbf{u}$ is parallel to the plane $x + 2y - 2z = 0$ and the other is perpendicular to it.*

The vector $\mathbf{v}$ should be the vector projection of $\mathbf{w}$ on a vector $\mathbf{n}$ normal to the plane. From the equation of the plane, we can take $\mathbf{n} = \mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$. Then

$$\mathbf{v} = \mathbf{w}_{\mathbf{n}} = \frac{\mathbf{w} \cdot \mathbf{n}}{|\mathbf{n}|^2}\mathbf{n} = \frac{1 - 2 - 8}{1 + 4 + 4}\mathbf{n} = -\mathbf{n} = -\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$$

and

$$\begin{aligned} \mathbf{u} = \mathbf{w} - \mathbf{v} &= (1 - (-1))\mathbf{i} + (-1 - (-2))\mathbf{j} + (4 - 2)\mathbf{k} \\ &= 2\mathbf{i} + \mathbf{j} + 2\mathbf{k} \end{aligned}$$

7. *Find the equations (in whichever form you prefer) of the line of intersection of the planes $3x - y + z = 2$ and $x + 2y - z = -4$.*

The two planes have normal vectors $\mathbf{n}_1 = 3\mathbf{i} - \mathbf{j} + \mathbf{k}$ and $\mathbf{n}_2 = \mathbf{i} + 2\mathbf{j} - \mathbf{k}$. The direction vector $\mathbf{v}$ of the line should be perpendicular to them both, so we take

$$\mathbf{v} = \mathbf{n}_1 \times \mathbf{n}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -1 & 1 \\ 1 & 2 & -1 \end{vmatrix} = -\mathbf{i} + 4\mathbf{j} + 7\mathbf{k}.$$

To find a point on the line, set e.g. $x = 0$ in the equations of both planes. We get $-y + z = 2$ and $2y - z = -4$. Adding these two equations we get $y = -2$, then from the first equation $z = y + 2 = 0$. Thus, $P = (0, -2, 0)$ is a point in the plane. (Of course there are other possibilities.)

Using P and $\mathbf{v}$ as above, the (vector parametric) equation of the line is

$$\mathbf{r} = \langle -t,\ -2 + 4t,\ 7t \rangle$$

8. *Find vectors* $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ *in* $\mathbb{R}^2$ *such that the set of points whose position vector* $\mathbf{r}$ *satisfies the inequalities*
$$\mathbf{r}\cdot\mathbf{a} \le 1,\ \ \mathbf{r}\cdot\mathbf{b} \le 1,\ \ \mathbf{r}\cdot\mathbf{c} \le 1 \tag{1.4.1}$$
is the triangle with vertices $(1,2)$, $(2,-2)$, $(-3,0)$.

The triangle is the common part of three closed half-planes:

- above the line through $(-3,0)$ and $(2,-2)$,
- below the line through $(1,2)$ and $(2,-2)$,
- below the line through $(-3,0)$ and $(1,2)$,

all with the boundary line included. We need to find the inequalities defining the three half-planes. Here we present the proof for the first one, the other two are similar. We first find the equation of the line through $(-3,0)$ and $(2,-2)$. The line must be parallel to $\langle 5,-2\rangle$, so its equation is
$$\frac{x+3}{5} - \frac{y}{-2}, \quad -2(x+3) = 5y.$$
The half-plane is then given by $5y \ge -2(x+3)$. Similarly, the other two half-planes are $y \le -4x+6$ and $2y \le x+3$.

We now need to rewrite these inequalities in a form that matches (1):
$$-2x-5y \le 6, \quad 4x+y\le 6, \quad -x+2y \le 3,$$
$$\frac{-2x-5y}{6} \le 1, \quad \frac{4x+y}{6} \le 1, \quad \frac{-x+2y}{3} \le 1.$$
So we can let
$$\mathbf{a} = \left\langle \frac{-1}{3}, \frac{-5}{6}\right\rangle, \quad \mathbf{b} = \left\langle \frac{2}{3}, \frac{1}{6}\right\rangle, \quad \mathbf{c} = \left\langle \frac{-1}{3}, \frac{2}{3}\right\rangle.$$
These are unique up to relabelling.

9. *Let* $\mathbf{v}$ *and* $\mathbf{w}$ *be two nonzero vectors in* $\mathbb{R}^3$ *that are perpendicular to each other. Prove that the set*
$$\{P = (x,y,z) \in \mathbb{R}^3 :\ \vec{OP}\times\mathbf{v} = \mathbf{w}\}$$
is a line.

We provide a proof based on the determinant formula for the cross product. Geometric proofs, if correct, are also fine.

Let $\mathbf{v} = A\mathbf{i} + B\mathbf{j} + C\mathbf{k}$ and $\mathbf{w} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$. Then
$$\vec{OP}\times\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x & y & z \\ A & B & C\end{vmatrix} = (Cy-Bz)\mathbf{i} - (Cx-Az)\mathbf{j} + (Bx-Ay)\mathbf{k}.$$

This is equal to $\mathbf{w}$ if and only if

$$Cy - Bz = a, \quad Cx - Az = -b. \quad Bx - Ay = c.$$

We assumed that $\mathbf{v} \neq \mathbf{0}$. Without loss of generality (interchanging the x, y, z coordinates if necessary), we may assume that $C \neq 0$. Then the first two equations above are equivalent to

$$y = C^{-1}(a + Bz), \quad x = C^{-1}(Az - b). \tag{1.4.2}$$

Substituting (1.4.2) for x and y, the last equation becomes

$$\begin{aligned} c &= Bx - Ay = BC^{-1}(Az - b) - AC^{-1}(a + Bz) \\ &= C^{-1}(ABz - bB - aA - ABz) = C^{-1}(-bB - aA), \\ cC &= -bB - aA. \end{aligned} \tag{1.4.3}$$

But we assumed that $\mathbf{v}$ and $\mathbf{w}$ are perpendicular to each other, so that

$$0 = \mathbf{v} \cdot \mathbf{w} = aA + bB + cC$$

and the last equation in (1.4.3) is always satisfied.

This means that (1.4.2) provides the complete set of solutions. We just have to notice that (1.4.2) describes a line. Letting $z = Ct$, we get

$$x = At - C^{-1}b, \quad y = Bt + C^{-1}a, \quad z = t,$$

a line with the direction vector $A\mathbf{i}+B\mathbf{j}+C\mathbf{k}$, passing through the point $(-C^{-1}b, C^{-1}a, 0)$.

(Notice that the line is parallel to $\mathbf{v}$! Can you give a geometric reason why this should be the case?)

Chapter 2

Functions of Several Variables

2.1 Graphs and Surfaces

1. *Let* $f(x,y) = \sqrt{y - x^2}$. *Find the domain of* f, *and find the equations of its level curves.*

   The domain is $\{(x,y) \in \mathbb{R}^2 : \; y \geq x^2\}$. The level curves $f(x,y) = c$ for $c \geq 0$ are parabolas $y = x^2 + c^2$.

2. *Find the function* $f(x,y,z)$ *if for each constant* D *the level surface* $f(x,y,z) = D$ *is the plane perpendicular to* $\mathbf{n} = -3\mathbf{i} + \mathbf{j} + 2\mathbf{k}$ *and passing through the point* $(0,0,D^5)$.

   First we find the equation of the plane described above. Since the plane is normal to $\mathbf{n}$ and passes through the point $(0,0,D^5)$, we know that any point $Q = (x,y,z)$ on $\mathcal{P}$ necessarily satisfies the equation

   $$0 = \langle x, y, z - D^5 \rangle \cdot \mathbf{n} = -3x + y + 2(z - D^5),$$

   which simplifies to $-3x + y + 2z = 2D^5$.

   Now, if (x,y,z) is such that $f(x,y,z) = D$, then for that (x,y,z) we have

   $$-3x + y + 2z = 2D^5 = 2(f(x,y,z))^5.$$

   We solve this to find

   $$f(x,y,z) = 2^{-1/5}(-3x + y + 2z)^{1/5}.$$

2.2 Limits and Continuity

1. *Evaluate the following limits, or prove that they do not exist. In this question, you do not have to use the $\epsilon - \delta$ definition of limits.*

(a) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{\sin(xy)}{x^2-y^2}$

Note that the function in the limit is defined where $x \neq \pm y$. If we approach $(0,0)$ along the line $x=0$, we have

$$\lim_{y\to 0} \frac{0}{-y^2} = \lim_{y\to 0} 0 = 0.$$

But if $x = 2y \to 0$, we have

$$\lim_{y\to 0} \frac{\sin(2y^2)}{3y^2} = \frac{2}{3}$$

using the limit $\lim_{t\to 0} \frac{\sin t}{t} = 1$ from single-variable calculus. Therefore the limit in both variables does not exist.

(b) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy^5\cos(1/y)}{x^4+y^4}$

The function is only defined for $y \neq 0$, so we will assume $y \neq 0$ in what follows. Then $|\cos(1/y)| \leq 1$ and $x^4 + y^4 > 0$, so that

$$\begin{aligned} 0 \leq \left|\frac{xy^5\cos(1/y)}{x^4+y^4}\right| &= \frac{|x||y|^5\,|\cos(1/y)|}{x^4+y^4} \\ &\leq \frac{|x||y|^5}{x^4+y^4} \\ &= |x||y|\,\frac{|y|^4}{x^4+y^4} \\ &\leq |x||y|. \end{aligned}$$

At the last step, we used that $|y|^4 = y^4 \leq x^4 + y^4$. As $(x,y) \to (0,0)$, we have $|x||y| \to 0$. By the squeeze theorem, our limit exists and is equal to 0.

(c) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^4+y^4}$

This limit does not exist. For example, if we let $x = y \to 0$, we get

$$\lim_{y\to 0} \frac{y^3}{2y^4} = \lim_{y\to 0} \frac{1}{2y},$$

which does not exist (the one-sided limits are $\pm\infty$, depending on the sign of y).

(d) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2y^4}{x^2+y^6}$

We have

$$0 \leq \frac{x^2y^4}{x^2+y^6} = \frac{x^2}{x^2+y^6}y^4 \leq y^4.$$

Since both 0 and y^4 have limit 0 at $(0,0)$, by the squeeze theorem our limit exists and is equal to 0.

2. *Evaluate the following limits, or prove that they do not exist. In this question, you do not have to use the $\epsilon - \delta$ definition of limits.*

(a) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}}$.

We use polar coordinates $x = r\cos\theta, y = r\sin\theta$. Then

$$\frac{xy}{\sqrt{x^2+y^2}} = \frac{r^2\cos\theta\sin\theta}{r} = r\cos\theta\sin\theta.$$

Since $|\sin\theta| \leq 1$ and $|\cos\theta| \leq 1$ for all θ, we have

$$\left|\frac{xy}{\sqrt{x^2+y^2}}\right| = |r\cos\theta\sin\theta| \leq r.$$

As $(x,y)\to 0$, we have $r\to 0$, so that our double limit is 0 by the Squeeze Theorem.

(This could also be solved using the Squeeze Theorem without the polar coordinates, similar to several other examples here. We include a solution with polar coordinates to indicate that this is another valid approach.)

(b) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{(x-y)^2}{x^2+y^2}$

This limit does not exist. If we approach $(0,0)$ along the line $x = y$, we get $\lim_{x\to 0} 0 = 0$. But if we let $y = 0$, we get $\lim_{x\to 0}\frac{x^2}{x^2} = 1$.

(c) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^3}{|x|+|y|}$

Using that $|x|+|y| \geq |x|$, we write

$$0 \leq \left|\frac{x^3}{|x|+|y|}\right| \leq \left|\frac{x^3}{|x|}\right| = x^2.$$

Since $x^2 \to 0$ as $(x,y) \to (0,0)$, our limit is 0 by the squeeze theorem.

(d) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x+y}{\sqrt{x^2+y^2}}$.

When $y=0$ and $x \neq 0$, we have

$$\frac{x+y}{\sqrt{x^2+y^2}} = \frac{x}{\sqrt{x^2}} = \frac{x}{|x|},$$

which is 1 for $x>0$ and -1 for $x<0$. Therefore the one-sided limits as $y=0$ and $x \to 0$ are different (1 when $x \to 0$ through positive values, and -1 when $x \to 0$ through negative values). This means that the double limit does not exist.

(e) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{\sin^2 x \sin^2 y}{x^2+y^2}$

We will use the inequality $|\sin x| \leq |x|$ from single-variable calculus. For $(x,y) \neq (0,0)$, we have

$$\left|\frac{\sin^2 x \sin^2 y}{x^2+y^2}\right| = \frac{\sin^2 x \sin^2 y}{x^2+y^2} \leq \frac{x^2y^2}{x^2+y^2} \leq y^2,$$

where at the last step we used that $x^2 \leq x^2+y^2$. Since $y^2 \to 0$ as $(x,y) \to 0$, our double limit is 0 by the Squeeze Theorem.

3. Prove using the $\epsilon - \delta$ definition that $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^3y^3}{(x^2+y^2)^2} = 0$. (Given $\epsilon > 0$, find $\delta > 0$ such that...)

We will first prove that

$$\left|\frac{x^3y^3}{(x^2+y^2)^2}\right| \leq \frac{1}{2}(x^2+y^2) \text{ for all } (x,y) \neq (0,0). \tag{2.2.1}$$

Since $(x^2+y^2)^2 = x^4+2x^2y^2+y^4 \geq 2x^2y^2$, we have

$$\begin{aligned}\left|\frac{x^3y^3}{(x^2+y^2)^2}\right| &= \frac{x^2y^2}{(x^2+y^2)^2}|xy| \leq \frac{1}{2}|xy| \\ &\leq \frac{1}{2}\sqrt{x^2+y^2}\,\sqrt{x^2+y^2} \\ &= \frac{1}{2}(x^2+y^2).\end{aligned}$$

Now we can complete the proof. Given $\varepsilon > 0$, we let $\delta = \sqrt{2\varepsilon} > 0$. For any (x, y) such that $0 < \sqrt{x^2 + y^2} < \delta$, we have by (2.2.1)

$$\begin{aligned}\left|\frac{x^3y^3}{(x^2+y^2)^2}\right| &\leq \frac{1}{2}(x^2+y^2)\\ &< \frac{1}{2}\delta^2 = \varepsilon,\end{aligned}$$

as required.

4. *Prove using the $\epsilon - \delta$ definition that*

$$\lim_{(x,y)\to(0,0)} \frac{x^3y}{x^2+y^4} = 0.$$

Given $\epsilon > 0$, we need to find $\delta > 0$ such that if $0 < \sqrt{x^2+y^2} < \delta$ then

$$\left|\frac{x^3y}{x^2+y^4}\right| < \epsilon.$$

We have

$$0 \leq \left|\frac{x^2y^2}{x^2+y^4}\right| = \left|\frac{x^2}{x^2+y^4}\right| |x|\,|y| \leq \sqrt{x^2+y^2}\,\sqrt{x^2+y^2} \leq x^2+y^2.$$

Therefore, let $\delta = \sqrt{\epsilon} > 0$. If $0 < \sqrt{x^2+y^2} < \delta = \sqrt{\epsilon}$, then by the above we have

$$0 \leq \left|\frac{x^3y}{x^2+y^4}\right| \leq x^2+y^2 < \delta^2 = \epsilon,$$

so the $\epsilon - \delta$ condition is satisfied.

5. *Give an $\epsilon - \delta$ proof that* $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2+4y^2}{|x|+|y|} = 0$. *(Hint: you might find it useful to consider three cases: when $x, y \neq 0$, when $x = 0$, and when $y = 0$.)*

We have to prove the following: for every $\epsilon > 0$, there is a $\delta > 0$ such that if $0 < \sqrt{x^2+y^2} < \delta$, then

$$\left|\frac{x^2+4y^2}{|x|+|y|}\right| < \epsilon.$$

Let $\delta = \epsilon/5$. Let (x, y) be a point with $0 < \sqrt{x^2+y^2} < \delta$. Then $|x| \leq \sqrt{x^2+y^2} < \delta$ and $|y| \leq \sqrt{x^2+y^2} < \delta$. We consider three cases:

- If $x, y \neq 0$, we have

$$\left|\frac{x^2+4y^2}{|x|+|y|}\right| \leq \left|\frac{x^2}{|x|+|y|}\right| + \left|\frac{4y^2}{|x|+|y|}\right| \leq \left|\frac{x^2}{|x|}\right| + \left|\frac{4y^2}{|y|}\right|$$
$$= |x| + 4|y| < \frac{\epsilon}{5} + \frac{4\epsilon}{5} = \epsilon.$$

- If $x = 0$, then
$$\left|\frac{x^2+4y^2}{|x|+|y|}\right| = \frac{4y^2}{|y|} = 4|y| < \frac{4\epsilon}{5} < \epsilon.$$

- Similarly, if $y = 0$, then

$$\left|\frac{x^2}{|x|}\right| = \frac{x^2}{|x|} = |x| < \frac{\epsilon}{5} < \epsilon.$$

Hence $\delta = \epsilon/5$ works in all cases.

6. *(10 marks) Prove using the $\epsilon - \delta$ definition that* $\lim_{(x,y)\to(0,0)} \frac{2xy^3}{x^2+y^2} = 0$. *(Given $\epsilon > 0$, find $\delta > 0$ such that...)*

Let $f(x,y) = \frac{2xy^3}{x^2+y^2}$. Let $\epsilon > 0$. We need to find $\delta > 0$ such that if $0 < \sqrt{x^2+y^2} < \delta$, then $|f(x,y)| < \epsilon$. We first simplify the problem: using that $y^2 \leq x^2 + y^2$, we can estimate for $(x,y) \neq (0,0)$,

$$|f(x,y)| = |2xy| \frac{y^2}{x^2+y^2} \leq |2xy|.$$

Let $\delta = \sqrt{\epsilon/2}$. Then $\delta > 0$. Let (x,y) satisfy $0 < \sqrt{x^2+y^2} < \delta$, then $|x| \leq \sqrt{x^2+y^2} < \delta$ and $|y| \leq \sqrt{x^2+y^2} < \delta$, so that

$$|f(x,y)| \leq |2xy| < 2\delta^2 = 2(\epsilon/2) = \epsilon$$

and the $\epsilon - \delta$ condition is satisfied.

7. *Assume that the following is true for some function $f(x,y)$, defined for all x and y: for every $\epsilon > 0$ there is a $\delta > 0$ such that for some (x,y) such that $0 < \sqrt{(x-a)^2+(y-b)^2} < \delta$ we have $|f(x,y) - L| < \epsilon$ (make sure to read this carefully!). Does this imply that $\lim_{(x,y)\to(a,b)} f(x,y) = L$? If yes, prove it. If not, give an example of a function $f(x,y)$ that satisfies the above condition but does not have a limit at (a,b).*

It does not. The condition in the definition of the limit is "<u>for all</u> $(x,y)\dots$" not "<u>for some</u> (x,y)". For example, let

$$f(x,y) = \begin{cases} \frac{(x-y)^2}{x^2+y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

Then $f(x,y)$ does not have a limit at $(0,0)$, as proved in 2(b). But if $(a,b) = (0,0)$ and $L = 0$, then the statement in the problem is true: for every $\epsilon > 0$ there is a $\delta > 0$ such that $|f(x,y)| < \epsilon$ for *some* (x,y) such that $0 < \sqrt{x^2+y^2} < \delta$. For example, we could take $x = y = \delta/2$. Then $\sqrt{x^2+y^2} = \delta/\sqrt{2} < \delta$ and $|f(\delta/2, \delta/2)| = 0 < \epsilon$.

8. *Prove using the $\epsilon - \delta$ definition that if $f(x,y)$ is a function such that*

$$\lim_{x\to 0} f(x,0) = 4, \quad \lim_{y\to 0} f(0,y) = 1,$$

then the double limit $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist. (You need to prove that the "for every $\epsilon > 0$ there exists $\delta > 0$..." statement **cannot** *be true for a function as above.)*

We will prove this by contradiction. Suppose that $\lim_{(x,y)\to(0,0)} f(x,y) = L$ for some number L. By the triangle inequality,

$$|4 - L| + |L - 1| \geq |4 - 1| = 3$$

so that at least one of $|4-L|$ and $|L-1|$ is $\geq 3/2$. We will consider the case when $|L-4| \geq 3/2$; the case $|L-1| \geq 3/2$ is similar.

We will prove that the $\epsilon - \delta$ condition fails for $\epsilon = 1/2$. If $\lim_{(x,y)\to(0,0)} f(x,y) = L$, then the $\epsilon - \delta$ condition with $\epsilon = 1/2$ tells us this: there exists a $\delta > 0$ such that for all (x,y) with $0 < \sqrt{x^2+y^2} < \delta$ we have $|f(x,y) - L| < 1/2$. In particular, if we use the points on the x-axis, we get that

$$|f(x,0) - L| < 1/2 \text{ for all } |x| < \delta.$$

But also, we have $\lim_{x\to 0} f(x,0) = 4$. Applying the $\epsilon - \delta$ condition with $\epsilon = 1/2$ to that limit, we get the following: there exists $\delta_1 > 0$ such that if $0 < |x| < \delta_1$, then $|f(x,0) - 4| < 1/2$. Take $\delta' = \min(\delta, \delta_1)$, then for all x with $0 < |x| < \delta'$ we have both $|f(x,0) - L| < 1/2$ and $|f(x,0) - 4| < 1/2$. Therefore

$$|L - 4| = |(L - f(x,0)) + (f(x,0) - 4)| \leq |L - f(x,0)| + |f(x,0) - 4| < \frac{1}{2} + \frac{1}{2} = 1,$$

contradicting our assumption that $|L-4| \geq 3/2$. The case $|L-1| \geq 3/2$ is similar, but we would use the limit along the y-axis instead.

2.3 Partial Derivatives

1. *Let*

$$f(x,y) = \begin{cases} \dfrac{\sin^3 x}{x^2+y^2} & \textit{if } (x,y) \neq (0,0) \\ 0 & \textit{if } (x,y) = (0,0) \end{cases}$$

Find the first order partial derivatives of f at $(0,0)$ using the definition of partial derivatives.

Using that $\lim_{x\to 0} \frac{\sin x}{x} = 1$, we get

$$\begin{aligned} f_1(0,0) &= \lim_{x\to 0} \frac{f(x,0)-f(0,0)}{x} = \lim_{x\to 0} \frac{\sin^3 x - 0}{x^3} \\ &= \lim_{x\to 0} \left(\frac{\sin x}{x}\right)^3 = 1, \end{aligned}$$

$$f_2(0,0) = \lim_{y\to 0} \frac{f(0,y)-f(0,0)}{y} = \lim_{y\to 0} \frac{0}{y} = 0.$$

2. *Let*

$$f(x,y) = \begin{cases} \dfrac{x^3+xy-y^3}{x^2+y^2} & \textit{if } (x,y) \neq (0,0) \\ 0 & \textit{if } (x,y) = (0,0) \end{cases}$$

Find $\dfrac{\partial f}{\partial x}(0,0)$ *and* $\dfrac{\partial f}{\partial y}(0,0)$.

We have $f(x,0) = \dfrac{x^3}{x^2} = x$ and $f(0,y) = \dfrac{-y^3}{y^2} = -y$ for $x, y \neq 0$, and it is also true that $f(x,0) = 0 = x$ at $x = 0$ and $f(0,y) = 0 = -y$ at $y = 0$. Hence

$$\begin{aligned} \frac{\partial f}{\partial x}(0,0) &= \frac{\partial}{\partial x}(x)\Big|_{x=0} = 1, \\ \frac{\partial f}{\partial y}(0,0) &= \frac{\partial}{\partial y}(-y)\Big|_{y=0} = -1. \end{aligned}$$

2.4 Tangent Planes

1. *Find the equation of the tangent plane to the graph of the function* $f(x,y) = ye^{xy^2}$ *at the point* $(0,2)$.

   We have

$$\begin{aligned} f(0,2) &= 2 \cdot 1 = 2 \\ f_1(x,y) &= y \cdot y^2 e^{xy^2} = y^3 e^{xy^2}, \quad f_1(0,2) = 8, \\ f_2(x,y) &= e^{xy^2} + y \cdot 2xye^{xy^2} = (1+2xy^2)e^{xy^2}, \quad f_2(0,2) = 1. \end{aligned}$$

   Hence the equation of the tangent plane is

$$z = 2 + 8(x-0) + (y-2) = 8x + y.$$

2. *Find the equation of the tangent plane to the surface* $z = 5x^2y - y^2$ *at the point where* $x = -1$, $y = 3$.

   We have $\dfrac{\partial z}{\partial x} = 10xy$ and $\dfrac{\partial z}{\partial y} = 5x^2 - 2y$. At $(x,y) = (-1,3)$, we get

$$z = 15 - 9 = 6, \quad \frac{\partial z}{\partial x} = -30, \quad \frac{\partial z}{\partial y} = 5 - 6 = -1.$$

   The equation of the tangent plane is

$$z = 6 - 30(x+1) - (y-3)$$

   which simplifies to $z = -30x - y - 21$.

3. *Find all points* $(x, y, f(x,y))$ *where the tangent plane to the graph of the function* $f(x,y) = x^2 + 3xy - 2y^2$ *is parallel to the plane* $x + 10y - z = 3$. *For each such point, find the equation of the tangent plane.*

   We have $f_x(x,y) = 2x + 3y$ and $f_y(x,y) = 3x - 4y$, so the equation of the tangent plane at $(x,y) = (a,b)$ is

$$z = f(a,b) + (2a+3b)(x-a) + (3a-4b)(y-b),$$

   with the normal vector $\mathbf{N} = \langle -(2a+3b), -(3a-4b), 1) \rangle$. We want to find all (a,b) where this plane is parallel to the plane $x + 10y - z = 3$, with the normal vector $\mathbf{n} = \langle 1, 10, -1 \rangle$. This means that $\mathbf{N} = \lambda \mathbf{n}$ for some $\lambda \in \mathbb{R}$, so that

$$\begin{aligned} -(2a+3b) &= \lambda \\ -(3a-4b) &= 10\lambda \\ 1 &= -\lambda. \end{aligned}$$

The last equation says that $\lambda = -1$. From the first two equations, we then get

$$\begin{aligned} 2a + 3b &= 1 \\ 3a - 4b &= 10. \end{aligned}$$

We now solve this system. Here is one way to do it: adding the two equations, we get $5a - b = 11$, so that $b = 5a - 11$. Plugging this into the first equation, we get

$$\begin{aligned} 2a + 3b = 2a + 3(5a - 11) &= 2a + 15a - 33 = 1, \\ 17a &= 34, \end{aligned}$$

so that $a = 2$ and $b = 5a - 11 = 10 - 11 = -1$. We have $f(2, -1) = 4 - 6 - 2 = -4$. Thus the only point where the tangent plane is parallel to the plane $x + 10y - z = 3$ is $(2, -1, -4)$, and the equation of that tangent plane is $z = -4 + (x - 2) + 10(y + 1) = x + 10y + 4$.

4. *Find all points $P = (1, b, c)$ such that P lies on the paraboloid $z = x^2 + y^2$ and the tangent plane to this paraboloid at P passes through the point $(2, 4, 18)$.*

The partial derivatives of $f(x, y) = x^2 + y^2$ are $f_x(x, y) = 2x$ and $f_y(x, y) = 2y$. At $(x, y) = (a, b)$, the equation of the tangent plane is $z - (a^2 + b^2) = 2a(x - a) + 2b(y - b) = 2ax + 2by - 2a^2 - 2b^2$. Letting $a = 1$, we get the equation

$$z = 2x + 2by - 1 - b^2.$$

We are looking for all values of b such that the point $(2, 4, 18)$ satisfies that equation: $18 = 4 + 8b - 1 - b^2$, $b^2 - 8b + 15 = 0$, $b = 3$ or 5. The corresponding points on the paraboloid are $(1, 3, 10)$ and $(1, 5, 26)$.

5. *Find all values of (a, b) such that the tangent plane to the surface $z = 4x^2 - y^2$ at $(a, b, 4a^2 - b^2)$ is parallel to the line with parametric equations $x = t + 1$, $y = 2t$, $z = -4t + 9$.*

Let $f(x, y) = 4x^2 - y^2$, then $f_x(a, b) = 8a$ and $f_y(a, b) = -2b$. Hence the tangent plane at $(a, b, 4a^2 - b^2)$ is perpendicular to the vector $\mathbf{n} = -8a\mathbf{i} + 2b\mathbf{j} + \mathbf{k}$. We want $\mathbf{n}$ to be perpendicular to the direction vector $\mathbf{v}$ of the given line. From the equations of the line, $\mathbf{v} = \mathbf{i} + 2\mathbf{j} - 4\mathbf{k}$. We therefore want $\mathbf{n} \cdot \mathbf{v} = 0$, so that $-8a + 4b - 4 = 0$, or simplifying, $-2a + b = 1$. All values (a, b) with $b = 1 + 2a$ satisfy this condition.

2.5 Higher Order Derivatives

1. *Let* $f(x, y) = x\sin(xy^2)$.

   (a) Find all first and second order partial derivatives of f.

$$\begin{aligned}
f_1(x, y) &= \sin(xy^2) + x\cos(xy^2)\cdot y^2 \\
f_2(x, y) &= x\cos(xy^2)\cdot 2xy = 2x^2y\cos(xy^2) \\
f_{11}(x, y) &= y^2\cos(xy^2) + \cos(xy^2)\cdot y^2 + xy^2(-\sin(xy^2)\cdot y^2) \\
&= 2y^2\cos(xy^2) - xy^4\sin(xy^2) \\
f_{12}(x, y) &= 4xy\cos(xy^2) + 2x^2y(-\sin(xy^2)\cdot y^2) \\
&= 4xy\cos(xy^2) - 2x^2y^3\sin(xy^2) \\
f_{22}(x, y) &= 2x^2\cos(xy^2) + 2x^2y(-\sin(xy^2)\cdot 2xy) \\
&= 2x^2\cos(xy^2) - 4x^3y^2\sin(xy^2)
\end{aligned}$$

   Since f and all its derivatives are continuous, $f_{12} = f_{21}$.

   (b) Find the equation of the tangent plane to the surface $z = f(x, y)$ *at the point where* $x = \pi$, $y = 1$.

   From (a), $f_1(x, y) = \sin(xy^2) + x\cos(xy^2)\cdot y^2$ and $f_2(x, y) = x\cos(xy^2)\cdot 2xy$, so that $f_1(\pi, 1) = \sin\pi + \pi\cos\pi\cdot 1 = 0 - \pi = -\pi$ and $f_2(\pi, 1) = \pi\cos\pi\cdot 2\pi = -2\pi^2$. We also have $f(\pi, 1) = \pi\sin\pi = 0$. Therefore the tangent plane is

$$z = -\pi(x - \pi) - 2\pi^2(y - 1) = -\pi x - 2\pi^2y + 3\pi^2.$$

2. *A function* $f(x_1, ..., x_n)$ *of* n*-many variables is said to be* ***harmonic*** *if it satisfies the equation,*

$$\frac{\partial^2 f}{\partial x_1^2} + \cdots + \frac{\partial^2 f}{\partial x_n^2} = 0.$$

   Demonstrate that the following functions are harmonic.

   (a) $f(x, y) = \frac{x}{x^2+y^2}$ *for* $(x, y) \neq 0$;

   We need to calculate the derivatives $f_{xx}(x, y)$ and $f_{yy}(x, y)$. The first-order partials can be calculated by applying the quotient rule, giving

$$f_x(x, y) = \frac{y^2 - x^2}{(x^2 + y^2)^2},\ f_y(x, y) = \frac{-2xy}{(x^2 + y^2)^2}.$$

   A second application of the quotient rule gives,

$$f_{xx}(x, y) = \frac{2x(x^2 - 3y^2)}{(x^2 + y^2)^3},\ f_{yy}(x, y) = -\frac{2x(x^2 - 3y^2)}{(x^2 + y^2)^3}.$$

Since $f_{xx}(x,y) = -f_{yy}(x,y)$ for any $(x,y) \neq (0,0)$, we see that $f(x,y)$ is indeed harmonic away from the origin.

(b) $f(x,y,z) = \frac{x}{(x^2+y^2+z^2)^{3/2}}$ *for* $(x,y,z) \neq (0,0,0)$ *(a longer calculation);*

We again need to calculate the partial derivatives $f_{xx}(x,y,z)$, $f_{yy}(x,y,z)$ and $f_{zz}(x,y,z)$ for $(x,y,z) \neq (0,0,0)$. The differentiation in x is the most complicated; the Quotient Rule gives

$$\begin{aligned}
f_x(x,y,z) &= \frac{(x^2+y^2+z^2)^{3/2} - x \cdot 2x \cdot 3/2\sqrt{x^2+y^2+z^2}}{(x^2+y^2+z^2)^3} \\
&= \frac{(x^2+y^2+z^2)^{3/2} - 3x^2\sqrt{x^2+y^2+z^2}}{(x^2+y^2+z^2)^3} \\
\text{(cancelling common factors)} &= \frac{x^2+y^2+z^2-3x^2}{(x^2+y^2+z^2)^{5/2}} \\
&= \frac{-x^2+y^2+z^2}{(x^2+y^2+z^2)^{5/2}}
\end{aligned}$$

The differentiation in y and z are symmetric (since the numerator is not a function of y or z). Using the Quotient Rule (or simply the Chain Rule, if you are clever with negative exponents), one arrives at

$$f_y(x,y) = -\frac{3xy}{(x^2+y^2+z^2)^{5/2}}, \ f_z(x,y) = -\frac{3xz}{(x^2+y^2+z^2)^{5/2}}$$

which is again symmetric in the variables y and z. In particular, if we know $f_{yy}(x,y,z)$ we can perform the change of variables $y \mapsto z$ and $z \mapsto y$ to find $f_{zz}(x,y,z)$. Using the Quotient Rule again gives

$$\begin{aligned}
f_{yy}(x,y,z) &= -\frac{3x(x^2+y^2+z^2)^{5/2} - (3xy)\cdot(2y)\cdot 5/2(x^2+y^2+z^2)^{3/2}}{(x^2+y^2+z^2)^5} \\
&= -\frac{3x(x^2+y^2+z^2)^{5/2} - 15xy^2(x^2+y^2+z^2)^{3/2}}{(x^2+y^2+z^2)^5} \\
&= -\frac{3x^3+3xy^2+3xz^2-15xy^2}{(x^2+y^2+z^2)^{7/2}} \\
&= -\frac{3x(x^2-4y^2+z^2)}{(x^2+y^2+z^2)^{7/2}}
\end{aligned}$$

Using the aforementioned change of variables $(y,z) \mapsto (z,y)$, we have

$$f_{zz}(x,y,z) = -\frac{3x(x^2+y^2-4z^2)}{(x^2+y^2+z^2)^{7/2}}$$

Finally, we could calculate $f_{xx}(x,y,z)$ using the Quotient Rule, which would give

$$f_{xx}(x,y,z) = \frac{6x^3 - 9x(y^2+z^2)}{(x^2+y^2+z^2)^{7/2}}.$$

Notice that f_{xx}, f_{yy} and f_{zz} all share a common denominator. So, to verify that $f(x,y,z)$ is harmonic, we need only add together the numerators. This yields

$$\underbrace{6x^3 - 9x(y^2+z^2)}_{\text{num. of } f_{xx}} + \underbrace{(-3x(x^2-4y^2+z^2))}_{\text{num. of } f_{yy}} + \underbrace{(-3x(x^2+y^2-4z^2))}_{\text{num. of } f_{zz}}$$

and combining like terms shows that $f_{xx}+f_{yy}+f_{zz}=0$ for any $(x,y,z)\neq(0,0,0)$.

(c) $f(x_1,...,x_n) = \frac{1}{(x_1^2+\cdots+x_n^2)^{n/2-1}}$ *for* $(x_1,...,x_n)\neq(0,...,0)$ *and* $n\geq 3$.

We rewrite

$$f(x_1,...,x_n) = \frac{1}{(x_1^2+\cdots+x_n^2)^{n/2-1}} = (x_1^2+\cdots+x_n^2)^{-n/2+1},$$

and use the Chain Rule to determine for $(x_1,...,x_n)\neq(0,...,0)$

$$\begin{aligned} f_j(x_1,...,x_n) &= \frac{\partial}{\partial x_j}\left[(x_1^2+\cdots+x_n^2)^{-n/2+1}\right] \\ &= -\frac{n-2}{2}\frac{2x_j}{(x_1^2+\cdots+x_n^2)^{n/2}} \\ &= -\frac{(n-2)x_j}{(x_1^2+\cdots+x_n^2)^{n/2}}. \end{aligned}$$

Applying the derivative in x_j again gives

$$\begin{aligned} f_{jj}(x_1,...,x_n) &= -\frac{(n-2)(x_1^2+\cdots+x_n^2)^{n/2} - ((n-2)x_j)(nx_j)(x_1^2+\cdots+x_n^2)^{(n-2)/2}}{(x_1^2+\cdots+x_n^2)^n} \\ &= -\frac{(n-2)(x_1^2+\cdots+x_n^2) - n(n-2)x_j^2}{(x_1^2+\cdots+x_n^2)^{(n+2)/2}} \\ &= -\frac{n-2}{(x_1^2+\cdots+x_n^2)^{(n+2)/2}}\left[(x_1^2+\cdots+x_n^2) - nx_j^2\right] \end{aligned}$$

Notice that, the terms outside the large brackets are symmetric in the variables $x_1,...,x_n$. That is, regardless of the choice of $1\leq j\leq n$, the same term will appear on the outside when we calculate f_{jj}. It then follows that

$$f_{11}+\cdots+f_{nn}=0$$

because one has

$$\sum_{j=1}^{n}\left[(x_1^2+\cdots+x_n^2) - nx_j^2\right] = n(x_1^2+\cdots+x_n^2) - n(x_1^2+\cdots+x_n^2) = 0.$$

Remark 2.5.1. *As you surely know by know: it is often an absolute pain to determine by brute calculation whether a given function $f(x_1, ..., x_n)$ is harmonic. Fortunately, mathematicians have developed more nuanced tools in areas like Complex Analysis and Partial Differential Equations which given necessary or sufficient conditions for a function f to be harmonic on some domain. More on this in later courses!*

3. *Let*

$$F(x,y) = \begin{cases} \dfrac{xy(x^2-y^2)}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

Demonstrate carefully that both $F_{12}(0,0)$ and $F_{21}(0,0)$ exist and <u>*are not equal*</u>*. Why does this not violate Theorem 2.5.2?*

Hint: we advise that you proceed by obtaining the following in order:

- *$F_1(0,y)$ for all $y \in \mathbb{R}$ and $F_2(x,0)$ for all $x \in \mathbb{R}$,*
- *$F_{12}(0,0)$ and $F_{21}(0,0)$.*

If done correctly, you will obtain that $F_{21}(0,0) = 1$ and $F_{12}(0,0) = -1$.

For $(x,y) \neq (0,0)$, we have

$$\begin{aligned} F_1(x,y) &= \frac{\partial}{\partial x}\left[\frac{xy(x^2-y^2)}{x^2+y^2}\right] \\ &= \frac{y(x^2-y^2)}{x^2+y^2} + \frac{xy \cdot 2x}{x^2+y^2} - \frac{xy(x^2-y^2)\cdot 2x}{(x^2+y^2)^2} \\ &= \frac{y(x^2+y^2-2x^2)(x^2-y^2)}{(x^2+y^2)^2} + \frac{2x^2y}{x^2+y^2} \\ &= -\frac{y(x^2-y^2)^2}{(x^2+y^2)^2} + \frac{2x^2y}{x^2+y^2}. \end{aligned}$$

When $x = 0$ and $y \neq 0$, we get

$$F_1(0,y) = -\frac{y(-y^2)^2}{(y^2)^2} + \frac{0}{y^2} = -\frac{y^5}{y^4} = -y.$$

We also have $F(x,y) = -F(y,x)$, so that by symmetry, we have for $x \neq 0$

$$\begin{aligned} F_2(x,y) &= -F_1(y,x) = \frac{x(y^2-x^2)^2}{(x^2+y^2)^2} - \frac{2xy^2}{x^2+y^2}, \\ F_2(x,0) &= \frac{x(-x^2)^2}{(x^2)^2} - \frac{0}{x^2} = \frac{x^5}{x^4} = x. \end{aligned}$$

Also, we have $F(0,x) = F(y,0) = F(0,0) = 0$. so that $F_1(0,0) = F_2(0,0) = 0$. Therefore the formulas $F_1(0,y) = -y$ and $F_2(x,0) = x$ are in fact valid for all x and y, including $x = 0$ and $y = 0$. It follows that

$$F_{12}(0,0) = \frac{d}{dy}F_1(0,y)\Big|_{y=0} = \frac{d}{dy}(-y)\Big|_{y=0} = -1,$$

$$F_{21}(0,0) = \frac{d}{dx}F_2(x,0)\Big|_{x=0} = \frac{d}{dx}(x)\Big|_{x=0} = 1.$$

(We used $\frac{d}{dy}$ notation here because $F_1(0,y)$ has only one variable left, but $\frac{\partial}{\partial y}$ would also be acceptable in this context.)

To explain why this function $f(x,y)$ does not violate Theorem 2.5.2, we calculate the mixed partial derivative $F_{12}(x,y)$ for $(x,y) \neq (0,0)$:

$$\begin{aligned}
F_{12}(x,y) &= -\frac{(x^2-y^2)^2}{(x^2+y^2)^2} - \frac{y(x^2-y^2)(-2y)}{(x^2+y^2)^2} - (-1)\frac{2y(x^2-y^2)^2(2y)}{(x^2+y^2)^3} \\
&\quad + \frac{2x^2(x^2+y^2) - 2x^2y\cdot 2y}{(x^2+y^2)^3} \\
&= \frac{(x^2-y^2)\big(-(x^2-y^2)(x^2+y^2) + 2y^2(x^2+y^2) + 2y^2(x^2-y^2)\big)}{(x^2+y^2)^3} \\
&\quad + \frac{2x^2(x^2-y^2)}{(x^2+y^2)^3} \\
&= \frac{(x^2-y^2)\big(y^4+5x^2y^2+x^4\big)}{(x^2+y^2)^3} = \frac{\big(x^6+9x^4y^2-9x^2y^4-y^6\big)}{(x^2+y^2)^3}
\end{aligned}$$

and similarly,

$$F_{21}(x,y) = -F_{12}(y,x) = -\frac{(y^2-x^2)\big(y^4+10x^2y^2+x^4\big)}{(x^2+y^2)^3} = F_{12}(x,y)$$

The mixed partials F_{12} and F_{21} are equal for all $(x,y) \neq (0,0)$, but they are not continuous at $(0,0)$, e.g. $F_{12}(x,x) = 0 \not\to -1$ as $x \to 0$. Therefore the fact that $F_{12}(0,0) \neq F_{21}(0,0)$ does not contradict Theorem 2.5.2.

Optional: Let us try to visualize the above function. If we make the change of variables to polar coordinates, notice that

$$f(r\cos\theta, r\sin\theta) = \frac{r^4\sin\theta\cos\theta(\cos^2\theta - \sin^2\theta)}{r^2} = r^2\sin\theta\cos\theta(\cos^2\theta - \sin^2\theta).$$

Using the trigonometric identities $2\sin\theta\cos\theta = \sin(2\theta)$ and $\cos^2\theta - \sin^2\theta = \cos(2\theta)$, this becomes:

$$f(r\cos\theta, r\sin\theta) = r^2\frac{\sin 2\theta\cos 2\theta}{2} = r^2\frac{\sin 4\theta}{4}$$

This leads to the graph shown below.

Consider also what happens when we try to evaluate the limit

$$\lim_{(x,y)\to(0,0)} \frac{Q(0,0,x,y)}{xy} = \lim_{(x,y)\to(0,0)} \frac{1}{xy}\Big(f(x,y) - f(0,y) - f(x,0) + f(0,0)\Big)$$

from Section 2.5. We have $f(0,y) = f(x,0) = f(0,0) = 0$, so the above limit is

$$\lim_{(x,y)\to(0,0)} \frac{f(x,y)}{xy} = \lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}.$$

This limit does not exist (consider the lines $x = 0$ and $y = 0$). This does not actually prove that the mixed partials are not equal (that's why we did the long calculation above!), but it does provide some intuition as to what might be going wrong.

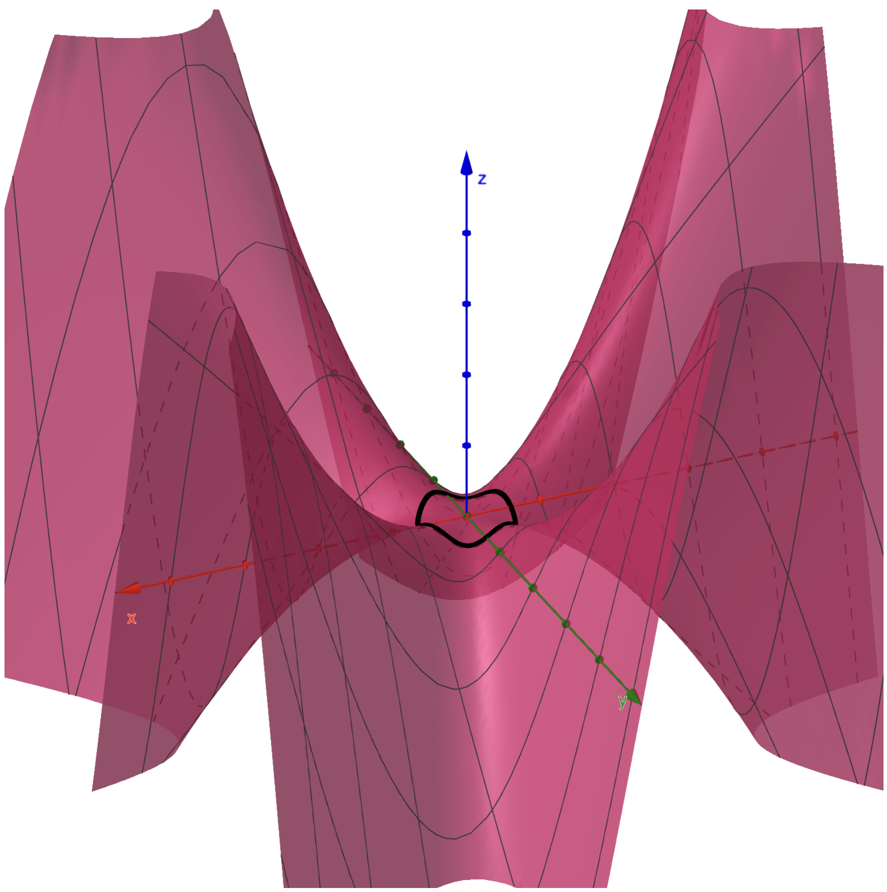

Figure 2.1: A graph of the function $f(x,y)$ from Problem 3.

2.6 The Chain Rule

1. *Let* $z = f(x,y)$*, where* $f(x,y)$ *is* C^1 *for all* x, y*. Suppose that* $f_x(\sqrt{3},1) = 3$ *and*

$f_y(\sqrt{3},1) = 5$. If we switch to polar coordinates r,θ so that $x = r\cos\theta$, $y = r\sin\theta$, what is $\frac{\partial z}{\partial \theta}$ at $(r,\theta) = (2,\pi/6)$? (Hint: $\sin(\pi/6) = 1/2$, $\cos(\pi/6) = \sqrt{3}/2$.)

At $(r,\theta) = (2,\pi/6)$, we have $x = 2\cos(\pi/6) = \sqrt{3}$, $y = 2\sin(\pi/6) = 1$. Also, at the same point,

$$\frac{\partial x}{\partial \theta} = -r\sin\theta = -1, \quad \frac{\partial y}{\partial \theta} = r\cos\theta = \sqrt{3}.$$

Therefore

$$\frac{\partial z}{\partial \theta} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial \theta} = 3(-1) + 5(\sqrt{3}) = 5\sqrt{3} - 3.$$

2. *Suppose that the temperature at a point with coordinates (x,y,z) and at time t is given by $T(x,y,z,t) = 20 + 3x - 2y + 4z^2 + xye^{0.03t}$. At time $t = 3$, a drone is at the point $(5,2,1)$ and flies along a trajectory such that $x'(3) = 7$, $y'(3) = 2$, $z'(3) = -1$. What is the rate of change of the temperature measured by the drone at $t = 3$?*

   We have

$$\begin{aligned}
&T_x(x,y,z,t) = 3 + ye^{0.03t}, \quad T_x(5,2,1,3) = 3 + 2e^{0.09},\\
&T_y(x,y,z,t) = -2 + xe^{0.03t}, \quad T_y(5,2,1,3) = -2 + 5e^{0.09},\\
&T_z(x,y,z,t) = 4z, \quad T_z(5,2,1,3) = 8,\\
&T_t(x,y,z,t) = 0.03xye^{0.03t}, \quad T_t(5,2,1,3) = 10\cdot 0.03e^{0.09} = 0.3e^{0.09}.
\end{aligned}$$

   Therefore at $t = 3$, we have

$$\begin{aligned}
\frac{dT}{dt} &= T_x(5,2,1,3)x'(3) + T_y(5,2,1,3)y'(3) + T_z(5,2,1,3)z'(3) + T_t(5,2,1,3)\\
&= (3 + 2e^{0.09})\cdot 7 + (-2 + 5e^{0.09})\cdot 2 + 8(-1) + 0.3e^{0.09}\\
&= 21 - 4 - 8 + (14 + 10 + 0.3)e^{0.09} = 9 + 24.3e^{0.09}.
\end{aligned}$$

3. *Let $z = f(u,v)$, where $f(u,v)$ is C^1 for all u,v. Let $u = xy$, $v = y/x$.*

   (a) Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ in terms of f_1 and f_2.

   We compute

$$\begin{aligned}
\frac{\partial z}{\partial x} &= \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x}\\
&= f_1(xy,y/x)\cdot y + f_2(xy,y/x)\cdot\Big(-\frac{y}{x^2}\Big)\\
\frac{\partial z}{\partial y} &= \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y}\\
&= f_1(xy,y/x)\cdot x + f_2(xy,y/x)\cdot\Big(\frac{1}{x}\Big)
\end{aligned}$$

(b) If $f_1\Big(-6,-\dfrac{3}{2}\Big)=4$ *and* $f_2\Big(-6,-\dfrac{3}{2}\Big)=-1$, *find* $\dfrac{\partial z}{\partial x}$ *and* $\dfrac{\partial z}{\partial y}$ *at* $(x,y)=(-2,3)$.

For $(x,y)=(-2,3)$, we have $(u,v)=(xy,y/x)=(-6,-\frac{3}{2})$, so that

$$\begin{aligned}\frac{\partial z}{\partial x} &= f_1(-6,-\frac{3}{2})\cdot 3+f_2(-6,-\frac{3}{2})\cdot\Big(-\frac{3}{4}\Big)=4\cdot 3+(-1)\cdot\Big(-\frac{3}{4}\Big)\\ &=12+\frac{3}{4}=12.75\\ \frac{\partial z}{\partial y} &= f_1(-6,-\frac{3}{2})\cdot(-2)+f_2(-6,-\frac{3}{2})\cdot\Big(-\frac{1}{2}\Big)=4\cdot(-2)+(-1)\cdot\Big(-\frac{1}{2}\Big)\\ &=-8+\frac{1}{2}=-7.5\end{aligned}$$

4. *Suppose that a function $f(x,y)$ is defined everywhere in the plane and has continuous partial derivatives of all orders. Assume that $f(2,3)=2$, $f_1(2,3)=-1$, $f_2(2,3)=4$, $f_{11}(2,3)=0$, $f_{12}(2,3)=f_{21}(2,3)=3$, $f_{22}(2,3)=1$, Find*

$$\frac{\partial}{\partial x}f(2x^2,x+y)\ \textit{and}\ \frac{\partial^2}{\partial x^2}f(2x^2,x+y)\ \textit{at}\ (x,y)=(1,2).$$

We start with

$$\frac{\partial}{\partial x}f(2x^2,x+y)=f_1(2x^2,x+y)\cdot(4x)+f_2(2x^2,x+y)\cdot 1.$$

At $x=1,y=2$, we have $2x^2=2$, $x+y=3$, and

$$\frac{\partial}{\partial x}f(2x^2,x+y)\Big|_{(x,y)=(1,2)}=f_1(2,3)\cdot 4+f_2(2,3)\cdot 1=-1\cdot 4+4\cdot 1=0.$$

Next, using Chain Rule again,

$$\begin{aligned}\frac{\partial^2}{\partial x^2}f(2x^2,x+y)&=\frac{\partial}{\partial x}\Big(f_1(2x^2,x+y)\cdot(4x)\Big)+\frac{\partial}{\partial x}f_2(2x^2,x+y)\\ &=\Big(f_{11}(2x^2,x+y)\cdot(4x)+f_{12}(2x^2,x+y)\cdot 1\Big)(4x)+f_1(2x^2,x+y)\cdot 4\\ &\quad+f_{21}(2x^2,x+y)\cdot(4x)+f_{22}(2x^2,x+y)\cdot 1.\end{aligned}$$

Therefore

$$\begin{aligned}\frac{\partial^2}{\partial x^2}f(2x^2,x+y)\Big|_{(x,y)=(1,2)}&=\Big(f_{11}(2,3)\cdot 4+f_{12}(2,3)\Big)\cdot 4\\ &\quad+f_1(2,3)\cdot 4+f_{21}(2,3)\cdot 4+f_{22}(2,3)\\ &=(0\cdot 4+3)\cdot 4+(-1)\cdot 4+3\cdot 4+1\\ &=12-4+12+1=21.\end{aligned}$$

Chapter 3

Differentiability

3.1 Linear Approximation and Differentiability

1. *Let* $f(x, y) = ye^{xy^2}$.

   (a) Find the equation of the tangent plane to the graph of the function $f(x, y)$ *at the point* $(x, y) = (0, 2)$.

   We compute

$$\begin{aligned} f(0,2) &= 2 \cdot 1 = 2 \\ f_1(x,y) &= y \cdot y^2 e^{xy^2} = y^3 e^{xy^2}, \quad f_x(0,2) = 8, \\ f_2(x,y) &= e^{xy^2} + y \cdot 2xye^{xy^2} = (1 + 2xy^2)e^{xy^2}, \quad f_y(0,2) = 1. \end{aligned}$$

   Hence the equation of the tangent plane is

$$z = 2 + 8(x - 0) + (y - 2) = 8x + y.$$

   (b) Use linear approximation at $(0, 2)$ *to give an approximate value of* $f(-0.2, 2.01)$.

   From (a), the linear approximation at $(0, 2)$ is

$$L(x, y) = 2 + 8(x - 0) + (y - 2) = 8x + y.$$

   Hence

$$f(-0.2, 2.01) \approx L(-0.2, 2.01) = 2 + 8(-0.2) + (0.01) = 2 - 1.6 + .01 = 0.41$$

   (c) Find $f_{12}(x, y)$.

Differentiate f_1 in y:

$$\begin{aligned} f_{12}(x,y) &= \frac{\partial}{\partial y} y^3 e^{xy^2} \\ &= 3y^2 e^{xy^2} + y^3 \cdot 2xye^{xy^2} \\ &= (3y^2 + 2xy^4)e^{xy^2}. \end{aligned}$$

2. *Let* $z = \frac{\sin(x)}{\cos(y)} = \sin(x)\sec(y)$.

(a) Find the general expression for the differential of $z(x,y)$ *for* x *and* y *in the interval* $[0, \frac{\pi}{2})$.

We first calculate the first-order derivatives,

$$\frac{\partial z}{\partial x} = \cos(x)\sec(y), \quad \frac{\partial z}{\partial y} = \sin(x)\sec(y)\tan(y).$$

Hence, the formula for our differential of z is

$$z(x,y) = \frac{\partial z}{\partial x}\, dx + \frac{\partial z}{\partial y}\, dy = \Big[\cos(x)\sec(y)\Big] dx + \Big[\sin(x) sec(y)\tan(y)\Big] dy.$$

(b) Now, utilizing the differential calculated in Part (a), approximate the value of the function at $(a,b) = (0.1, 0.3)$ *using a nearby point.*

The nearest convenient point of approximation is $(0,0)$. We calculate

$$z(0,0) = 0, \quad \frac{\partial z}{\partial x}(0,0) = 1, \quad \frac{\partial z}{\partial y}(0,0) = 0.$$

Hence, our linearization at $(0,0)$ is simply:

$$L(x,y)(x,y) = z(0,0) + \frac{\partial z}{\partial x}(0,0)\cdot x + \frac{\partial z}{\partial y}(0,0)\cdot y = x,$$

and so we take our approximation to be:

$$z(0.1, 0.3) \approx L(0.1, 0.3) = 0.1$$

(c) Rather than utilizing the the linearization formula from Lecture Notes 3.1, utilize first order Taylor approximation for sin *and* cos *to give another estimation of the function at* $(0.1, 0.3)$. *Do your approximations agree?*

Using single-variable first-order Taylor approximation for $x, y \approx y$, we have:

$$\sin(x) \approx x, \quad \cos(y) \approx 1$$

In particular, we see that:

$$z(x,y) = \frac{\sin(x)}{\cos(y)} \approx x,$$

which agrees with our previous approximation.

3. *Suppose that $f(x,y)$ is a differentiable function of x and y, and that $x = x(s,t)$ and $y = y(s,t)$ are themselves differentiable functions of s, t.*

(a) Using the Multivariable Chain Rule (see Lecture Notes 2.6), find a general formula for the linearization of $f(x(s,t), y(s,t))$ for (s,t) near a point (a,b) in terms of the first-order partial derivatives of the functions f, x and y.

Let $x_0 = x(a,b)$ and $y_0 = y(a,b)$. The linearization of f near a point $(s,t) = (a,b)$ then takes form

$$L(s,t) := f(x_0,y_0) + \frac{\partial f}{\partial s}(x_0,y_0)(s-a) + \frac{\partial f}{\partial t}(x_0,y_0)(t-b).$$

The multivariable chain rule states that

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s}$$

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t}.$$

Hence, our (very long) formula for the linearization of f then becomes

$$\begin{aligned} L(s,t) \quad = \quad & f(x_0,y_0) + \left[\frac{\partial f}{\partial x}(x_0,y_0)\frac{\partial x}{\partial s}(a,b) + \frac{\partial f}{\partial y}(x_0,y_0)\frac{\partial y}{\partial s}(a,b)\right](s-a) \\ & + \left[\frac{\partial f}{\partial x}(x_0,y_0)\frac{\partial x}{\partial t}(a,b) + \frac{\partial f}{\partial y}(x_0,y_0)\frac{\partial y}{\partial t}(a,b)\right](t-b) \end{aligned}$$

Definition 3.1.1. *A function $g(x,y)$ of two-variables is called **Lipschitz** if there exists a number $C > 0$ such that, given any pair of points (x_0,y_0) and (x_1,y_1) in $\mathbb{R}^2$, one has*

$$\left|g(x_1,y_1) - g(x_0,y_0)\right| \leq C\sqrt{(x_1-x_0)^2 + (y_1-y_0)^2}.$$

*The smallest value of $C > 0$ for which the above inequality holds is known as the **Lipschitz constant** of g.*

(b) Suppose that $f(x,y)$, $x(s,t)$ and $y(s,t)$ are all functions whose first-order partial derivatives are bounded by some number $M > 0$. By ***bounded****, we mean that:*

$$\Big|\frac{\partial f}{\partial x}(x,y)\Big|,\ \Big|\frac{\partial f}{\partial y}(x,y)\Big| \leq M,\ \textit{for every pair } (x,y) \in \mathbb{R}^2,$$

$$\Big|\frac{\partial x}{\partial s}(s,t)\Big|,\ \Big|\frac{\partial x}{\partial t}(s,t)\Big|, \leq M,\ \textit{for every pair } (s,t) \in \mathbb{R}^2.$$

$$\Big|\frac{\partial y}{\partial s}(s,t)\Big|,\ \Big|\frac{\partial y}{\partial t}(s,t)\Big|, \leq M,\ \textit{for every pair } (s,t) \in \mathbb{R}^2.$$

Prove that for any $(a,b) \in \mathbb{R}^2$, the linearization of $f(x(s,t),y(s,t))$ for (s,t) near (a,b) is Lipschitz, with Lipschitz constant $C \leq 4M^2$.

Hint: this is tricky! You will need four tools to prove this. They are (in order of likely use): the formula you obtained from Part (a.), the triangle inequality (used twice), the bound on the first-order partial derivatives, and the inequality

$$|U| + |V| \leq 2\max\{|U|,|V|\} \leq 2\sqrt{|U|^2 + |V|^2},$$

which is valid for any two real numbers U,V (prove this to yourself!)

Let (a,b) be given and again let $(x_0,y_0) := (x(a,b),y(a,b))$. To show that the linearization L near (a,b) is Lipschitz, we consider points (s_0,t_0) and (s_1,t_1), and prove that:

$$\big|L(s_1,t_1) - L(s_0,t_0)\big| \leq 4M^2\sqrt{(s_1-s_0)^2 + (t_1-t_0)^2}.$$

We first use the formula from Part (a.), which tells us that for any $(s,t) \in \mathbb{R}^2$

$$\begin{aligned} L(s,t) \quad = \quad & f(x_0,y_0) + \underbrace{\left[\frac{\partial f}{\partial x}(x_0,y_0)\frac{\partial x}{\partial s}(a,b) + \frac{\partial f}{\partial y}(x_0,y_0)\frac{\partial y}{\partial s}(a,b)\right]}_{A_s}(s-a) \\ & + \underbrace{\left[\frac{\partial f}{\partial x}(x_0,y_0)\frac{\partial x}{\partial t}(a,b) + \frac{\partial f}{\partial y}(x_0,y_0)\frac{\partial y}{\partial t}(a,b)\right]}_{A_t}(t-b), \end{aligned}$$

and notice that the terms A_s and A_t do not depend upon any of the variables s,t,x or y! In particular, this means that:

$$\begin{aligned} \big|L(s_1,t_1) - L(s_0,t_0)\big| \quad &= \quad \bigg|\Big[f(x_0,y_0) + A_s(s_1-a) + A_t(t_1-b)\Big] \\ &\qquad - \Big[f(x_0,y_0) + A_s(s_0-a) + A_t(t_0-b)\Big]\bigg| \\ &= \quad \big|A_s(s_1-s_0) + A_t(t_1-t_0)\big|. \end{aligned}$$

Now, use the triangle inequality for the first time, giving

$$\big|L(s_1,t_1)-L(s_0,t_0)\big| \le |A_s||s_1-s_0|+|A_t||t_1-t_0|. \tag{3.1.1}$$

We now look at the terms A_s and A_t, which involve the partial derivatives of $f(x,y)$, $x(s,t)$ and $y(s,t)$. Using the triangle inequality again, we see that

$$\begin{aligned}
\big|A_s\big| &\le \left|\frac{\partial f}{\partial x}(x_0,y_0)\frac{\partial x}{\partial s}(a,b)\right| + \left|\frac{\partial f}{\partial y}(x_0,y_0)\frac{\partial y}{\partial s}(a,b)\right| \\
&= \left|\frac{\partial f}{\partial x}(x_0,y_0)\right|\left|\frac{\partial x}{\partial s}(a,b)\right| + \left|\frac{\partial f}{\partial y}(x_0,y_0)\right|\left|\frac{\partial y}{\partial s}(a,b)\right| \\
&\le 2M^2,
\end{aligned}$$

where, in the final inequality, we used the fact that all partial derivatives are uniformly bounded by the number $M>0$. Substituting this estimate into inequality (3.1.1) shows us that

$$\big|L(s_1,t_1)-L(s_0,t_0)\big| \le 2M^2\big(|s_1-s_0|+|t_1-t_0|\big).$$

We finish by bounding the sum by twice the maximum, which is bounded by square root of the sum of squares

$$|s_1-s_0|+|t_1-t_0| \le 2\max\{\cdots\} \le 2\sqrt{|s_1-s_0|^2+|t_1-t_0|^2}$$

Since this inequality holds for two arbitrary pairs $(s_1,t_1),(s_0,t_0)\in\mathbb{R}^2$, we have shown that $L(s,t)$ is Lipschitz with the appropriate constant.

3.2 More on Differentiability

1. *(In this question, you have to use the definitions of partial derivatives and differentiability.) Let*

$$f(x,y)=\begin{cases}\dfrac{x^3-\sin^3 x}{x^2+y^2} & \textit{if } (x,y)\neq(0,0)\\ 0 & \textit{if } (x,y)=(0,0)\end{cases}$$

(a) Find the first order partial derivatives of f at $(0,0)$.

Using that $\lim_{x\to 0}\frac{\sin x}{x}=1$, we get

$$\begin{aligned}
f_1(0,0) &= \lim_{x\to 0}\frac{f(x,0)-f(0,0)}{x} = \lim_{x\to 0}\frac{x^3-\sin^3 x}{x^3} \\
&= \lim_{x\to 0}\left(1-\frac{\sin^3 x}{x^3}\right) = 1-1=0, \\
f_2(0,0) &= \lim_{y\to 0}\frac{f(0,y)-f(0,0)}{y} = \lim_{y\to 0}\frac{0}{y^3}=0.
\end{aligned}$$

(b) Is f differentiable at $(0,0)$? Prove your answer.

Based on part (a), the linear approximation to f at $(0,0)$ is $L(x,y)=0$. For f to be differentiable at $(0,0)$, we need

$$\lim_{(x,y)\to(0,0)} \frac{f(x,y)-L(x,y)}{\sqrt{x^2+y^2}} = \lim_{(x,y)\to(0,0)} \frac{x^3-\sin^3 x}{(x^2+y^2)^{3/2}} = 0.$$

We have

$$\lim_{(x,y)\to(0,0)} \left|\frac{x^3-\sin^3 x}{(x^2+y^2)^{3/2}}\right| \le \lim_{(x,y)\to(0,0)} \left|\frac{x^3-\sin^3 x}{x^3}\right|$$

and that limit is 0 as in part (a). Therefore the function is differentiable.

2. *(a) Prove that there exist positive constants c, C (independent of x, y) such that*

$$c(x^2+y^2) \le x^2 - xy + 4y^2 \le C(x^2+y^2).$$

Since $x^2+y^2-2xy=(x-y)^2 \ge 0$, we have $-2xy \ge -(x^2+y^2)$, so that

$$x^2-xy+4y^2 \ge x^2 - \frac{x^2+y^2}{2} + 4y^2 = \frac{x^2+7y^2}{2} \ge \frac{1}{2}(x^2+y^2).$$

Similarly, since $x^2+y^2+2xy=(x+y)^2 \ge 0$, we have $-2xy \le x^2+y^2$, so that

$$x^2-xy+4y^2 \le x^2 + \frac{x^2+y^2}{2} + 4y^2 = \frac{3x^2+9y^2}{2} \le \frac{9}{2}(x^2+y^2).$$

(b) Let

$$f(x,y) = \begin{cases} \dfrac{xy^3}{x^2-xy+4y^2} & \textit{if } (x,y)\neq(0,0) \\ 0 & \textit{if } (x,y)=(0,0) \end{cases}$$

Prove that f is differentiable at $(0,0)$. (Hint: You have to use the definition of differentiability. You may also have to use the estimate from (a).)

We have $f(x,0)=\frac{0}{x^2}=0$ for $x\neq 0$, and $f(0,y)=\frac{0}{4y^2}=0$ for $y\neq 0$. Combining this with $f(0,0)=0$, we get $f(x,0)=0$ and $f(0,y)=0$ for all x,y. Differentiating in x and y respectively, we get $f_1(0,0)=f_2(0,0)=0$. Hence the linear approximation at $(0,0)$ is $L(x,y)=0$.

For differentiability, we need

$$\lim_{(x,y)\to(0,0)} \frac{f(x,y)-L(x,y)}{|(x,y)-(0,0)|} = \lim_{(x,y)\to(0,0)} \frac{f(x,y)}{\sqrt{x^2+y^2}} = 0.$$

We have for $x \neq 0$, $y \neq 0$,

$$\frac{f(x,y)}{\sqrt{x^2+y^2}} = \frac{xy^3}{(x^2-xy+4y^2)\sqrt{x^2+y^2}}$$

Using the estimate from (a), we get

$$\left|\frac{xy^3}{(x^2-xy+4y^2)\sqrt{x^2+y^2}}\right| \leq \left|\frac{2xy^3}{(x^2+y^2)^{3/2}}\right| \leq \left|\frac{2xy^3}{(y^2)^{3/2}}\right| = 2|x| \to 0$$

as $x \to 0$. Hence $\dfrac{f(x,y)}{\sqrt{x^2+y^2}} \to 0$ as $(x,y) \to (0,0)$, so that f is differentiable at $(0,0)$.

3. *Let*

$$f(x,y) = \begin{cases} \dfrac{x^3-y^3+2xy}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

(a) Find the partial derivatives $f_1(0,0)$, $f_2(0,0)$.

We have $f(x,0) = \frac{x^3}{x^2} = x$ for $x \neq 0$, and $f(0,y) = -\frac{y^3}{y^2} = -y$ for $y \neq 0$. SInce these formulas match $f(0,0) = 0$, we have $f(x,0) = x$ and $f(0,y) = -y$ for all x, y. Therefore $f_1(0,0) = 1$ and $f_2(0,0) = -1$.

(b) Is $f(x,y)$ *differentiable at* $(0,0)$*? Prove your answer.*

Let $x = y$, then $f(x,x) = \frac{2x^2}{2x^2} = 1$. As $x \to 0$, this does not approach $f(0,0) = 1$. This means that f is not continuous at $(0,0)$ and therefore not differentiable at $(0,0)$.

(Note: if f were continuous at $(0,0)$, this proof would not work, but that would still not mean differentiability at $(0,0)$. You would have to check the limit in the definition of differentiability.)

4. *Let*

$$f(x,y) = \begin{cases} \dfrac{(x-e^x+1)y^2}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

Is f *differentiable at* $(0,0)$*? Prove your answer.*

We start by computing $L(x,y)$, the linear approximation to f at $(0,0)$. We have $f(x,0) = 0$ for all $x \in \mathbb{R}$, so $f_x(0,0) = 0$. Similarly, $f(0,y) = 0$ for all $y \in \mathbb{R}$, so $f_y(0,0) = 0$. Hence $L(x,y) = 0$ for all (x,y). To determine whether f is differentiable at $(0,0)$, we need to check whether

$$\lim_{(x,y)\to(0,0)} \frac{f(x,y)}{(x^2+y^2)^{1/2}} = 0.$$

We compute

$$\begin{aligned}
\left|\frac{f(x,y)}{(x^2+y^2)^{1/2}}\right| &= \frac{|x-e^x+1|\,y^2}{(x^2+y^2)^{3/2}} \\
&\le \frac{|x-e^x+1|(x^2+y^2)}{(x^2+y^2)^{3/2}} \\
&= \frac{|x-e^x+1|}{(x^2+y^2)^{1/2}} \\
&\le \left|\frac{x-e^x+1}{x}\right| \\
&= \left|1-\frac{e^x-1}{x}\right|.
\end{aligned}$$

But

$$\lim_{x\to 0}\left(1-\frac{e^x-1}{x}\right) = 1 - \lim_{x\to 0}\frac{e^x-1}{x} = 1-1 = 0.$$

The last limit can be checked for example using l'Hopital's rule:

$$\lim_{x\to 0}\frac{e^x-1}{x} = \lim_{x\to 0}\frac{e^x}{1} = e^0 = 1.$$

By the Squeeze Theorem,

$$\lim_{(x,y)\to(0,0)} \frac{f(x,y)}{(x^2+y^2)^{1/2}} = 0.$$

Therefore f is differentiable at $(0,0)$.

5. *Let*

$$f(x,y) = \begin{cases} x^2\sin(1/y) & \text{if } y \neq 0 \\ 0 & \text{if } y = 0 \end{cases}$$

Prove that:

(a) f is differentiable at $(0,0)$ (you will have to use the definition of differentiability).

We have $f(x,0) = \frac{0}{x^2} = 0$ for all x by the second part of the formula, and $f(0,y) = 0$ for $y \neq 0$ by the first part. Combining this with $f(0,0) = 0$, we get $f(x,0) = 0$

and $f(0, y) = 0$ for all x, y. Differentiating in x and y respectively, we get $f_1(0, 0) = f_2(0, 0) = 0$. Hence the linear approximation at $(0, 0)$ is $L(x, y) = 0$.

For differentiability, we need

$$\lim_{(x,y)\to(0,0)} \frac{f(x, y) - L(x, y)}{|(x, y) - (0, 0)|} = \lim_{(x,y)\to(0,0)} \frac{f(x, y)}{\sqrt{x^2 + y^2}} = 0.$$

We have for $y \neq 0$,

$$\left|\frac{f(x, y)}{\sqrt{x^2 + y^2}}\right| = \left|\frac{x^2 \sin(1/y)}{\sqrt{x^2 + y^2}}\right| \leq \sqrt{x^2 + y^2},$$

using that $x^2 \leq x^2 + y^2$ and $|\sin(1/y)| \leq 1$. Since $f(x, 0) = 0 \leq |x|$, we actually have $|\frac{f(x,y)}{\sqrt{x^2+y^2}}| \leq \sqrt{x^2 + y^2}$ for all (x, y). But $\sqrt{x^2 + y^2} \to 0$ as $x \to 0$. By the Squeeze Theorem, $\dfrac{f(x, y)}{\sqrt{x^2 + y^2}} \to 0$ as $(x, y) \to (0, 0)$, so that f is differentiable at $(0, 0)$.

(b) f is not continuous in any neighbourhood of $(0, 0)$. (For every neighbourhood of $(0, 0)$, prove that there is a point in that neighbourhood where f is not continuous.)

Let $r > 0$, and let $B_r = B_r(0, 0) = \{(x, y) : x^2 + y^2 < r^2$. We have to prove that there is a point in B_r where f is not continuous. We will show that this is true at the point $(r/2, 0) \in B_r$.

Consider the limit $\lim_{(x,y)\to(r/2,0)} f(x, y)$. Along the path where $x = r/2$ and $y \to 0$, we get

$$\lim_{y\to 0} x^2 \sin(1/y) = x^2 \lim_{y\to 0} \sin(1/y),$$

but the last limit does not exist by single-variable calculus. Therefore f is not continuous at $(r/2, 0)$.

6. *Let*

$$f(x, y) = \begin{cases} xy & \textit{if } x, y \textit{ are both rational,} \\ -xy & \textit{if at least one of } x, y \textit{ is irrational.} \end{cases}$$

(a) Find all points where f is continuous. Prove your answer using the epsilon-delta definition of continuity. (You may have to consider several cases separately.)

The function is continuous at the points $(a, 0)$ and $(0, b)$ with $a, b \in \mathbb{R}$ arbitrary, and is not continuous at any other points.

Proof that f is continuous at $(a,0)$: Let $\epsilon > 0$. We have to prove that there is a $\delta > 0$ such that

$$\sqrt{(x-a)^2+y^2} < \delta \quad \Rightarrow \quad |f(x,y) - f(a,0)| < \epsilon.$$

We have $f(a,0) = 0$, regardless of whether a is rational or not, and

$$|f(x,y) - f(a,0)| = |f(x,y)| = |xy|.$$

Let $\sqrt{(x-a)^2+y^2} < \delta$, with $\delta > 0$ to be chosen in a moment. Then $|y| < \delta$ and $|x-a| < \delta$ so that $|x| < |a| + \delta$. Therefore

$$|xy| \leq (|a| + \delta)\delta.$$

- If $a = 0$, let $\delta = \sqrt{\epsilon}$, then $|xy| < \delta^2 = \epsilon$, so we have a $\delta > 0$ that works in this case.
- If $a \neq 0$, let $\delta = \min(|a|, \epsilon/2|a|)$. Then

  $$|xy| < (2|a|)(\epsilon/2|a|) = \epsilon,$$

  and we are also done in this case.

Proof that f is continuous at $(0,b)$: since $f(x,y) = f(y,x)$, the proof is identical to the above.

Proof that f is not continuous at (a,b) with $a \neq 0$ and $b \neq 0$: We will use that any neighbourhood of (a,b) contains both points (x,y) with x, y rational and (x,y) with x, y irrational. To prove this, let $r > 0$. Each of the intervals $(a - r/2, a + r/2)$ and $(b - r/2, b + r/2)$ contains both rational and irrational points. Therefore the square

$$Q_r = (a - r/2, a + r/2) \times (b - r/2, b + r/2)$$

contains both points (x,y) with x, y rational and (x,y) with x, y irrational. But $Q_r \subset B_r(a,b)$, because for $(x,y) \in Q_r$ we have

$$\sqrt{(x-a)^2+(y-b)^2} \leq \sqrt{(r/2)^2+(r/2)^2} = \sqrt{r^2/2} = \frac{r}{\sqrt{2}} < r.$$

Let $\epsilon = |ab| > 0$, and let $\delta > 0$. Let $r = \min(\delta, |x|, |y|)$.

- If a, b are both rational, choose a point $(x,y) \in B_r(a,b) \subseteq B_\delta(a,b)$ such that x, y are both irrational. Then $f(a,b) = ab \neq 0$, and $f(x,y) = -xy$ has the opposite sign from $f(a,b)$, so that $|f(x,y) - f(a,b)| \geq |f(a,b)| = \epsilon$.
- Similarly, if at least one of a, b is irrational, choose a point $(x,y) \in B_r(a,b) \subseteq B_\delta(a,b)$ such that x, y are both rational. Then $f(a,b) = -ab \neq 0$, and $f(x,y) = xy$ has the opposite sign from $f(a,b)$, so that $|f(x,y) - f(a,b)| \geq |f(a,b)| = \epsilon$.

(b) Find all points where f is differentiable. Prove your answer. (You may have to consider several cases separately, and you will have to use the definition of differentiability at least once. Epsilon-delta proofs are not required in this part, but may be useful depending on how you solve the question.)

If f is not continuous at (a, b), then it is not differentiable at (a, b). Therefore f is not differentiable at any (a, b) with $a \neq 0$ and $b \neq 0$.

At points $(a, 0)$ and $(0, b)$ with $a, b \in \mathbb{R}$, we have to do some further testing. Since $f(x, y) = f(y, x)$, we will focus on the points $(a, 0)$, and then switch the variables to get the answer for $(0, b)$. We first note that $f(x, 0) = 0$ for all x, so that $f_1(a, 0) = 0$. For the rest of the calculation, we consider the following cases.

- If a is rational but $a \neq 0$, we try to evaluate $f_2(a, 0)$:

$$\begin{aligned} f_2(a,0) &= \lim_{h\to 0} \frac{1}{h}(f(a,h) - f(a,0)) \\ &= \lim_{h\to 0} \frac{1}{h} f(a,h) \\ &= \lim_{h\to 0} \frac{1}{h} \begin{cases} ah & \text{if } h \text{ is rational,} \\ -ah & \text{if } h \text{ is irrational} \end{cases} \\ &= \lim_{h\to 0} \begin{cases} a & \text{if } h \text{ is rational,} \\ -a & \text{if } h \text{ is irrational.} \end{cases} \end{aligned}$$

  This limit does not exist, hence f is not differentiable here.

- If a is irrational, we have

$$\begin{aligned} f_2(a,0) &= \lim_{h\to 0} \frac{1}{h}(f(a,h) - f(a,0)) \\ &= \lim_{h\to 0} \frac{1}{h} f(a,h) \\ &= \lim_{h\to 0} \frac{-ah}{h} = -a. \end{aligned}$$

  Therefore the tentative linear approximation is $L(x, y) = 0 + 0\cdot(x-a) + (-a)y = -ay$. We check the differentiability condition. Consider the limit

$$\begin{aligned} \lim_{(x,y)\to(a,b)} \frac{|f(x,y) - L(x,y)|}{\sqrt{(x-a)^2 + y^2}} &= \lim_{(x,y)\to(a,b)} \frac{|xy + ay|}{\sqrt{(x-a)^2 + y^2}} \\ &= \lim_{(x,y)\to(a,b)} \frac{|x+a||y|}{\sqrt{(x-a)^2 + y^2}}. \end{aligned}$$

  Let x, y be rational numbers such that $0 < y < |a|/2$ and $|x - a| < y$. Then

$$|x + a| = |(x - a) + 2a| \geq |2a| - |x - a| \geq |2a| - |y| \geq 3|a|/2$$

and

$$\sqrt{(x-a)^2+y^2} \le \sqrt{y^2+y^2} = \sqrt{2}\,y,$$

so that

$$\frac{|x+a||y|}{\sqrt{(x-a)^2+y^2}} \ge \frac{(3|a|/2)y}{\sqrt{2}\,y} = \frac{3|a|}{2\sqrt{2}}.$$

Therefore the limit as $(x,y) \to (0,0)$ is not 0, and the function is not differentiable here.

- Finally, let $a = 0$. Since $f(x,0) = f(0,y) = 0$, we have $f_1(0,0) = f_2(0,0) = 0$ and $L(x,y) = 0$. We check the differentiability condition:

$$\begin{aligned}\lim_{(x,y)\to(a,b)} \frac{|f(x,y)-L(x,y)|}{\sqrt{x^2+y^2}} &= \lim_{(x,y)\to(a,b)} \frac{|xy|}{\sqrt{x^2+y^2}} \\ &= \lim_{(x,y)\to(a,b)} |x|\frac{|y|}{\sqrt{x^2+y^2}} \\ &\le \lim_{(x,y)\to(a,b)} |x| = 0.\end{aligned}$$

The function is differentiable here.

Hence $f(x,y)$ is differentiable at $(0,0)$.

3.3 Gradients and Directional Derivatives

1. *Find all points (x,y,z) on the surface $xy+z^2 = 20$ where the tangent plane to that surface is perpendicular to the vector $2\mathbf{i}+3\mathbf{j}+4\mathbf{k}$.*

Let $f(x,y,z) = xy+z^2$. The tangent plane at (x,y,z) is perpendicular to $\nabla f(x,y,z) = \langle y,x,2z\rangle$. We also want it to be perpendicular to the vector $2\mathbf{i}+3\mathbf{j}+4\mathbf{k}$. This means that for some λ we have

$$\langle y,x,2z\rangle = \lambda\langle 2,3,4\rangle,$$

therefore $y = 2\lambda$, $x = 3\lambda$, $2z = 4\lambda$ so that $z = 2\lambda$. The point also has to lie on the surface $f(x,y,z) = xy+z^2 = 20$, so that

$$(3\lambda)(2\lambda)+(2\lambda)^2 = 6\lambda^2+4\lambda^4 = 10\lambda^2 = 20,$$

$$\lambda^2 = 2, \quad \lambda = \pm\sqrt{2}.$$

This corresponds to two points, $(3\sqrt{2},2\sqrt{2},2\sqrt{2})$ and $(-3\sqrt{2},-2\sqrt{2},-2\sqrt{2})$.

2. *Let* $f(x, y) = x \sin(xy^2)$.

(a) Find the equation of the tangent plane to the surface $z = f(x, y)$ *at the point where* $x = \pi$, $y = 1$.

We have $f_1(x, y) = \sin(xy^2) + x\cos(xy^2)\cdot y^2$ and $f_2(x, y) = x\cos(xy^2)\cdot 2xy$, so that $f_1(\pi, 1) = \sin\pi + \pi\cos\pi\cdot 1 = 0 - \pi = -\pi$ and $f_2(\pi, 1) = \pi\cos\pi\cdot 2\pi = -2\pi^2$. We also have $f(\pi, 1) = \pi\sin\pi = 0$. Therefore the tangent plane is

$$z = -\pi(x - \pi) - 2\pi^2(y - 1) = -\pi x - 2\pi^2 y + 3\pi^2.$$

(b) Find the directional derivative of f *at* $(\pi, 1)$ *in the direction of the vector* $\mathbf{v} = \mathbf{i} - 3\mathbf{j}$.

The unit vector in the direction of $\mathbf{v}$ is $\mathbf{u} = \dfrac{\mathbf{v}}{|\mathbf{v}|} = \dfrac{1}{\sqrt{10}}\mathbf{i} - \dfrac{3}{\sqrt{10}}\mathbf{i}$. From (a), we have $\nabla f(\pi, 1) = -\pi\mathbf{i} - 2\pi^2\mathbf{j}$. Therefore

$$D_{\mathbf{u}}f(\pi, 1) = \nabla f(\pi, 1)\cdot\mathbf{u} = \frac{1}{\sqrt{10}}(-\pi) - \frac{3}{\sqrt{10}}(-2\pi^2) = \frac{\pi}{\sqrt{10}}(6\pi - 1)$$

(c) Find a vector $\mathbf{w}$ *in the* xy*-plane tangent to the level curve of* f *at* $(\pi, 1)$. *(The answer might not be unique. Any non-zero tangent vector will suffice.)*

The tangent vector should be perpendicular to $\nabla f(\pi, 1) = -\pi\mathbf{i} - 2\pi^2\mathbf{j}$. For example, we could take $\mathbf{w} = 2\pi^2\mathbf{i} - \pi\mathbf{j}$.

3. *Let* $f(x, y) = xe^{x-2y}$.

(a) Find the directional derivative of f *at* $(x, y) = (2, 1)$ *in the direction of the vector* $3\mathbf{i} - \mathbf{j}$.

We have

$$\begin{aligned} f_1(x, y) &= e^{x-2y} + xe^{x-2y}, \quad f_1(2, 1) = e^0 + 2e^0 = 3, \\ f_2(x, y) &= -2xe^{x-2y}, \quad f_2(2, 1) = -4e^0 = -4, \end{aligned}$$

so that $\nabla f(2, 1) = 3\mathbf{i} - 4\mathbf{j}$. We also need a unit vector in the direction of $3\mathbf{i} - \mathbf{j}$:

$$\mathbf{v} = \frac{3\mathbf{i} - \mathbf{j}}{\sqrt{3^2 + 1^2}} = \frac{3\mathbf{i} - \mathbf{j}}{\sqrt{10}}.$$

Therefore

$$D_{\mathbf{v}}f(2, 1) = (3\mathbf{i} - 4\mathbf{j})\cdot\frac{3\mathbf{i} - \mathbf{j}}{\sqrt{10}} = \frac{3\cdot 3 + (-4)(-1)}{\sqrt{10}} = \frac{13}{\sqrt{10}}.$$

(b) Find a unit vector $\mathbf{u}$ *such that* $D_{\mathbf{u}}f(2,1) = 0$.

The vector $\mathbf{u}$ should be perpendicular to $\nabla f(2,1) = 3\mathbf{i} - 4\mathbf{j}$, and should have length 1. We can take

$$\mathbf{u} = \frac{4\mathbf{i} + 3\mathbf{j}}{\sqrt{4^2 + 3^2}} = \frac{4\mathbf{i} + 3\mathbf{j}}{5}.$$

4. *Let* $f(x, y)$ *be a differentiable function such that* $f(2, 5) = 3$, $f_x(2, 5) = 4$, *and* $D_{\mathbf{u}}f(2, 5) = 1$, *where* $\mathbf{u}$ *is the unit vector in the direction of* $\mathbf{i} - \sqrt{3}\mathbf{j}$. *Find the equation of the plane tangent to the graph of* $z = f(x, y)$ *at* $(x, y) = (2, 5)$.

We are given that $f(2, 5) = 3$, $f_x(2, 5) = 4$, and $D_{\mathbf{u}}f(2, 5) = 1$, where

$$\mathbf{u} = \frac{\mathbf{i} - \sqrt{3}\mathbf{j}}{|\mathbf{i} - \sqrt{3}\mathbf{j}|} = \frac{\mathbf{i} - \sqrt{3}\mathbf{j}}{\sqrt{1+3}} = \frac{1}{2}\mathbf{i} - \frac{\sqrt{3}}{2}\mathbf{j}.$$

From the gradient formula,

$$1 = D_{\mathbf{u}}f(2, 5) = \nabla f(2, 5) \cdot \mathbf{u} = \frac{1}{2}(4) - \frac{\sqrt{3}}{2} f_y(2, 5),$$

so that

$$\frac{\sqrt{3}}{2} f_y(2, 5) = -1 + 2 = 1, \quad f_y(2, 5) = \frac{2}{\sqrt{3}}.$$

Therefore the equation of the tangent plane is

$$z = 3 + 4(x - 2) + \frac{2}{\sqrt{3}}(y - 5).$$

5. *The tangent plane to the graph of* $z = f(x, y)$ *at* $(x, y) = (3, 4)$ *has the equation* $x - 6y - 2z = 1$.

(a) Find the directional derivative $D_{\mathbf{u}}f(3, 4)$ *if* $\mathbf{u} = \frac{\sqrt{3}}{2}\mathbf{i} - \frac{1}{2}\mathbf{j}$.

We rewrite the equation of the tangent plane as $z = \frac{1}{2}x - 3y - \frac{1}{2}$. Reading the partial derivatives from this, we get $\nabla f(3, 4) = \langle 1/2, -3\rangle$. Therefore

$$D_{\mathbf{u}}f(3, 4) = \frac{1}{2} \cdot \frac{\sqrt{3}}{2} - 3(-\frac{1}{2}) = \frac{\sqrt{3}}{4} + \frac{3}{2}.$$

(b) Is there a unit vector $\mathbf{v}$ *such that* $D_{\mathbf{v}}f(3, 4) = 4$*? If yes, find it. If no, explain why.*

The largest possible value of $D_{\mathbf{v}}f(3, 4)$ is $|\nabla f(3, 4)| = \sqrt{(1/2)^2 + 3^2} = \sqrt{9.25}$. This is less than 4 (since $9.25 < 16$). Hence there is no such $\mathbf{v}$.

6. *The surfaces* $4x^2 + y^2 = z^2$ *and* $xy = z + 1$ *intersect in a curve that contains the point* $P = (2, 3, 5)$*. Find the equation of the line tangent to that curve at* P*.*

The direction vector $\mathbf{v}$ of the line has to be tangent to both surfaces, therefore perpendicular to the normal vectors to both surfaces. The surfaces can be represented as level surfaces $f(x, y, z) = 0$ and $g(x, y, z) = 0$, where $f(x, y, z) = 4x^2 + y^2 - z^2$ and $g(x, y, z) = xy - z$. The normal vectors at P are

$$\mathbf{n}_1 = \nabla f(2, 3, 5), \quad \mathbf{n}_2 = \nabla g(2, 3, 5).$$

We have

$$\nabla f(x, y, z) = \langle 8x, 2y, -2z\rangle, \quad \nabla g(x, y, z) = \langle y, x, -1\rangle,$$

so that

$$\mathbf{n}_1 = \langle 16, 6, -10\rangle, \quad \mathbf{n}_2 = \langle 3, 2, -1\rangle.$$

A vector perpendicular to both normals is

$$\frac{1}{2}\mathbf{n}_1 \times \mathbf{n}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 8 & 3 & -5 \\ 3 & 2 & -1 \end{vmatrix} = (-3 + 10)\mathbf{i} - (-8 + 15)\mathbf{j} + (16 - 9)\mathbf{k} = 7\mathbf{i} - 7\mathbf{j} + 7\mathbf{k}.$$

For convenience, we will use $\mathbf{i} - \mathbf{j} + \mathbf{k}$ for the direction vector $\mathbf{v}$. The line is parallel to $\mathbf{v}$ and passes through $(2, 3, 5)$, so it has the parametric equations

$$x = 2 + t, \quad y = 3 - t, \quad z = 5 + t.$$

7. *Let*

$$g(x, y) = \begin{cases} \dfrac{x^2 y}{x^6 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

Find the directional derivative $D_{\mathbf{u}}g(0, 0)$*, if* $\mathbf{u}$ *is the unit vector in the direction of* $3\mathbf{i}+4\mathbf{j}$*. (This function is not differentiable or even continuous at* $(0, 0)$*, so you have to use the definition of the directional derivative.)*

We have $\mathbf{u} = \frac{1}{5}\big(3\mathbf{i} + 4\mathbf{j}\big)$. Then

$$\begin{aligned} D_{\mathbf{u}} g(0,0) &= \lim_{h\to 0^+} \frac{g(0+3h/5, 0+4h/5) - g(0,0)}{h} \\ &= \lim_{h\to 0^+} \frac{g(3h/5, 4h/5)}{h} \\ &= \lim_{h\to 0^+} \frac{(\frac{3}{5}h)^2(\frac{4}{5}h)}{h\big((\frac{3}{5}h)^6 + (\frac{4}{5}h)^2\big)} \\ &= \lim_{h\to 0^+} \frac{(\frac{3}{5}h)^2(\frac{4}{5}h)}{h^3\big((\frac{3}{5})^6 h^4 + (\frac{4}{5})^2\big)} \\ &= \lim_{h\to 0^+} \frac{(\frac{3}{5})^2(\frac{4}{5})}{\big((\frac{3}{5})^6 h^4 + (\frac{4}{5})^2\big)} \\ &= \frac{9}{20}. \end{aligned}$$

Additional notes:

1) To see that g is not continuous at $(0,0)$, let $y = x^3$, then $g(x, x^3) = \frac{x^5}{2x^6} = \frac{1}{2x}$ does not have a limit as $x \to 0$. In fact, g is not even bounded near $(0,0)$!

2) And yet, the directional derivative $D_{\mathbf{u}} g(0,0)$ exists for every unit vector $\mathbf{u}$. The proof is similar to the above, but with general u_1 and u_2 instead of $3/5$ and $4/5$).

8. *Let*

$$f(x,y) = \begin{cases} \dfrac{3xy^2 - y^3}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

(a) Is f differentiable at $(0,0)$? Prove your answer.

We have $f(x,0) = \frac{0}{x^2} = 0$ for $x \neq 0$, and $f(0,y) = \frac{-y^3}{y^2} = -y$ for $y \neq 0$. Combining this with $f(0,0) = 0$, we get $f(x,0) = 0$ and $f(0,y) = -y$ for all x, y. Differentiating in x and y respectively, we get $f_1(0,0) = 0$ and $f_2(0,0) = -1$. Hence the linear approximation at $(0,0)$ is $L(x,y) = -y$.

For differentiability, we need

$$\lim_{(x,y)\to(0,0)} \frac{f(x,y) - L(x,y)}{|(x,y) - (0,0)|} = \lim_{(x,y)\to(0,0)} \frac{f(x,y) + y}{\sqrt{x^2+y^2}} = 0.$$

We have for $x \neq 0$, $y \neq 0$,

$$\frac{f(x,y) + y}{\sqrt{x^2+y^2}} = \frac{3xy^2 - y^3 + y(x^2+y^2)}{(x^2+y^2)\sqrt{x^2+y^2}} = \frac{3xy^2 + x^2 y}{(x^2+y^2)\sqrt{x^2+y^2}}.$$

When $x = y$, the last fraction is equal to

$$\frac{3x^3 + x^3}{2x^2\sqrt{2x^2}} = \frac{2x}{\sqrt{2x^2}} = \frac{\sqrt{2}\,x}{|x|}.$$

This has a limit $\sqrt{2}$ as $x \to 0^+$ and $-\sqrt{2}$ as $x \to 0^-$. In particular, the limit is not 0, so that the function is not differentiable at $(0, 0)$.

(b) Find the directional derivative $D_{\mathbf{u}}f(0, 0)$ for a general unit vector $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j}$. If it does not exist, explain why.

By definition,

$$\begin{aligned} D_{\mathbf{u}}f(0,0) &= \lim_{h\to 0^+} \frac{f(hu_1, hu_2) - f(0,0)}{h} \\ &= \lim_{h\to 0^+} \frac{1}{h} \cdot \frac{3h^3u_1u_2^2 - h^3u_2^3}{h^2u_1^2 + h^2u_2^2} \\ &= \lim_{h\to 0^+} \frac{3u_1u_2^2 - u_2^3}{u_1^2 + u_2^2} \\ &= \frac{3u_1u_2^2 - u_2^3}{u_1^2 + u_2^2}. \end{aligned}$$

9. *Let*

$$f(x,y) = \begin{cases} \dfrac{x^5}{x^4 + y^4} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

Find the general formula for $D_{\mathbf{u}}f(0, 0)$, where $\mathbf{u} = (u_1, u_2)$ is a unit vector, in terms of u_1, u_2.

We calculate from the definition:

$$\begin{aligned} D_{\mathbf{u}}f(0,0) = \lim_{t\to 0^+} \frac{1}{t}(f(t\mathbf{u}) - f(0,0)) &= \lim_{t\to 0^+} \frac{1}{t}\Big(\frac{t^5u_1^5}{t^4u_1^4 + t^4u_2^4} - 0\Big) \\ = \lim_{t\to 0^+} \frac{1}{t}\frac{tu_1^5}{u_1^4 + u_2^4} &= \frac{u_1^5}{u_1^4 + u_2^4}. \end{aligned}$$

10. *Suppose that $f(x, y)$ is a function defined for all (x, y) in a neighbourhood of (a, b). Suppose also that*

$$D_{\mathbf{u}}f(a,b) = 2, \quad D_{\mathbf{v}}f(a,b) = 3,$$

where

$$\mathbf{u} = \frac{\sqrt{3}}{2}\,\mathbf{i} + \frac{1}{2}\,\mathbf{j}, \quad \mathbf{v} = \frac{\sqrt{3}}{2}\,\mathbf{i} - \frac{1}{2}\,\mathbf{j},$$

and that $f_1(a,b) = -1$. Prove that f is not differentiable at (a,b).

We will prove this by contradiction. If f were differentiable at (a,b), we would have the formula

$$D_{\mathbf{w}}f = f_1(a,b)w_1 + f_2(a,b)w_2 = -w_1 + f_2(a,b)w_2$$

for all unit vectors $\mathbf{w} = \langle w_1, w_2 \rangle$. Applying this formula to the vectors $\mathbf{u}$ and $\mathbf{v}$, we would have

$$D_{\mathbf{u}}f(a,b) = -\frac{\sqrt{3}}{2} + \frac{1}{2} f_2(a,b) = 2,$$

$$D_{\mathbf{v}}f(a,b) = -\frac{\sqrt{3}}{2} - \frac{1}{2} f_2(a,b) = 3.$$

Therefore $f_2(a,b) = 4 + \sqrt{3}$ from the first equation and $f_2(a,b) = -6 - \sqrt{3}$ from the second equation, a contradiction. It follows that f cannot be differentiable at (a,b).

11. *Is there a differentiable function $f : \mathbb{R}^2 \to \mathbb{R}$ such that $D_{\mathbf{u}}f(0,0) = u_1^2 - u_2^2$ for every unit vector $\mathbf{u} = (u_1, u_2)$? If yes, find it. If no, explain why.*

If there were such a function, we would have $f_x(0,0) = D_{\mathbf{i}}f(0,0) = 1^2 - 0^2 = 1$ and $f_y(0,0) = D_{\mathbf{j}}f(0,0) = 0^2 - 1^2 = -1$, hence

$$\nabla f(0,0) = (1, -1).$$

From the dot product formula, we would then have $D_{\mathbf{u}}f(0,0) = \nabla f(0,0) \cdot \mathbf{u} = u_1 - u_2$ for all $\mathbf{u}$. But this is not consistent with $D_{\mathbf{u}}f(0,0) = u_1^2 - u_2^2$. For example, if $\mathbf{u} = -\mathbf{i}$, the dot product formula gives $D_{-\mathbf{i}}f(0,0) = -1 - 0 = -1$, and the formula in the text of the problem gives $D_{-\mathbf{i}}f(0,0) = 1^2 - 0^2 = 1$. Therefore there is no such function.

3.4 Implicit Functions

1. *A surface is given by the equation $e^x(y+1) - e^y z = (e^z - 1)x$.*

(a) Prove that the point $P = (1,0,1)$ lies on the surface.

We check that $(x,y,z) = (1,0,1)$ satisfies the equation of the surface: $e^1(0+1) - e^0 \cdot 1 = (e^1 - 1) \cdot 1$, $e - 1 = e - 1$, which is true.

(b) Find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ at P.

We rewrite the equation of the surface as $F(x,y,z)=0$, where $F(x,y,z)=e^x(y+1)-e^yz-(e^z-1)x$. Then F has continuous partial derivatives of all orders, and

$$\begin{aligned} F_x(x,y,z) &= e^x(y+1)-(e^z-1) \\ F_y(x,y,z) &= e^x-e^yz \\ F_z(x,y,z) &= -e^y-e^zx \end{aligned}$$

so that

$$\begin{aligned} F_x(1,0,1) &= e^1(0+1)-(e^1-1)=1 \\ F_y(1,0,1) &= e^1-e^0\cdot 1=e-1 \\ F_z(1,0,1) &= -e^0-e^1\cdot 1=-1-e \end{aligned}$$

Since $F_z(1,0,1)\neq 0$, the equation defines z as a function of x,y in a neighbourhood of P, and we have at P

$$\frac{\partial z}{\partial x}=-\frac{F_x(1,0,1)}{F_z(1,0,1)}=\frac{1}{e+1},\quad \frac{\partial z}{\partial y}=-\frac{F_y(1,0,1)}{F_z(1,0,1)}=\frac{e-1}{e+1}.$$

2. *Suppose that z is a function of x,y such that $\sin(xz)-yz=0$. Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ where possible.*

We first differentiate the equation in x:

$$\cos(xz)\left(z+x\frac{\partial z}{\partial x}\right)-y\frac{\partial z}{\partial x}=0$$

$$\frac{\partial z}{\partial x}\left(x\cos(xz)-y\right)=-z\cos(xz)$$

$$\frac{\partial z}{\partial x}=-\frac{z\cos(xz)}{x\cos(xz)-y}$$

Now differentiate in y:

$$\cos(xz)\left(x\frac{\partial z}{\partial y}\right)-z-y\frac{\partial z}{\partial y}=0$$

$$\frac{\partial z}{\partial y}\left(x\cos(xz)-y\right)=z$$

$$\frac{\partial z}{\partial y}=\frac{z}{x\cos(xz)-y}$$

Both derivatives exist whenever $x\cos(xz)-y\neq 0$.

3. *Suppose that the functions $x(u,v)$ and $y(u,v)$ solve the equations*

$$\begin{cases} u &= 3x + 2e^y \\ v &= e^x - y \end{cases}$$

in some neighbourhood of the point where $(x,y) = (1,0)$. Find $\frac{\partial x}{\partial u}$ and $\frac{\partial y}{\partial u}$ where possible, and evaluate them at $(x,y) = (1,0)$.

We differentiate the equations in u:

$$\begin{cases} 1 &= 3\frac{\partial x}{\partial u} + 2e^y \frac{\partial y}{\partial u} \\ \\ 0 &= e^x \frac{\partial x}{\partial u} - \frac{\partial y}{\partial u} \end{cases}$$

From the second equation, $\frac{\partial y}{\partial u} = e^x \frac{\partial x}{\partial u}$. Plugging this into the first equation, we get

$$3\frac{\partial x}{\partial u} + 2e^y e^x \frac{\partial x}{\partial u} = (3 + e^{x+y})\frac{\partial x}{\partial u} = 1,$$

$$\frac{\partial x}{\partial u} = \frac{1}{3 + 2e^{x+y}}, \quad \frac{\partial y}{\partial u} = \frac{e^x}{3 + 2e^{x+y}}.$$

Finally, plugging in $(x,y) = (1,0)$, we get that at that point,

$$\frac{\partial x}{\partial u} = \frac{1}{3+2e}, \quad \frac{\partial y}{\partial u} = \frac{e}{3+2e}.$$

4. *The equations*

$$u = x^3 - y^2$$
$$v = 2xy^3$$

define x, y implicitly as functions of u, v near $x = -1$, $y = 1$. Find $\frac{\partial x}{\partial u}$ and $\frac{\partial y}{\partial u}$ where possible, and evaluate them at $x = -1$, $y = 1$.

Differentiating the two equations with respect to u, we get

$$1 = 3x^2\frac{\partial x}{\partial u} - 2y\frac{\partial y}{\partial u}$$
$$0 = 2y^3\frac{\partial x}{\partial u} + 6xy^2\frac{\partial y}{\partial u}$$

so that

$$\frac{\partial x}{\partial u} = \frac{\begin{vmatrix} 1 & -2y \\ 0 & 6xy^2 \end{vmatrix}}{\begin{vmatrix} 3x^2 & -2y \\ 2y^3 & 6xy^2 \end{vmatrix}} = \frac{6xy^2}{18x^3y^2 + 4y^4} = \frac{3x}{9x^3 + 2y^2}$$

$$\frac{\partial y}{\partial u} = \frac{\begin{vmatrix} 3x^2 & 1 \\ 2y^3 & 0 \end{vmatrix}}{\begin{vmatrix} 3x^2 & -2y \\ 2y^3 & 6xy^2 \end{vmatrix}} = \frac{-2y^3}{18x^3y^2 + 4y^4} = \frac{-y}{9x^3 + 2y^2}$$

At $(x, y) = (-1, 1)$, we have

$$\frac{\partial x}{\partial u} = \frac{-3}{-9+2} = \frac{3}{7}, \quad \frac{\partial y}{\partial u} = \frac{-1}{-9+2} = \frac{1}{7}.$$

5. *The quantities u, v are defined as functions of x, y so that they satisfy the equations $x = u\cos(u+v)$, $y = v\sin u$. Find $\frac{\partial u}{\partial x}, \frac{\partial v}{\partial x}$ at the point where $u = \pi/3$, $v = 0$. (Hint: $\sin(\pi/3) = \frac{\sqrt{3}}{2}$, $\cos(\pi/3) = \frac{1}{2}$.)*

   Differentiating the equations $x = u\cos(u+v)$, $y = v\sin u$ in x, we get

$$\begin{aligned} 1 &= u_x \cos(u+v) - u\sin(u+v)\cdot(u_x + v_x), \\ 0 &= v_x \sin u + v\cos u \cdot u_x. \end{aligned}$$

   At $u = \pi/3$, $v = 0$, we get

$$\begin{aligned} 1 &= u_x \cos(\pi/3) - \frac{\pi}{3}\sin(\pi/3)\cdot(u_x + v_x) = \frac{u_x}{2} - \frac{\pi\sqrt{3}}{6}(u_x + v_x), \\ 0 &= v_x \sin(\pi/3) = \frac{\sqrt{3}}{2}v_x. \end{aligned}$$

   Hence $v_x = 0$, and

$$u_x = \frac{1}{\frac{1}{2} - \frac{\pi\sqrt{3}}{6}} = \frac{6}{3 - \pi\sqrt{3}}.$$

3.5 Taylor Polynomials

1. *Find the second order Taylor polynomial of the function $f(x, y) = \sin(x + y^2)$ at $(\pi, 0)$.*

   We calculate the necessary derivatives:

$$\begin{gathered} f(\pi, 0) = \sin\pi = 0, \\ f_x(x, y) = \cos(x + y^2), \;\; f_x(\pi, 0) = \cos\pi = -1, \\ f_y(x, y) = 2y\cos(x + y^2), \;\; f_y(\pi, 0) = 0, \\ f_{xx}(x, y) = -\sin(x + y^2), \;\; f_{xx}(\pi, 0) = -\sin\pi = 0, \end{gathered}$$

$$f_{xy}(x,y) = -2y\sin(x+y^2),\ f_{xy}(\pi,0) = 0,$$
$$f_{yy}(x,y) = -4y^2\sin(x+y^2) + 2\cos(x+y^2),\ f_{yy}(\pi,0) = 2\cos\pi = -2.$$

Therefore $p_2(x,y) = -(x-\pi) - y^2$.

2. *Suppose that the function $f(x,y)$ and all its partial derivatives are continuous for all x, y, and that its second order Taylor polynomial at (a,b) is*

$$p_2(x,y) = -7 + 4(x-a) - 13(x-a)^2 - 6(x-a)(y-b) + 10(y-b)^2.$$

Find the partial derivatives $f_1(a,b)$, $f_{11}(a,b)$, and $f_{12}(a,b)$.

We can just read the derivatives from the given formula: $f_1(a,b) = 4$, $f_{11}(a,b) = -26$, and $f_{12}(a,b) = -6$.

3. *Let $f(x,y) = Ax^2 + Bxy + Cy^2$ for some constants A, B, C. Prove that the second order Taylor polynomial of $f(x,y)$ at every point (a,b) in the xy-plane is equal to $f(x,y)$.*

We have

$$f_1(x,y) = 2Ax + By,\ f_2(x,y) = Bx + 2Cy,$$
$$f_{11}(x,y) = 2A,\ f_{12}(x,y) = f_{21}(x,y) = B,\ f_{22}(x,y) = 2C.$$

Thus at (a,b) the second order Taylor polynomial is

$$\begin{aligned}
p_2(x,y) &= (Aa^2 + Bab + Cb^2) + (2Aa + Bb)(x-a) + (2Cb + Ba)(y-b) \\
&\quad + A(x-a)^2 + B(x-a)(y-b) + C(y-b)^2 \\
&= Aa^2 + Bab + Cb^2 + 2Aax + Bbx - 2Aa^2 - Bab \\
&\quad + 2Cby + Bay - 2Cb^2 - Bab + Ax^2 - 2Aax + Aa^2 \\
&\quad + Bxy - Bay - Bbx + Bab + Cy^2 - 2Cby + Cb^2 \\
&= Ax^2 + Bxy + Cy^2.
\end{aligned}$$

4. *Let $F(x,y) = xf(x+y) + yg(x+y)$, where f, g are real-valued functions of one variable whose all derivatives are continuous.*

(a) Prove that $F(x,y)$ satisfies the equation $F_{xx}(x,y) - 2F_{xy}(x,y) + F_{yy}(x,y) = 0$ for all x, y.

We calculate the derivatives:

$$F_x(x,y) = f(x+y) + xf'(x+y) + yg'(x+y)$$

$$F_y(x,y) = xf'(x+y) + g(x+y) + yg'(x+y)$$
$$F_{xx}(x,y) = xf''(x+y) + 2f'(x+y) + yg''(x+y)$$
$$F_{xy}(x,y) = f'(x+y) + xf''(x+y) + g'(x+y) + yg''(x+y)$$
$$F_{yy}(x,y) = xf''(x+y) + 2g'(x+y) + yg''(x+y)$$

so that, with all derivatives evaluated at $x+y$,

$$\begin{aligned} &F_{xx} - 2F_{xy} + F_{yy} \\ &= (xf'' + 2f' + yg'') + (xf'' + 2g + yg'') - 2(f' + xf'' + g' + yg'') \\ &= 0. \end{aligned}$$

(b) Find the second order Taylor polynomial of F at the point $(0,0)$ in terms of the derivatives of f, g at 0.

At $(0,0)$, we have $F(0,0) = 0$ and

$$F_x(0,0) = f(0), \quad F_y(0,0) = g(0),$$

$$F_{xx}(0,0) = 2f'(0), \quad F_{xy}(0,0) = f'(0) + g'(0), \quad F_{yy}(0,0) = 2g'(0)$$

Thus

$$p_2(x,y) = f(0)x + g(0)y + f'(0)x^2 + (f'(0) + g'(0))xy + g'(0)y^2$$

5. *Let*

$$f(x,y) = \begin{cases} \dfrac{x^4}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

(a) Compute $f_1, f_2, f_{11}, f_{12}, f_{21}, f_{22}$ at $(0,0)$. Be careful and make sure to use the definition of the derivative where necessary. As an intermediate step, it may also be necessary to find the general expressions for some of these partials at points other than $(0,0)$.

- We have $f(x,0) = x^4/x^2 = x^2$ for $x \neq 0$, and this is also consistent with $f(0,0) = 0$, so that $f(x,0) = 0$ for all x. Therefore

$$f_1(x,0) = 2x, \quad f_{11}(x,0) = 2,$$

so that $f_1(0,0) = 0$, $f_{11}(0,0) = 2$.

- Similarly, $f(0,y) = 0$ for all y including $y = 0$. Therefore

$$f_2(0,y) = 0 \quad f_{22}(0,y) = 0,$$

so that $f_2(0,0) = 0$, $f_{22}(0,0) = 0$.

- To compute $f_{12}(0,0)$ and $f_{21}(0,0)$, we need to do more work. We have

$$f_{12}(0,0) = \lim_{y\to 0} \frac{f_1(0,y) - f_1(0,0)}{y}, \quad f_{21}(0,0) = \lim_{x\to 0} \frac{f_2(x,0) - f_2(0,0)}{x}.$$

To evaluate these, we have to compute $f_1(0,y)$ and $f_2(x,0)$ first. At $(x,y) \neq (0,0)$, we have

$$f_1(x,y) = \frac{4x^3(x^2+y^2) - x^4 \cdot 2x}{(x^2+y^2)^2}, \quad f_1(0,y) = 0,$$

$$f_2(x,y) = \frac{-x^4 \cdot 2y}{(x^2+y^2)^2}, \quad f_2(x,0) = 0,$$

Therefore $f_{12}(0,0) = f_{21}(0,0) = 0$.

(b) Based on your answer from (a), write the second order Taylor polynomial $p_2(x,y)$ for $f(x,y)$ at $(0,0)$. Is it true that

$$\lim_{(x,y)\to(0,0)} \frac{f(x,y) - p_2(x,y)}{x^2+y^2} = 0?$$

Prove your answer.

Based on the above, $p_2(x,y) = x^2$, We have

$$\begin{aligned}\lim_{(x,y)\to(0,0)} \frac{f(x,y) - p_2(x,y)}{x^2+y^2} &= \lim_{(x,y)\to(0,0)} \frac{x^4 - x^2(x^2+y^2)}{(x^2+y^2)^2} \\ &= \lim_{(x,y)\to(0,0)} \frac{-x^2y^2}{(x^2+y^2)^2}.\end{aligned}$$

This limit does not exist. To see this, let $g(x,y) = \dfrac{-x^2y^2}{(x^2+y^2)^2}$. If $x = 0$ and $y \to 0$, then $g(0,y) = 0 \to 0$. If on the other hand $x = y$, then $g(x,x) = \frac{-x^4}{4x^4} = \frac{-1}{4} \to -\frac{1}{4}$ as $x \to 0$. In particular, it is not true that the limit is 0.

Chapter 4

Extreme Values

4.1 Critical Points

1. *Let* $f(x,y) = \dfrac{y}{x} + \dfrac{1}{y} + x$.

   (a) This function has exactly one critical point. Find it.

   We have
   $$f_1(x,y) = -\frac{y}{x^2} + 1, \quad f_2(x,y) = \frac{1}{x} - \frac{1}{y^2}.$$
   For $f_1 = f_2 = 0$, we need $1 = y/x^2$, so that $y = x^2$, and $\frac{1}{x} = \frac{1}{y^2}$, so that
   $$x = y^2 = (x^2)^2 = x^4.$$
   We cannot have $x = 0$ since then f would not be defined, so we must have $x^3 = 1$, $x = 1$, and $y = 1^2 = 1$. The critical point is at $(1,1)$.

   (b) Find the second order Taylor polynomial of $f(x,y)$ at the critical point found in (a).

   We have $f(1,1) = 1 + 1 + 1 = 3$, and
   $$f_{11}(x,y) = \frac{2y}{x^3}, \quad f_{12}(x,y) = f_{21}(x,y) = -\frac{1}{x^2}, \quad f_{22}(x,y) = \frac{2}{y^3}.$$
   Therefore $f_{11}(1,1) = 2$, $f_{12}(1,1) = f_{21}(1,1) = -1$, and $f_{22}(1,1) = 2$. The Taylor polynomial is
   $$p_2(x,y) = 3 + \frac{1}{2}\Big(2(x-1)^2 - 2(x-1)(y-1) + 2(y-1)^2\Big).$$

   (c) Does $f(x,y)$ have a minimum, maximum, or a saddle point at this critical point?

We check the minors of the Hessian matrix:

$$\begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = 4 - 1 = 3 > 0, \quad f_{11}(1,1) = 2 > 0,$$

so by the second derivative test, f has a local minimum at $(1,1)$.

2. *Let $f(x,y) = x^3 + 3x^2y - y - y^3$. Find all critical points of f and classify them as local minima, maxima, or saddle points.*

   We have
   $$f_1(x,y) = 3x^2 + 6xy, \quad f_2(x,y) = 3x^2 - 1 - 3y^2.$$
   We look for points with $f_1 = f_2 = 0$. If $f_1 = 3x^2 + 6xy = 3x(x+2y) = 0$, then $x = 0$ or $x = -2y$. If $x = 0$, then from $f_2 = 0$ we get $3x^2 - 1 - 3y^2 = -1 - 3y^2 = 0$, so that $3y^2 = -1$, which is not possible. Therefore $x = -2y$. From $f_2 = 0$, we then get
   $$3x^2 - 1 - 3y^2 = 12y^2 - 1 - 3y^2 = 9y^2 - 1 = 0,$$
   so that $y = \pm\frac{1}{3}$. This corresponds to two critical points, $(-\frac{2}{3}, \frac{1}{3})$ and $(\frac{2}{3}, -\frac{1}{3})$.

   The second order partials are $f_{11}(x,y) = 6x + 6y$, $f_{12}(x,y) = f_{21}(x,y) = 6x$, $f_{22}(x,y) = -6y$.

   - At $(-\frac{2}{3}, \frac{1}{3})$,
     $$\begin{vmatrix} -2 & -4 \\ -4 & -2 \end{vmatrix} = 4 - 16 < 0,$$
     and $-2 < 0$, so by the second derivative test, f has a saddle point.
   - At $(\frac{2}{3}, -\frac{1}{3})$
     $$\begin{vmatrix} 2 & 4 \\ 4 & 2 \end{vmatrix} = 4 - 16 < 0,$$
     and $2 > 0$, so by the second derivative test, f has a saddle point.

3. *Let $f(x,y) = x^4 - 4xy + y^2$. Find all critical points of f and classify them as local minima, maxima, or saddle points.*

   We have
   $$f_1(x,y) = 4x^3 - 4y, \quad f_2(x,y) = -4x + 2y.$$
   For $f_1 = f_2 = 0$, we need $y = 2x$ and $4x^3 = 4y = 8x$, $x^3 = 2x$, so that $x = 0$ or $x = \pm\sqrt{2}$. This corresponds to three critical points $(0,0)$, $(\sqrt{2}, 2\sqrt{2})$, $(-\sqrt{2}, -2\sqrt{2})$.

   The second order partials are $f_{11}(x,y) = 12x^2$, $f_{12}(x,y) = f_{21}(x,y) = -4$, $f_{22}(x,y) = 2$.

- At $(0, 0)$,
$$\begin{vmatrix} 0 & -4 \\ -4 & 2 \end{vmatrix} = -16 < 0,$$
and $f_{11} = 0$, so by the second derivative test, f has a saddle point.

- At $(\sqrt{2}, 2\sqrt{2})$ and $(-\sqrt{2}, -2\sqrt{2})$,
$$\begin{vmatrix} 24 & -4 \\ -4 & 2 \end{vmatrix} = 48 - 16 > 0, \quad f_{11} = 24 > 0$$
and $f_{11} > 0$, so by the second derivative test, f has local minima at these points.

4. *Let $f(x, y) = x^3 - 3x^2y + xy - y^2$. Do there exist constants A, B such that the function $g(x, y) = f(x, y) - Ax - By$ has a local minimum or maximum at $(1, 2)$? If so, find all such A and B. If no, explain why.*

Suppose that such A, B exist. Then $g(x, y)$ has a critical point at $(1, 2)$. We have
$$\begin{aligned} g_1(x, y) &= 3x^2 - 6xy + y - A, \quad g_1(1, 2) = 3 - 12 + 2 - A = -7 - A, \\ g_2(x, y) &= -3x^2 + x - 2y - B, \quad g_2(1, 2) = -3 + 1 - 4 - B = -6 - B. \end{aligned}$$

If $g_1(1, 2) = g_2(1, 2) = 0$, then $A = -7$ and $B = -6$, and these are the only possible values of A and B.

Let us now check what type of a critical point this is. The second order partials are
$$\begin{aligned} g_{11}(x, y) &= 6x - 6y, \\ g_{12}(x, y) &= g_{21}(x, y) = -6x + 1, \\ g_{22}(x, y) &= -2. \end{aligned}$$

At $(1, 2)$, $g_{11}(1, 2) = -6$ and $g_{12}(1, 2) = -5$. We have
$$\begin{vmatrix} -6 & -5 \\ -5 & -2 \end{vmatrix} = 12 - 25 < 0,$$
and $-6 < 0$, so by the second derivative test, g has a saddle point. Therefore there are no constants A, B such that $g(x, y)$ has a local minimum or maximum at $(1, 2)$.

5. *Let $f(x, y, z) = x^2 - 2x + 2y^2 + 3z^2$.*

(a) This function has one critical point. Find it and classify it.

The gradient of f is

$$\nabla f(x,y,z) = (2x-2)\mathbf{i} + 4y\mathbf{j} + 6z\mathbf{k}.$$

We see that $\nabla f(x,y,z) = \mathbf{0}$ if and only if $x = 1$ and $y = z = 0$. So, f has a critical point at $(x,y,z) = (1,0,0)$, and this is the only critical point.

To classify this critical point, we will evaluate the principal minors of the Hessian matrix of f. For this function $f(x,y,z)$, our evaluation is simplified once we first notice that $f_{ij} = 0$ whenever $i \neq j$ (for $i,j = 1,2,3$). Another way of explaining this simplification is to observe that the Hessian matrix $\mathscr{H} f(x,y,z)$ is a diagonal matrix at all points (x,y,z).

Using this observation, the principal minors are

$$D_1(x,y,z) := f_{11}(x,y,z) = 2,$$

$$D_2(x,y,z) := \begin{vmatrix} f_{11} & 0 \\ 0 & f_{22} \end{vmatrix} = \begin{vmatrix} 2 & 0 \\ 0 & 4 \end{vmatrix} = 8,$$

$$D_3(x,y,z) := \begin{vmatrix} f_{11} & 0 & 0 \\ 0 & f_{22} & 0 \\ 0 & 0 & f_{33} \end{vmatrix} = \begin{vmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 8 \end{vmatrix} = 48.$$

All three principal minors D_1, D_2, and D_3 are positive constant functions. In particular, we see that the origin is a minimum of the function f.

(b) Why can we conclude that the critical point from part (a) is, in fact, a global minimum?

This follows from a direct evaluation of the function. Since $t^2 \geq 0$ for every real number t, we see that

$$f(x,y,z) = (x-1)^2 - 1 + 2y^2 + 3z^2 \geq (0)^2 - 1 + 2(0)^2 + 3(0)^2 = -1,$$

and $f(1,0,0) = -1$. Therefore $f(x,y,z) \geq f(1,0,0)$ for every point (x,y,z), so that f has a global minimum at $(1,0,0)$,

6. *Let $f(x,y,z) = x^2y^2\sin(z)$.*

(a) Determine all critical points of the function $f(x,y,z)$.

All critical points of f must satisfy the equation $\nabla f(x,y,z) = (0,0,0)$. This corresponds to the following system of equations

$$\begin{aligned} 2xy^2\sin(z) &= 0 \\ 2x^2y\sin(z) &= 0 \\ x^2y^2\cos(z) &= 0 \end{aligned}$$

Clearly, then, if (x_0, y_0, z_0) is any point satisfying $x_0 = 0$ or $y_0 = 0$, then $\nabla f(x_0, y_0, z_0) = (0, 0, 0)$. Hence, all points contained in both sets $C_1 := \{(x, y, z) : x = 0\}$ and $C_2 = \{(x, y, z) : y = 0\}$ are critical points of f.

Now, suppose that (x_0, y_0, z_0) is such that x_0 and y_0 are both nonzero. This is just another way of saying that (x_0, y_0, z_0) is *not* an element of either C_1 or C_2. If it were true that $\nabla f(x_0, y_0, z_0) = (0, 0, 0)$ for such a point (x_0, y_0, z_0), the above system of equations would then imply that $\sin(z_0) = \cos(z_0) = 0$. But this is impossible! Hence, the set $C = C_1 \cup C_2$ contains all of the critical points of f.

(b) Calculate the Hessian matrix $\mathscr{H}(x, y, z)$ for an arbitrary point (x, y, z).

We apply the formula from Section 1.2 of Lecture Notes 4.1:

$$\mathscr{H}(x, y, z) := \begin{vmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{vmatrix} = \begin{vmatrix} 2y^2 \sin(z) & 4xy \sin(z) & 2xy^2 \cos(z) \\ 4xy \sin(z) & 2x^2 \sin(z) & 2x^2 y \cos(z) \\ 2xy^2 \cos(z) & 2x^2 y \cos(z) & -x^2 y^2 \sin(z) \end{vmatrix}$$

(c) Based on your answers to (a) and (b), explain why the second-order test must necessarily fail at all critical points of this function.

From (a), we know that critical points (x, y, z) of f must satisfy $x = 0$ or $y = 0$. Using our formula from (b), we see that at any such point we have

$$\mathscr{H}(x, y, z) = \begin{vmatrix} 2y^2 \sin(z) & 0 & 0 \\ 0 & 2x^2 \sin(z) & 0 \\ 0 & 0 & 0 \end{vmatrix}$$

which has determinant 0. The second derivative test is thus *inconclusive* at all critical points of the function f.

7. *Let $f(x, y)$ be a function defined for all $(x, y) \in \mathbb{R}^2$. Assume that f and all its partial derivatives of all orders are continuous everywhere in the plane. Assume also that the second order Taylor polynomial of f at $(a, b) = (0, 0)$ is*

$$p_2(x, y) = 2x^2 + 3y^2,$$

and that the error $f(x, y) - p_2(x, y)$ satisfies

$$\lim_{(x,y)\to(0,0)} \frac{f(x, y) - p_2(x, y)}{x^2 + y^2} = 0.$$

Prove that f has a local minimum at $(0, 0)$.

We start with $f(0,0) = p_2(0,0) = 0$. Next, by the $\epsilon - \delta$ definition of the limit with $\epsilon = 1$, there is a $\delta > 0$ such that for all (x, y) with $0 < \sqrt{x^2 + y^2} < \delta$,

$$\left| \frac{f(x,y) - p_2(x,y)}{x^2 + y^2} \right| < 1.$$

We rewrite this as $|f(x,y) - p_2(x,y)| < x^2 + y^2$, or equivalently,

$$p_2(x,y) - (x^2 + y^2) < f(x,y) < p_2(x,y) + (x^2 + y^2).$$

The first inequality tells us that, for (x, y) as above, we have

$$f(x,y) > (2x^2 + 3y^2) - (x^2 + y^2) = x^2 + 2y^2 > 0 = f(0,0).$$

Therefore $f(x,y) \geq f(0,0)$ for all (x, y) in the δ-neighbourhood of $(0,0)$, meaning that f has a local minimum at $(0,0)$.

8. *Let $f(x,y)$ be a function defined for all $(x,y) \in \mathbb{R}^2$. Assume that f and all its partial derivatives of all orders are continuous everywhere in the plane. Assume also that*

$$f(x,y) = x^4 - 2y^4 + g(x,y),$$

where the function $g(x,y)$ satisfies $g(0,0) = 0$ and

$$\lim_{(x,y)\to(0,0)} \frac{g(x,y)}{x^4 + y^4} = 0.$$

Prove that f does not have a local minimum or maximum at $(0,0)$.

We start with $f(0,0) = p_2(0,0) = 0$. Next, by the $\epsilon - \delta$ definition of the limit with $\epsilon = 1/2$, there is a $\delta > 0$ such that for all (x, y) with $0 < \sqrt{x^2 + y^2} < \delta$,

$$\left| \frac{g(x,y)}{x^4 + y^4} \right| < \frac{1}{2}.$$

We rewrite this as $|g(x,y)| < \frac{1}{2}(x^4 + y^4)$.

Consider first what happens when $y = 0$ but $x \neq 0$. Then

$$f(x,0) = x^4 + g(x,0) \geq x^4 - \frac{x^4}{2} = \frac{x^4}{2} > 0 = f(0,0),$$

so that in any neighbourhood of $(0,0)$ there are points $(x,0)$ where $f(x,0) > f(0,0)$.

Now consider what happens when $x = 0$ but $y \neq 0$. Then

$$f(x,0) = -2y^4 + g(0,y) \leq -2y^4 + \frac{y^4}{2} = -\frac{3y^4}{2} < 0 = f(0,0),$$

so that in any neighbourhood of $(0,0)$ there are also points $(0,y)$ where $f(0,y) < f(0,0)$. Hence, there is no local minimum or maximum at $(0,0)$.

4.2 Global Extrema on Restricted Domains

1. *Find the largest and smallest values of the function* $f(x,y) = x^3 - 3xy + 3y^2$ *on the closed triangle with vertices* $(0,0), (1,0), (1,1)$.

   We first look for critical points. We have $\nabla f = \langle 3x^2 - 3y, -3x + 6y\rangle$. At a critical point we get

   $$0 = -3x + 6y \;\Rightarrow\; x = 2y,$$
   $$3x^2 - 3y = 12y^2 - 3y = 0 \;\Rightarrow\; 4y^2 - y = y(4y-1) = 0,$$

   which gives $y = 0$ or $y = 1/4$. We get two critical points $(0,0)$ and $(\frac{1}{2}, \frac{1}{4})$. We evaluate f at these points:

   $$f(0,0) = 0, f\left(\frac{1}{2}, \frac{1}{4}\right) = \frac{1}{8} - \frac{3}{8} + \frac{3}{16} = -\frac{1}{16}.$$

   Note that $(0,0)$ is one of the vertices and $(\frac{1}{2}, \frac{1}{4})$ is inside the triangle.

   Next, we look for the largest and smallest values on the boundary. We consider each of the three line segments:

   - If $y = 0$, $0 \le x \le 1$, then $f(x,0) = x^3$ has the minimum $f(0,0) = 0$ and maximum $f(1,0) = 1$.
   - If $y = x$, $0 \le x \le 1$, then $f(x,x) = x^3 - 3x^2 + 3x^2 = x^3$ has the minimum $f(0,0) = 0$ and maximum $f(1,1) = 1$.
   - If $x = 1$, $0 \le y \le 1$, then $f(1,y) = 1 - 3y + 3y^2$. To find the extremal values, we compute the derivative:

     $$\frac{d}{dy} f(1,y) = -3 + 6y.$$

     This is 0 when $-3 + 6y = 0$, so that $y = 1/2$, $f(1, \frac{1}{2}) = 1 - \frac{3}{2} + \frac{3}{4} = \frac{1}{4}$. We already computed $f(1,0) = 1$ and $f(1,1) = 1$.

   Finally, we choose the largest and smallest of these values. The largest value is $f(1,0) = f(1,1) = 1$, and the smallest value is $f(\frac{1}{2}, \frac{1}{4}) = -\frac{1}{16}$.

2. *Find the largest and smallest values of the function* $f(x,y) = 4x - 2xy + y^2$ *on the square* $\{(x,y) : \; 0 \le x \le 2, \; 0 \le y \le 2\}$.

   We first look for critical points inside the region. We have $f_x = 4 - 2y$ and $f_y = -2x + 2y$, hence if $f_x = f_y = 0$, then $y = 2$ and $x = 2$. At the critical point, $f(2,2) = 8 - 8 + 4 = 4$.

   Next, we look for possible minima and maxima on the boundary:

- $f(x,0) = 4x$ with $0 \le x \le 2$ has the minimum value $f(0,0) = 0$ and maximum value $f(2,0) = 8$;
- $f(x,2) = 4x - 4x + 4 = 4$;
- $f(0,y) = y^2$ with $0 \le x \le 2$ has the minimum value $f(0,0) = 0$ and maximum value $f(0,2) = 4$;
- $f(2,y) = 8 - 4y + y^2$. To find its extrema on $[0,2]$, we look for critical points: $-4 + 2y = 0$, $y = 2$. We have already evaluated $f(2,2) = 4$ and $f(2,0) = 8$.

Thus the smallest value is $f(0,0) = 0$ and the largest value is $f(2,0) = 8$.

3. *Find, with proof, the global minimum and maximum of the function* $f(x,y) = \frac{xy}{x^4+y^4+32}$ *defined for all* x, y, *by following the steps listed below.*

   (a) Find all critical points of f, *evaluate* $f(x,y)$ *at these points, and choose the largest and smallest value,* M *and* m.

   We have

$$f_x(x,y) = \frac{y(x^4+y^4+32) - xy(4x^3)}{(x^4+y^4+32)^2} = \frac{y(y^4+16-3x^4)}{(x^4+y^4+16)^2},$$

$$f_y(x,y) = \frac{x(x^4+y^4+32) - xy(4y^3)}{(x^4+y^4+32)^2} = \frac{x(x^4+16-3y^4)}{(x^4+y^4+16)^2}.$$

   For a critical point, we should have $f_x(x,y) = f_y(x,y) = 0$. From $f_x(x,y) = 0$, we get that either $y = 0$ or $y^4 + 32 = 3x^4$. From $f_y(x,y) = 0$, either $x = 0$ or $x^4 + 32 = 3y^4$.

   If $y = 0$, then $x^4 + 32 = 3y^4 = 0$ has no solutions, so we must have $x = 0$. Similarly, if $x = 0$, then $y = 0$. If on the other hand $y^4 + 32 = 3x^4$ and $x^4 + 32 = 3y^4$, then we can combine the two equations to get $x^4 = 3y^4 - 32 = 3(3x^4 - 32) - 32 = 9x^4 - 128$, so that

$$8x^4 = 128, \quad x^4 = 16, \quad x = \pm 2,$$

   and $y^4 = 3x^4 - 32 = 3 \cdot 16 - 32 = 16$, $y = \pm 2$. This gives us 5 critical points, with the corresponding values of f as follows:

$$\begin{aligned} &f(0,0) = 0, \\ &f(2,2) = f(-2,-2) = \frac{4}{16+16+32} = \frac{1}{16}, \\ &f(2,-2) = f(-2,2) = -\frac{1}{16}. \end{aligned}$$

   Thus $M = \frac{1}{16}$ and $m = -\frac{1}{16}$.

(b) Prove that $x^4 + y^4 \geq \frac{1}{2}(x^2 + y^2)^2$.

We have $x^4 + y^4 - 2x^2y^2 = (x^2 - y^2)^2 \geq 0$, so that $x^4 + y^4 \geq 2x^2y^2$. Therefore

$$2(x^4 + y^4) = x^4 + y^4 + (x^4 + y^4) \geq x^4 + y^4 + 2x^2y^2 = (x^2 + y^2)^2,$$

and the inequality we want follows when we divide by 2.

(c) Use (b) to prove that $|f(x, y)| \leq \frac{2}{x^2+y^2}$. Based on this, prove that the values M and m from (a) are in fact the global minimum and maximum values.

Using (b), we get

$$\begin{aligned} |f(x, y)| &= \frac{|xy|}{x^4 + y^4 + 32} \leq \frac{|xy|}{x^4 + y^4} \leq \frac{|xy|}{\frac{1}{2}(x^2 + y^2)^2} \\ &= \frac{2|xy|}{(x^2 + y^2)^2} \leq \frac{2\sqrt{x^2 + y^2}\sqrt{x^2 + y^2}}{(x^2 + y^2)^2} = \frac{2}{x^2 + y^2}, \end{aligned}$$

when we used that $|x|, |y| \leq \sqrt{x^2 + y^2}$. (Alternatively, one could use that $2|x||y| \leq x^2 + y^2$ by completing the square as in (b), and then we get a similar estimate without the factor of 2.)

Now to finish the problem, consider a large enough disc, e.g. $D = \{(x, y) : x^2 + y^2 \leq 100\}$. The only critical points in that disc are those found in (a). On the other hand, if (x, y) is either on the boundary of D or outside of it, then we have $|f(x, y)| \leq \frac{2}{x^2+y^2} \leq \frac{2}{100} = \frac{1}{50} < \frac{1}{16}$. Therefore M and m from (a) are the actual global maximum and minimum, respectively.

4. *Let $f(x, y) = 2x^2 - 7x - xy + y^2$. Find the global minimum of $f(x, y)$ on $\mathbb{R}^2$, by following the procedure below.*

(a) Find all critical points of f. Evaluate f at these points.

(b) Prove that if $|x|$ and $|y|$ are large enough, then $f(x, y)$ is greater than the values you found in (a).

(c) Conclude that the global minimum is attained at a point found in (a).

(a) The partial derivatives are

$$f_1(x, y) = 4x - 7 - y, \quad f_2(x, y) = -x + 2y.$$

For a critical point, we must have $f_2 = 0$ so that $x = 2y$, and then $f_1 = 0$ implies $0 = 4x - 7 - y = 7y - 7$, so that $y = 1$ and $x = 2$. At the critical point $(2, 1)$, we have

$$f(2, 1) = 2 \cdot 4 - 7 \cdot 2 - 2 + 1 = 8 - 14 - 2 + 1 = -7.$$

Now we need to check that this is a global minimum. The proof below is just one of many ways that this can be done, so that if you are using a different inequality, that should be fine as long as you are doing it correctly.

(b) We rewrite $f(x, y)$ as

$$\begin{aligned} f(x,y) &= \frac{x^2}{2} + (x^2 - 7x) + \left(\frac{x^2}{2} - xy + \frac{y^2}{2}\right) + \frac{y^2}{2} \\ &= \frac{x^2+y^2}{2} + \left(x^2 - 7x + \frac{49}{4}\right)^2 - \frac{49}{4} + \frac{(x-y)^2}{2} \\ &= \frac{x^2+y^2}{2} + \left(x - \frac{7}{2}\right)^2 - \frac{49}{4} + \frac{(x-y)^2}{2} \\ &\geq \frac{x^2+y^2}{2} - \frac{49}{4}. \end{aligned}$$

If $x^2 + y^2 \geq 100$, then $f(x, y) \geq 50 - \frac{49}{4} > 0 > f(2, 1)$.

(c) Combining (a) and (b), we see that the minimum of f on the compact region $x^2 + y^2 \leq 100$ is $f(2, 1) = -7$, and that $f(x, y) > 0 > f(2, 1)$ outside of the region. Therefore $f(2, 1) = -7$ is the global minimum on $\mathbb{R}^2$.

5. *Find the global minimum and maximum of* $f(x, y) = (2x^2 - y^2)e^{-x^2-y^2}$ *on* $\mathbb{R}^2$*, with proof.*

We first find the critical points. The partials are:

$$f_x = 4xe^{-x^2-y^2} + (2x^2 - y^2)e^{-x^2-y^2} \cdot (-2x) = 2x(2 - 2x^2 + y^2)e^{-x^2-y^2},$$

$$f_y = -2ye^{-x^2-y^2} + (2x^2 - y^2)e^{-x^2-y^2} \cdot (-2y) = -2y(1 + 2x^2 - y^2)e^{-x^2-y^2}.$$

If $f_x = 0$, then either $x = 0$ or $2 - 2x^2 + y^2 = 0$, and if $f_y = 0$, then either $y = 0$ or $1 + 2x^2 - y^2 = 0$.

- For $x = 0$, we get $y = 0$ or $1 - y^2 = 0$, so that $y = \pm 1$.
- For $y = 0$, we get $x = 0$ or $2 - 2x^2 = 0$, so that $x = \pm 1$.
- Finally, we cannot have $2-2x^2+y^2 = 0$ and $1+2x^2-y^2 = 3-(2-2x^2+y^2) = 0$ at the same time.

This gives us 5 critical points: $(0, 0), (0, \pm 1), (\pm 1, 0)$. We have $f(0, 0) = 0$, $f(0, \pm 1) = -e^{-1}$, $f(\pm 1, 0) = 2e^{-1}$.

Next, we have

$$\lim_{(x,y)\to\infty} f(x, y) = 0.$$

To prove this, we use single-variable calculus. Let $r = \sqrt{x^2+y^2}$ (as in polar coordinates), then $|2x^2 - y^2| \le 2x^2 + y^2 \le 3r^2$ and $e^{-x^2-y^2} = e^{-r^2}$, so that

$$|f(x,y)| = |2x^2 - y^2|e^{-x^2-y^2} \le 3r^2 e^{-r^2}.$$

To see that this tends to 0 as $r \to \infty$, we use l'Hôpital's Rule:

$$\lim_{r\to\infty} \frac{3r^2}{e^{r^2}} = \lim_{r\to\infty} \frac{6r}{2re^{r^2}} = \lim_{r\to\infty} \frac{3}{e^{r^2}} = 0.$$

In particular, if we take R large enough, then we have $|f(x,y)| \le (2e)^{-1}$ for all (x,y) with $r = \sqrt{x^2+y^2} \ge R$. Let

$$X = \{(x,y) : \ x^2 + y^2 \le R^2\}$$

(a closed disc of radius R centered at the origin). Then on the boundary of X we have $|f(x,y)| \le (2e)^{-1}$, hence the maximal and minimal values of f on X are, respectively, $f(\pm 1, 0) = 2e^{-1}$ and $f(0,\pm 1) = -e^{-1}$. But outside of X we also have $|f(x,y)| \le (2e)^{-1}$. Therefore, $f(\pm 1, 0) = 2e^{-1}$ and $f(0, \pm 1) = -e^{-1}$ are in fact the global extrema of f.

6. *Let $f(x,y) = x^2y - 2xy^2 + 3y$. Find the largest and smallest values of $f(x,y)$ on the following sets, or else prove that they do not exist:*

   (a) $D_1 = \{(x,y) : \ x \ge 0, 0 \le y \le 2\}$,

   (b) $D_2 = \{(x,y) : \ x \ge 0, y \ge 0, xy \le 16\}$.

   Make sure that you account for both regions being unbounded.

The partial derivatives are

$$f_1(x,y) = 2xy - 2y^2 = 2y(x-y), \quad f_2(x,y) = x^2 - 4xy + 3.$$

For a critical point, we must have $f_1 = 0$, so that $y = 0$ or $x = y$.

- If $y = 0$, then $f_2 = 0$ implies $x^2 + 3$, which has no solutions.
- If $x = y$, then $f_2 = 0$ implies $x^2 - 4x^2 + 3 = 0$, so that $x = \pm 1$. We have

  $$f(1,1) = 1 - 2 + 3 = 2,$$

  and the point $(-1,-1)$ is not in D_1 or D_2.

(a) We check the boundary of D_1. When $y = 0$, we have $f(x,0) = 0$. When $y = 2$, we have

$$f(x,2) = 2x^2 - 8x + 6 = 2(x-2)^2 - 2,$$

which has a minimum $f(2,2) = -2$ and goes to ∞ as $x \to \infty$. Finally, when $x = 0$ and $0 \leq y \leq 2$, we have $f(0,y) = 3y$, with minimum $f(0,0) = 0$ and maximum $f(0,2) = 6$.

This already tells us that no maximum value exists (since $f(x,2) \to \infty$ as $x \to \infty$), and the smallest value we found so far if $f(2,2) = -2$. We just need to check that f does not have any smaller values at the "unbounded end" of D_1. One way to do that is to notice that

$$f(x,y) = xy(x-2y) + 3y.$$

For $x \geq 4$ and $(x,y) \in D_1$, we have $x \geq 4 \geq 2y$, so that $f(x,y) \geq 0$. Thus any values of f that are smaller than -2 would have to lie in the compact region $\{(x,y) :\ 0 \leq x \leq 4,\ 0 \leq y \leq 2$. But we just checked that the smallest value in that region is $f(2,2) = -2$, so that this is also the global minimum of f on D_1.

(b) We now check the boundary of D_2. We have $f(x,0) = 0$ and $f(0,y) = 3y$. Since $3y \to \infty$ as $y \to \infty$, there is no global maximum on D_2.

On the hyperbola $xy = 16$, we have

$$f(x,y) = xy(x-2y) + 3y = 16(x-2y) + 3y = \frac{16^2}{y} - 32y + 3y = \frac{16^2}{y} - 29y.$$

When $y \to \infty$ (therefore $x \to 0$), we have $\frac{16^2}{y} \to 0$, so that $f(\frac{16}{y}, y) \to -\infty$ as $y \to \infty$. Hence f attains all values from $-\infty$ to ∞ on D_2, with no global minimum and no global maximum.

(Notice that as $y \to \infty$, $f \to \infty$ along the line $x = 0$ and $f \to -\infty$ along the hyperbola that approaches the same line! That's how polynomial functions can behave.)

4.3 The Method of Lagrange Multipliers

1. *Find the largest and smallest values of the function $f(x,y,z) = x + 2y$ in the closed disc $x^2 + y^2 \leq 25$.*

Since $\nabla f = \langle 1, 2 \rangle \neq 0$, there are no critical points. Therefore the largest and smallest values are attained on the circle $x^2 + y^2 = 25$.

We use Lagrange multipliers with $g(x,y) = x^2 + y^2 - 25$. We have $\nabla g = \langle 2x, 2y \rangle$, so that if $\nabla f = \lambda \nabla g$, then $1 = 2\lambda x$, $2 = 2\lambda y$. Hence $\lambda \neq 0$ and $2\lambda y = 2 = 4\lambda x$, $y = 2x$. Plugging this into the equation of the circle we get $x^2 + (2x)^2 = 5x^2 = 25$, $x^2 = 5$, $x = \pm\sqrt{5}$.

This yields two critical points, $(\sqrt{5}, 2\sqrt{5})$ and $(-\sqrt{5}, -2\sqrt{5})$. We have $f(\sqrt{5}, 2\sqrt{5}) = 5\sqrt{5}$, the largest value, and $f(-\sqrt{5}, -2\sqrt{5}) = -5\sqrt{5}$, the smallest value.

2. *Find the largest and smallest values of the function* $f(x,y) = x^2 - 3x - \frac{y^2}{2}$ *in the closed disc* $x^2 + y^2 \le 4$.

We first look for critical points. From $\nabla f = \langle 2x - 3, -y \rangle = 0$, we get $y = 0$ and $x = 3/2$. The corresponding value of f is

$$f\left(\frac{3}{2}, 0\right) = \frac{9}{4} - \frac{9}{2} = -\frac{9}{4}.$$

Next, we look for the largest and smallest values on the circle $x^2 + y^2 = 4$. We use Lagrange multipliers with $g(x,y) = x^2 + y^2 - 4$. Then $\nabla g = \langle 2x, 2y \rangle$, so that $\nabla f = \lambda \nabla g$ implies $2x - 3 = 2\lambda x$, $-y = 2\lambda y$. From the second equation, we have $y = 0$ or $2\lambda = -1$.

- If $y = 0$, then from the equation of the circle we get $x^2 = 4$, $x = \pm 2$, with the corresponding values of f:

$$f(2,0) = 4 - 6 = -2, \quad f(-2,0) = 4 + 6 = 10.$$

- If $2\lambda = -1$, then from the first equation we have $2x - 3 = -x$, $3x = 3$, $x = 1$. From the equation of the circle, we then have $y^2 = 3$, $y = \pm\sqrt{3}$. The corresponding values of f are

$$f(1, \pm\sqrt{3}) = 1 - 3 - \frac{3}{2} = -\frac{7}{2}.$$

Finally, we choose the largest and smallest of these values: the largest value is $f(-2,0) = 4 + 6 = 10$, and the smallest value is $f(1, \pm\sqrt{3}) = 1 - 3 - \frac{3}{2} = -\frac{7}{2}$.

3. *Find the largest and smallest values of the function* $f(x,y) = \frac{x}{y}$ *in the closed region* $x^2 + (y-2)^2 \le 1$.

Note that the function is continuous in the region: if $x^2 + (y-2)^2 \le 1$, then $(y-2)^2 \le 1$, $|y - 2| \le 1$, $1 \le y \le 3$, so that in particular $y \ne 0$.

We first check for critical points: from $\nabla f = \langle \frac{1}{y}, -\frac{x}{y^2} \rangle = 0$, we get no solutions. Therefore the largest and smallest values have to be attained on the boundary.

Next, we look for the largest and smallest values on the circle $x^2 + (y-2)^2 = 1$. We use Lagrange multipliers with $g(x,y) = x^2 + (y-2)^2$. We have $\nabla g = \langle 2x, 2(y-2) \rangle$. For convenience, we will use the equation $\nabla f = \frac{\lambda}{2} \nabla g$, so that

$$\frac{1}{y} = \lambda x, \quad -\frac{x}{y^2} = \lambda(y-2).$$

The first equation implies that $x \neq 0$ (otherwise we would have $1/y = 0$, which is impossible), and $\lambda = 1/(xy)$. From the second equation, we then have

$$-\frac{x}{y^2} = \frac{y-2}{xy}, \quad -\frac{x}{y} = \frac{y-2}{x}, \quad -x^2 = y(y-2) = y^2 - 2y.$$

But then from the equation of the circle,

$$1 = x^2 + (y-2)^2 = -y^2 + 2y + y^2 - 4y + 4 = -2y + 4.$$

Therefore $2y = 3$, $y = 3/2$, and $x^2 = -(y^2 - 2y) = -(\frac{9}{4} - 3) = \frac{3}{4}$, $x = \pm\frac{\sqrt{3}}{2}$. Finally, we have

$$f\left(\frac{\sqrt{3}}{2}, \frac{3}{2}\right) = \frac{1}{\sqrt{3}}, \quad f\left(-\frac{\sqrt{3}}{2}, \frac{3}{2}\right) = -\frac{1}{\sqrt{3}},$$

and these are the largest and smallest values, respectively.

4. *Let $A > 0$ be some real number. Find the point(s) on the surface $xyz = A^3$ that is closest to the origin, and calculate the minimum distance of this surface to the origin. Your answer will depend upon the constant A.*

We want to minimize the square of the distance to the origin, $f(x, y, z) = x^2+y^2+z^2$, with the constraint $g(x, y, z) = xyz = A^3$. We have $\nabla f = \langle 2x, 2y, 2z\rangle$ and $\nabla g = \langle yz, xz, xy\rangle$. Hence we need to solve

$$2x = \lambda yz,\ 2y = \lambda xz,\ 2z = \lambda xy,\ xyz = A^3.$$

Since $xyz = A^3 \neq 0$, all of x, y, z are nonzero. We therefore rewrite the first three equations as

$$\frac{\lambda}{2} = \frac{x}{yz} = \frac{y}{xz} = \frac{z}{xy}.$$

Cross-multiplying, we get $x^2z = y^2z$ and $xy^2 = xz^2$, so that (using again that x, y, z are all nonzero) we have $x^2 = y^2 = z^2$, so that $x = \pm y$ and $y = \pm z$. This gives up to 8 possible points. Now, from the final equation we have

$$A^3 = xyz = \pm x^3.$$

Now, since $A > 0$, we see that xyz *must be positive*! This will further restrict our choices, leaving us with four points:

$$(A, A, A),\ (A, -A, -A), (-A, A, -A),\ (-A, -A, A).$$

Clearly, all four points have the same distance from the origin, which is $\sqrt{3A^2}$.

Since the surface $xyz = A^3$ is not bounded, we should also explain why the distance function has a global minimum (and not just a critical point) at the four points listed above. This is because if any of x, y, z tend to infinity, so does the distance to the origin. The details could be worked out as in Notes 4.2. The two figures below illustrate the surface $xyz = A$ for two distinct values of A. Minimizing points for the distance to the origin are shown in red.

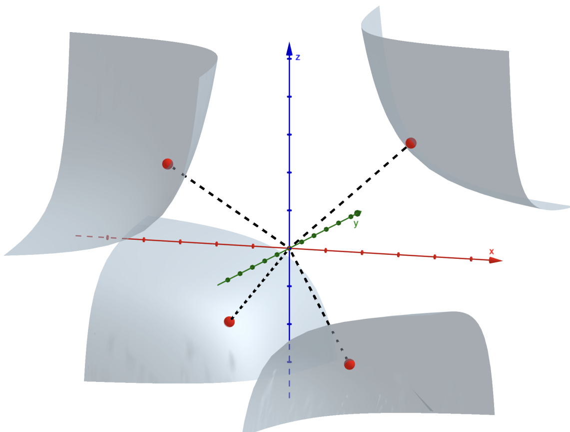

Figure 4.1: $A = 5$

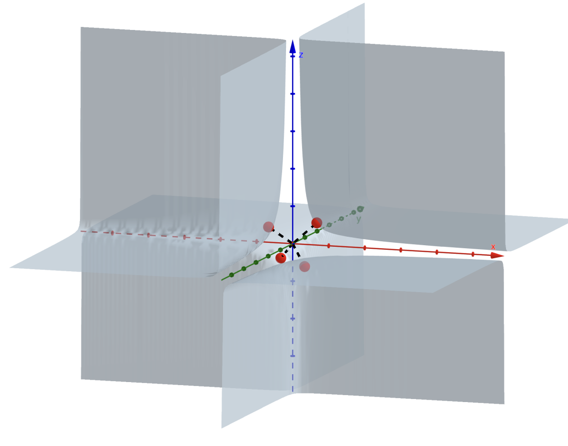

Figure 4.2: $A = 1$

5. *Find the maximum and minimum values of $xy^2 + yz^2$ in the ball $x^2 + y^2 + z^2 \leq 1$.*

Let $f(x, y, z) = xy^2 + yz^2$. We first look for critical points: $\nabla f = \langle y^2, 2xy + z^2, 2yz\rangle$, so that at a critical point we must have $y = 0, z = 0$. There is no equation for x, so we have a family of critical points $(x, 0, 0)$ with $f(x, 0, 0) = 0$.

Next, we look for the extreme values on the surface of the ball. We use Lagrange multipliers. Let $g(x, y, z) = x^2 + y^2 + z^2$, then $\nabla g = \langle 2x, 2y, 2z\rangle$. We get equations

$$y^2 = 2\lambda x, \quad 2xy + z^2 = 2\lambda y, \quad 2yz = 2\lambda z.$$

From the last equation, $z = 0$ or $\lambda = y$.

- If $z = 0$, then from the second equation $2xy = 2\lambda y$, so that $y = 0$ (a case we've covered already) or $\lambda = x$. In the latter case, from the first equation $y^2 = 2x^2$, and plugging it into the equation of the sphere, $x^2+y^2+z^2 = 3x^2 = 1$, $x = \pm 1/\sqrt{3}$. We get four critical points
$$f\Big(\sqrt{\frac{1}{3}}, \pm\sqrt{\frac{2}{3}}, 0\Big) = \frac{2}{3\sqrt{3}}, \quad f\Big(-\sqrt{\frac{1}{3}}, \pm\sqrt{\frac{2}{3}}, 0\Big) = -\frac{2}{3\sqrt{3}}.$$
- If $\lambda = y$, then from the first two equations we have $y^2 = 2xy$ and $2xy + z^2 = 2y^2$, so that $y^2 + z^2 = 2y^2$, $y^2 = z^2$. Also, from $y^2 = 2xy$ we have $y = 0$ or $y = 2x$. If $y = 0$, then also $z = 0$ and we've covered this case already. If $y = 2x$, then $x^2 + y^2 + z^2 = y^2 + 4x^2 + 4x^2 = 9x^2 = 1$, so that $x = \pm 1/3$. We get four critical points:
$$f\Big(\frac{1}{3}, \frac{2}{3}, \pm\frac{2}{3}\Big) = \frac{4}{9}, \quad f\Big(-\frac{1}{3}, -\frac{2}{3}, \pm\frac{2}{3}\Big) = -\frac{4}{9}.$$

Since $\frac{4}{9} > \frac{2}{3\sqrt{3}}$, the maximum and minimum values are $4/9$ and $-4/9$, at the above points.

Chapter 5

Integration

5.1 Introduction to the Riemann integral

1. *Let $f(x, y) = x + y$, and let $D = \{(x, y) : 0 \leq x \leq 1, 0 \leq y \leq 1\}$. Prove that for any $\epsilon > 0$, there is a $\delta > 0$ such that the following holds:*

   > *Let $\mathscr{P}_1$, $\mathscr{P}_2$ be two partitions of D together with choices of sampling points. If $\|\mathscr{P}_1\| < \delta$ and $\|\mathscr{P}_2\| < \delta$, then*
   >
   > $$|\mathcal{R}(f, \mathscr{P}_1) - \mathcal{R}(f, \mathscr{P}_2)| < \epsilon.$$

   (This property is used in the proof that f is integrable on D.)

   Hint: let $\mathscr{P}$ be a common refinement of $\mathscr{P}_1$ and $\mathscr{P}_2$ – that is, a partition that includes all the partition points of both $\mathscr{P}_1$ and $\mathscr{P}_2$. What can you say about $\mathcal{R}(f, \mathscr{P}_1) - \mathcal{R}(f, \mathscr{P})$ and $\mathcal{R}(f, \mathscr{P}_2) - \mathcal{R}(f, \mathscr{P})$?

   It suffices to prove the following. Let $\epsilon > 0$. Then there is a $\delta > 0$ such that if $\mathscr{P}_1$ is a partition of D with $\|\mathscr{P}_1\| < \delta$, and if $\mathscr{P}$ is a refinement of $\mathscr{P}_1$, then

   $$|\mathcal{R}(f, \mathscr{P}_1) - \mathcal{R}(f, \mathscr{P})| < \epsilon/2. \tag{5.1.1}$$

   Applying this to $\mathscr{P}_1$, $\mathscr{P}_2$ and their common refinement $\mathscr{P}$, we get

   $$\begin{aligned} &|\mathcal{R}(f, \mathscr{P}_1) - \mathcal{R}(f, \mathscr{P}_2)| \\ &\quad \leq |\mathcal{R}(f, \mathscr{P}_1) - \mathcal{R}(f, \mathscr{P})| + |\mathcal{R}(f, \mathscr{P}) - \mathcal{R}(f, \mathscr{P}_2)| \\ &\quad < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

   We now prove (5.1.1). Let $D_{ij} = \{(x, y) : \ x_{i-1} \leq x \leq x_i, \ y_{j-1} \leq y \leq y_j\}$ be a small rectangle in the partition $\mathscr{P}_1$, with the evaluation point P_{ij}. In the partition $\mathscr{P}$, this rectangle is subdivided further into smaller rectangles that we will call $R_{ijk\ell}$, with evaluation points $Q_{ijk\ell}$.

The contribution to $\mathcal{R}(f, \mathscr{P}_1)$ from D_{ij} is

$$f(P_{ij})(\text{area of } D_{ij}) = \sum_{k,\ell} f(P_{ij})(\text{area of } R_{ijk\ell}).$$

The contribution to $\mathcal{R}(f, \mathscr{P})$ from D_{ij} is

$$\sum_{k,\ell} f(Q_{ijk\ell})(\text{area of } R_{ijk\ell}).$$

The difference between the two contributions is

$$\sum_{k,\ell} \Big(f(P_{ij}) - f(Q_{ijk\ell})\Big)(\text{area of } R_{ijk\ell}).$$

But if $\mathscr{P}_1$ has diameter less than δ, then the diagonal of each D_{ij} is less than δ. Since P_{ij} and $Q_{ijk\ell}$ are two points in D_{ij}, their x-coordinates can differ by at most δ, and similarly for their y-coordinates. Therefore

$$|f(P_{ij}) - f(Q_{ijk\ell})| \leq 2\delta$$

so that

$$\begin{aligned}
&\left|\sum_{k,\ell} \Big(f(P_{ij}) - f(Q_{ijk\ell})\Big)(\text{area of } R_{ijk\ell})\right| \\
&\qquad \leq 2\delta \sum_{k,\ell} (\text{area of } R_{ijk\ell}) \\
&\qquad = 2\delta(\text{area of } D_{ij}).
\end{aligned}$$

Adding up the D_{ij} rectangles, we get

$$|\mathcal{R}(f, \mathscr{P}_1) - \mathcal{R}(f, \mathscr{P})| \leq 2\delta \sum_{i,j} (\text{area of } D_{ij}) = 2\delta.$$

For this to be less that $\epsilon/2$, it suffices to take $\delta < \epsilon/4$.

2. *Let $D = \{(x, y) : 0 \leq x \leq 1, 0 \leq y \leq 1\}$, and let $S = \{P_1, \ldots, P_n\} \subset D$ be a set of finitely many points. Lef $f(x, y)$ be a function such that $f(P_1) = c_1$, $\ldots$, $f(P_n) = c_n$ for some arbitrary (but finite) values $c_1, \ldots, c_n \in \mathbb{R}$, and $f(x, y) = 0$ for all $(x, y) \notin S$.*

   Let $\mathcal{P}$ be a partition of D, with partition rectangles $D_{ij} = \{(x, y) : x_{i-1} \leq x \leq x_i, y_{j-1} \leq y \leq y_j\}$. Let $\mathcal{R}$ be the Riemann sum for f associated with this partition.

Let $d_{ij} = \sqrt{(x_i - x_{i-1})^2 + (y_j - y_{j-1})^2}$ *be the length of the diagonal of* D_{ij}*. Assume that* $0 < d_{ij} < \epsilon$ *for all* i, j*. Prove that*

$$|\mathcal{R}| < C\epsilon^2$$

for some constant C *that may depend on* $n, c_1, \ldots, c_n$*, but is independent of* ϵ*. What does this say about the integral* $\iint_D f(x, y)\, dA$*?*

Let $M = \max(|c_1|, |c_2|, \ldots, |c_n|)$.
Since each rectangle of the partition has diameter less than ϵ, we have $0 < x_i - x_{i-1} < \epsilon$ and $0 < y_j - y_{j-1} < \epsilon$, so that each rectangle has area at most $(x_i - x_{i-1})(y_i - y_{j-1}) < \epsilon^2$. Each point P_k can belong to at most 4 rectangles of the partition (4 rectangles if P_k is a "corner point", 2 if it lies on a common edge but is not a corner point, 1 otherwise). Therefore there are at most 4n rectangles that contain one or more points of S. The contribution to $\mathcal{R}$ from each such rectangle is at most $M\epsilon^2$, and the contricutions fron all other rectangles are zero. Therefore

$$|\mathcal{R}| \leq 4n \cdot M\epsilon^2.$$

Since this goes to zero as $\epsilon \to 0$, we have $\iint_D f(x, y)\, dA = 0$.

5.2 Integration in Two Real Variables

1. *Evaluate the integral* $\displaystyle\iint_D 3dA$*, if* D *is the region bounded by the parabola* $y^2 - x - 5 = 0$ *and the line* $x + 2y = 3$*.*

   We first find the points where the parabola intersects the line: if $x = y^2 - 5$ and $x + 2y = 3$ then $y^2 - 5 = 3 - 2y$, so that $y^2 + 2y - 8 = 0$, $y = 2, -4$. It will be more convenient to integrate in x first (draw a picture!):

$$\iint_D 3dA = \int_{-4}^{2} \int_{y^2-5}^{3-2y} 3dxdy = \int_{-4}^{2} 3(3 - 2y - y^2 + 5)dy$$

$$= 3\int_{-4}^{2} (-y^2 - 2y + 8)dy = 3(-\frac{y^3}{3} - y^2 + 8y)\Big|_{-4}^{2} = 108.$$

2. *Reverse the order of integration to evaluate the following integrals.*

   (a) $\displaystyle\int_0^1 \int_{\sqrt{x}}^1 \cos(y^3 + 1)dydx = \int_0^1 \int_0^{y^2} \cos(y^3 + 1)dxdy = \int_0^1 y^2 \cos(y^3 + 1)dy$

$$= \frac{\sin(y^3 + 1)}{3}\Big|_0^1 = \frac{\sin(2) - \sin(1)}{3}.$$

(b) $$\int_0^1\int_{x^2}^1 x^3\sin(y^3)dydx = \int_0^1\int_0^{\sqrt{y}} x^3\sin(y^3)dxdy = \int_0^1 \Big(\frac{x^4}{4}\sin(y^3)\Big)\Big|_{x=0}^{\sqrt{y}}dy$$
$$= \int_0^1 \frac{y^2}{4}\sin(y^3)dy = -\frac{\cos(y^3)}{12}\Big|_0^1 = \frac{-\cos(1)+1}{12}.$$

3. *Change the order of integration in the integral*

$$\int_0^4\int_{x/2}^x \cos(y^2)\,dy\,dx + \int_4^8\int_{x/2}^4 \cos(y^2)\,dy\,dx.$$

Then evaluate the integral.

$$\int_0^4\int_y^{2y} \cos(y^2)\,dx\,dy = \int_0^4 y\cos(y^2)\,dy$$
$$= \frac{\sin(y^2)}{2}\Big|_{y=0}^4 = \frac{\sin(16)}{2}.$$

4. *Evaluate the double integrals. Use any method you like.*

(a) $\iint_D y\,dA$*, D is the triangle in the xy-plane with vertices* $(0,0)$*,* $(1,2)$*,* $(3,0)$*.*

$$\int_0^2\int_{y/2}^{3-y} y\,dx\,dy = \int_0^2 y\Big(3-y-\frac{y}{2}\Big)\,dy$$
$$= \int_0^2 \Big(3y-\frac{3y^2}{2}\Big)\,dy = \frac{3}{2}y^2 - \frac{y^3}{2}\Big|_0^2$$
$$= \frac{3}{2}\cdot 4 - \frac{8}{2} = 6-4 = 2.$$

(b) $$\int_1^2\int_{1/x}^1 ye^{xy}dydx = \int_{1/2}^1\int_{1/y}^2 ye^{xy}dxdy = \int_{1/2}^1 e^{xy}|_{x=1/y}^2 dy$$
$$= \int_{1/2}^1 (e^{2y}-e)dy = \frac{e^{2y}}{2} - ey\Big|_{1/2}^1 = \frac{e^2}{2} - e.$$

5. *Find the volume under the plane $z = x$ and above the region in the xy-plane bounded by $y = x$ and $y = x^2 - x$.*

$$\begin{aligned}\iint_D x\,dA &= \int_0^2 \int_{x^2-x}^{x} x dy dx = \int_0^2 x(x - x^2 + x)dx \\ &= \int_0^2 (2x^2 - x^3)dx = \frac{2x^3}{3} - \frac{x^4}{4}\Big|_{x=0}^{2} = \frac{16}{3} - \frac{16}{4} = \frac{16}{3} - 4 = \frac{4}{3}.\end{aligned}$$

6. *Evaluate the double integral $\iint_D x\,dA$, where D is the triangle with vertices $(0,0)$, $(2,1)$, $(4,0)$.*

$$\begin{aligned}\iint_D x\,dA &= \int_0^1 \int_{2y}^{4-2y} x dx dy = \int_0^1 \frac{x^2}{2}\Big|_{2y}^{4-2y} dy \\ &= \int_0^1 2\big((2-y)^2 - y^2\big)dy = \int_0^1 (8 - 8y)dy \\ &= 8y - 4y^2\Big|_0^1 = 8 - 4 = 4.\end{aligned}$$

An alternative way to do it is to notice that $\iint_D x\,dA =$ (area of D) $\times$ (the average value of x on D). The average value of x on D is, by symmetry, 2. The area of D is $\frac{1}{2} \cdot 4 \cdot 1 = 2$. Therefore $\iint_D x\,dA = 2 \cdot 2 = 4$.

7. *Let T be the triangle with vertices $(0,0)$, $(s,0)$ and (s,t) for some $s, t > 0$. Determine the centroid of T.*

The centroid of T is the point $(\bar{x}, \bar{y})$, with $\bar{x}$ and $\bar{y}$ determined by the formulas

$$\bar{x} = \frac{1}{\text{Area}(T)} \iint_T x\,dA, \quad \bar{y} = \frac{1}{\text{Area}(T)} \iint_T y\,dA.$$

The triangle T has a right angle at $(s,0)$, and

$$\text{Area}(T) = \frac{st}{2}.$$

We can represent T both as x-simple and as y-simple:

$$\begin{aligned}T &:= \{(x,y) \in \mathbb{R}^2 : \ 0 \le x \le s, \ 0 \le y \le \frac{tx}{s}\} \\ &= \{(x,y) \in \mathbb{R}^2 : \ 0 \le y \le t, \ \frac{sy}{t} \le x \le s\}.\end{aligned}$$

We now calculate the points $\bar{x}$ and $\bar{y}$ using iterated integrals. For $\bar{y}$, we have

$$\begin{aligned} \bar{y} &= \frac{2}{st}\int_0^s \int_0^{tx/s} y\,dydx = \frac{1}{ts}\int_0^s \Big[y^2\Big]_{y=0}^{y=tx/s} dx \\ &= \frac{t}{s^3}\int_0^s x^2 dx = \frac{t}{3s^3}\Big[x^3\Big]_{x=0}^{x=s} \\ &= \frac{t}{3}. \end{aligned}$$

For $\bar{x}$, we have

$$\begin{aligned} \bar{x} &= \frac{2}{st}\int_0^t \int_{sy/t}^{s} x\,dxdy = \frac{1}{st}\int_0^t \Big[x^2\Big]_{x=sy/t}^{x=s} dy \\ &= \frac{s}{t^3}\int_0^t (t^2-y^2)dy = \frac{s}{t^3}\left[t^2y - \frac{y^3}{3}\right]_{y=0}^{y=t} \\ &= \frac{2s}{3}. \end{aligned}$$

Hence, the centroid of T is $(\bar{x},\bar{y}) = (\frac{2s}{3}, \frac{t}{3})$.

5.3 Improper Integrals

1. *Determine whether the integral*

$$\iint_Q \frac{x}{(1+x)(1+y)} dA,$$

where Q is the first quadrant of the xy-plane, converges or diverges. If it converges, evaluate it.

We first try to get some intuition for whether the integral should be convergent or divergent, since that will influence our choice of strategy. Notice that for any fixed y, the function $\frac{x}{(1+x)(1+y)}$ approaches $\frac{1}{1+y}$ at infinity, so we expect the integral in x to be infinite. We now make this rigorous using inequalities.

We have

$$\iint_Q \frac{x}{(1+x)(1+y)} dA \geq \int_0^\infty dy \int_1^\infty \frac{x}{(1+x)(1+y)} dx,$$

since the function is nonnegative and the second region of integration is smaller. Therefore, if we prove that the second integral is divergent, it follows that so is the first one. We now estimate the second integral from below. For $x \geq 1$, we have

$$\frac{x}{1+x} \geq \frac{x}{2x} = \frac{1}{2},$$

so that

$$\int_0^\infty dy \int_1^\infty \frac{x}{(1+x)(1+y)} dx \geq \int_0^\infty dy \int_1^\infty \frac{1}{2(1+y)} dx = \int_0^\infty \infty dy = \infty$$

and the integral is divergent.

2. *Determine whether the integral*

$$\iint_Q e^{y-x} dA$$

converges or diverges, if

$$Q := \{(x, y) : \; x \geq 1, \; 0 \leq y \leq \ln(x^2 + x + 1)\}.$$

If it converges, evaluate it.

We will prove that the integral converges by direct evaluation. First, we write the integral over Q as the iterated integral

$$\int_1^\infty \int_0^{\ln(x^2+x+1)} e^{y-x} dy dx = \int_1^\infty e^{-x} \left(\int_0^{\ln(x^2+x+1)} e^y dy \right) dx.$$

Evaluating the inner integral gives

$$\int_0^{\ln(x^2+x+1)} e^y dy = e^y \Big|_{y=0}^{\ln(x^2+x+1)} = (x^2 + x + 1) - 1 = x^2 + x.$$

Therefore

$$\iint_Q e^{y-x} dA = \int_1^\infty (x^2 + x) e^{-x} dx = \int_1^\infty x^2 e^{-x} dx + \int_1^\infty x e^{-x} dx$$

We will evaluate both of these integrals via integration by parts. First notice that

$$\int_1^\infty x^2 e^{-x} dx = -x^2 e^{-x} \Big|_{x=1}^\infty + 2 \int_1^\infty x e^{-x} dx = e^{-1} + 2 \int_1^\infty x e^{-x} dx,$$

so that

$$\iint_Q e^{y-x} dA = e^{-1} + 3 \int_1^\infty x e^{-x} dx.$$

Now, we again use integration by parts. This gives that

$$\int_1^\infty x e^{-x} dx = -x e^{-x} \Big|_1^\infty + \int_1^\infty e^{-x} dx = e^{-1} - e^{-x} \Big|_1^\infty = 2e^{-1}.$$

Putting together the previous steps, we see that

$$\iint_Q e^{y-x} dA = e^{-1} + 3 \cdot (2e^{-1}) = 7e^{-1}.$$

3. *Determine whether the integral*

$$\iint_D \frac{e^y}{y^{3/2}}\, dA$$

is convergent, if D is the triangle in the xy-plane with vertices $(0,0)$, $(1,0)$, $(1,1)$. (You do not have to evaluate the integral.)

We estimate:

$$\begin{aligned}\iint_D \frac{e^y}{y^{3/2}} dA \geq \iint_D \frac{1}{y^{3/2}} dA &= \int_0^1 \int_0^x y^{-3/2} dy dx \\ &= \int_0^1 \frac{y^{-1/2}}{-1/2}\Big|_{y=0}^{x} dx = \int_0^1 \infty\, dx = \infty,\end{aligned}$$

so that the integral is divergent.

4. *Let D be the region $\{(x,y):\ x \geq 2,\ 0 \leq y \leq \pi\}$. Prove that the integral*

$$\iint_D \frac{1+e^{-x}}{x^2 - \sin^2 y} dA$$

is convergent, by comparing it to a different integral that is easier to evaluate.

We estimate, using that $e^{-x} \leq 1$ for $x \geq 0$, that $\sin^2 y \leq 1$, and that $x^2 - 1 \geq x^2/2$ for $x \geq 2$:

$$\begin{aligned}\iint_D \frac{1+e^{-x}}{x^2 - \sin^2 y} dA &\leq \iint_D \frac{2}{x^2 - \sin^2 y} dA \\ &\leq \int_2^\infty \int_0^\pi \frac{2}{x^2-1} dy dx \leq \int_2^\infty \frac{2\pi}{x^2-1} dx \\ &\leq \int_2^\infty \frac{2\pi}{x^2/2} dx = \int_2^\infty \frac{4\pi}{x^2} dx = -\frac{4\pi}{x}\Big|_2^\infty = 2\pi.\end{aligned}$$

Therefore the integral converges.

5.4 Change of Variables

1. *Evaluate the double integrals.*

   (a) $\iint_D y\, dA$, *where* $D = \{(x,y):\ y \geq 0,\ 1 \leq x^2 + y^2 \leq 9\}$.

$$\int_0^\pi \int_1^3 r\sin\theta\, r dr\, d\theta = \int_0^\pi \int_1^3 r^2 \sin\theta\, dr\, d\theta$$
$$= \int_0^\pi \frac{1}{3} r^3 \sin\theta \Big|_{r=1}^3 d\theta = \int_0^\pi (9 - \frac{1}{3}) \sin\theta\, d\theta$$
$$= \int_0^\pi \frac{26}{3}(-\cos\theta)\Big|_0^\pi = \frac{26}{3} \cdot 2 = \frac{52}{3}$$

(b)

$$\iint_{\mathbb{R}^2} \frac{dA}{(4+x^2+y^2)^5} = \int_0^{2\pi} \int_0^\infty \frac{r\, dr\, d\theta}{(4+r^2)^5}$$
$$= \int_0^{2\pi} \int_0^\infty \frac{1}{2}(4+r^2)^{-5}(2r)\, dr\, d\theta$$
$$= \int_0^{2\pi} \frac{-1}{8}(4+r^2)^{-4}\Big|_0^\infty d\theta = \int_0^{2\pi} \frac{1}{8} 4^{-4}\, d\theta = \frac{2\pi}{8} 4^{-4} = \frac{\pi}{4^5}.$$

2. *Evaluate the integral $\iint_D (\frac{x}{y} - 1) dA$, if D is the region in the first quadrant bounded by the hyperbolas $x^2 - y^2 = 1$, $x^2 - y^2 = 4$, and the lines $x = 2y$, $x = 3y$.*

Let $u = x^2 - y^2$, $v = x/y$, then the region of integration is given by $1 \le u \le 4$, $2 \le v \le 3$. To find the Jacobian, we first compute

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} 2x & -2y \\ 1/y & -x/y^2 \end{vmatrix} = -2\frac{x^2}{y^2} + 2 = 2 - 2v^2,$$

so that

$$\frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{2-2v^2}.$$

Hence

$$\int\int_D (\frac{x}{y} - 1) dA = \int_1^4 \int_2^3 (v-1)\left|\frac{1}{2-2v^2}\right| dvdu$$
$$= \int_1^4 \int_2^3 \frac{v-1}{2(v^2-1)} dvdu = \int_1^4 \int_2^3 \frac{1}{2(v+1)} dvdu$$
$$= \int_1^4 \frac{\ln(v+1)}{2}\Big|_2^3 du = \int_1^4 \frac{\ln 4 - \ln 3}{2} du$$
$$= \frac{3}{2}(\ln 4 - \ln 3).$$

We used that $|2 - 2v^2| = 2v^2 - 2$ for $v \in [2,3]$.

3. *Use a change of variables to evaluate the integral $\iint_D (x-y)dxdy$, where D is the region in the first quadrant bounded by the hyperbolas $x^2-y^2=1$, $x^2-y^2=4$, and the lines $x+y=3$, $x+y=6$.*

   Let $u=x^2-y^2$, $v=x+y$, then the region of integration is $1\le u\le 4$, $3\le v\le 6$. (We can solve this for x,y: $x-y=u/v$, $x=\frac{1}{2}(v+\frac{u}{v})$, $y=\frac{1}{2}(v-\frac{u}{v})$. This is, however, optional.) We compute the Jacobian:

$$\frac{\partial(u,v)}{\partial(x,y)}=\begin{vmatrix}2x & -2y\\ 1 & 1\end{vmatrix}=2x+2y=2v,\ \frac{\partial(x,y)}{\partial(u,v)}=\frac{1}{2v}.$$

   (Alternatively, one can compute $\frac{\partial(x,y)}{\partial(u,v)}$ directly. This leads to a slightly longer computation, but is equally correct.)

   Thus our integral is

$$\int_1^4\int_3^6 \frac{u}{v}\cdot\frac{1}{2v}dvdu=\int_1^4\int_3^6\frac{u}{2v^2}dvdu=\int_1^4\frac{-u}{2v}\Big|_{v=3}^6 du=\int_1^4\frac{u}{2}\Big(\frac{1}{3}-\frac{1}{6}\Big)du$$
$$=\int_1^4\frac{u}{12}du=\frac{u^2}{24}\Big|_1^4=\frac{15}{24}=\frac{5}{8}.$$

4. *Evaluate the area of the region in the first quadrant bounded by the hyperbolas $xy=1$, $xy=5$, and the parabolas $y=x^2$, $y=4x^2$. (Hint: introduce new coordinates in which the region of integration is a rectangle.)*

   Let $u=xy$ and $v=y/x^2$, so that

$$R=\{(u,v):1\le u\le 5,\ 1\le v\le 4\}.$$

   We also need the Jacobian $\frac{\partial(x,y)}{\partial(u,v)}$. It is easiest to compute the Jacobian of the inverse transformation $\frac{\partial(u,v)}{\partial(x,y)}$ and then take its reciprocal. We find that

$$\begin{aligned}\frac{\partial(u,v)}{\partial(x,y)}&=\begin{vmatrix} y & x\\ -2y/x^3 & 1/x^2\end{vmatrix}\\ &=\frac{y}{x^2}+2\frac{y}{x^2}=3\frac{y}{x^2}=3v,\end{aligned}$$

   so that

$$\frac{\partial(x,y)}{\partial(u,v)}=\left(\frac{\partial(u,v)}{\partial(x,y)}\right)^{-1}=\frac{1}{3v}.$$

Then the area of our region is

$$\begin{aligned}
\text{Area}(R) &= \int_R dA = \int_1^5 \int_1^4 \left| \frac{\partial(x,y)}{\partial(u,v)} \right| dv\, du \\
&= \int_1^5 \int_1^4 \frac{1}{3v}\, dv\, du \\
&= \int_1^5 \frac{1}{3} \ln v \Big|_1^4 du \\
&= \int_1^5 \frac{1}{3} \ln 4\; du = \frac{4}{3} \ln 4.
\end{aligned}$$

We used $\ln v$ and not $\ln |v|$ for the antiderivative of $1/v$ because $v > 0$ on R.

5. *Determine whether the integral* $\displaystyle\iint_{\mathbb{R}^2} \frac{|y|}{(1+x^2+y^2)^3}\, dA$ *is convergent.*

We write

$$\begin{aligned}
\iint_{\mathbb{R}^2} \frac{|y|}{(1+x^2+y^2)^3}\, dA &= \iint_{x^2+y^2\le 1} \frac{|y|}{(1+x^2+y^2)^3}\, dA \\
&\quad + \iint_{x^2+y^2\ge 1} \frac{|y|}{(1+x^2+y^2)^3}\, dA,
\end{aligned}$$

and estimate each part separately:

$$\iint_{x^2+y^2\le 1} \frac{|y|}{(1+x^2+y^2)^3}\, dA \le \iint_{x^2+y^2\le 1} |y|\, dA \le \iint_{x^2+y^2\le 1} 1\, dA = \pi,$$

$$\begin{aligned}
\iint_{x^2+y^2\ge 1} \frac{|y|}{(1+x^2+y^2)^3}\, dA &\le \iint_{x^2+y^2\ge 1} \frac{\sqrt{x^2+y^2}}{(1+x^2+y^2)^3}\, dA \\
&= \int_1^\infty \int_0^{2\pi} \frac{r}{(1+r^2)^3} r\, d\theta dr = 2\pi \int_1^\infty \frac{r}{(1+r^2)^3} r\, dr \\
&= 2\pi \int_1^\infty \frac{r^2}{(1+r^2)^3} dr \le 2\pi \int_1^\infty \frac{r^2}{(r^2)^3} dr \\
&= 2\pi \int_1^\infty r^{-4} dr = -\frac{2\pi}{3} r^{-3} \Big|_{r=1}^\infty = \frac{2\pi}{3}.
\end{aligned}$$

Hence the integral is convergent.

5.5 Integration in Three Variables

1. *Let R be the solid region in $\mathbb{R}^3$ bounded by the planes $x = 0$, $y = 0$, $z = 0$, and $x + 2y + z = 6$. Write $\iiint_R f(x, y, z)dV$ as iterated integrals where the order of integration is (a) $dzdydx$, (b) $dydxdz$.*

(a)

$$\int_0^6 \int_0^{3-x/2} \int_0^{6-x-2y} f(x, y, z)dzdydx$$

(b)

$$\int_0^6 \int_0^{6-z} \int_0^{3-\frac{x+z}{2}} f(x, y, z)dydxdz$$

2. *Evaluate the integral $\iiint_D y\,dV$, if D is the tetrahedron with vertices $(0, 0, 0)$, $(4, 2, 0)$, $(0, 2, 0)$, and $(0, 2, 4)$.*

$$\begin{aligned}
\iiint_D y\,dV &= \int_0^2 dy \int_0^{2y} dx \int_0^{2y-x} ydz \\
&= \int_0^2 dy \int_0^{2y} dx\,[yz]_{z=0}^{2y-x} = \int_0^2 dy \int_0^{2y} (2y^2 - yx)dx \\
&= \int_0^2 dy \left[2y^2x - \frac{yx^2}{2}\right]_{x=0}^{2y} = \int_0^2 (4y^3 - 2y^3)dy \\
&= \int_0^2 2y^3\,dy = \left[\frac{y^4}{2}\right]_{y=0}^{2} = 8.
\end{aligned}$$

3. *Let D be the spatial region above the xy-plane, inside the cylinder $x^2 + y^2 = 1$, and inside the sphere $x^2 + y^2 + z^2 = 4$. Evaluate the integral $\iiint_D zdV$.*

We use cylindrical coordinates:

$$\begin{aligned}
\iiint_D z dV &= \int_0^1 \int_0^{2\pi} \int_0^{\sqrt{4-r^2}} zr\, dz\, d\theta\, dr \\
&= \int_0^1 \int_0^{2\pi} r\, \frac{z^2}{2}\bigg|_0^{\sqrt{4-r^2}} d\theta\, dr \\
&= \int_0^1 \int_0^{2\pi} r\, \frac{4-r^2}{2}\, d\theta\, dr \\
&= \int_0^1 \pi(4r - r^3)\, dr \\
&= \pi\Big(2r^2 - \frac{r^4}{4}\Big)\bigg|_0^1 = \pi\Big(2 - \frac{1}{4}\Big) = \frac{7\pi}{4}.
\end{aligned}$$

4. *Evaluate the integral* $\iiint_R x\, dV$, *if* $R = \{(x,y,z):\ 0 \le z \le y \le x,\ x^2+y^2 \le 1\}$.

Let $D = \{(x,y):\ 0 \le y \le x,\ x^2+y^2 \le 1\}$. Then

$$\begin{aligned}
\iiint_R x\, dV &= \iint_D \int_0^y x\, dz\, dA = \iint_D xy\, dA \\
&= \int_0^{\pi/4} \int_0^1 (r\cos\theta)(r\sin\theta) r\, dr\, d\theta \\
&= \int_0^{\pi/4} \int_0^1 r^3 \cos\theta \sin\theta\, dr\, d\theta \\
&= \int_0^{\pi/4} \frac{r^3}{4}\bigg|_{r=0}^1 \cos\theta\sin\theta\, d\theta \\
&= \int_0^{\pi/4} \frac{\cos\theta\sin\theta}{4}\, d\theta \\
&= \frac{\sin^2\theta}{8}\bigg|_{\theta=0}^{\pi/4} = \frac{(\sqrt{2}/2)^2}{8} = \frac{1}{16}.
\end{aligned}$$

5.6 Spherical Coordinates

1. *A surface in* $\mathbb{R}^3$ *has the equation* $x^2 + y^2 = z^2 - 4$. *Convert the equation of this surface to cylindrical and spherical coordinates.*

- In cylindrical coordinates, we have $r^2 = x^2 + y^2$, so the equation of the surface is $r^2 = z^2 - 4$.

- In spherical coordinates, we have $r = R\sin\phi$ and $z = R\cos\phi$, so our equation is $R^2\sin^2\phi = R^2\cos^2\phi - 4$. (Optional: This can be simplified to $R^2(\cos^2\phi - \sin^2\phi) = 4$, or $R^2\cos(2\phi) = 4$.)

2. *A surface in $\mathbb{R}^3$ has the equation $R^2 - 4R\sin\phi + 3 = 0$ in spherical coordinates. Find its equation in cylindrical coordinates.*

We have $r = R\sin\theta$ and $R^2 = r^2 + z^2$, hence our equation in cylindrical coordinates is $r^2 + z^2 - 4r + 3 = 0$. (Optional: To see what kind of a surface this is, we rewrite the equation as $(r-2)^2 + z^2 = 1$. This is the torus obtained by rotating the circle $(x-2)^2 + z^3 = 1$ in the xz-plane around the z-axis.)

3. *Evaluate the integral $\iiint_D \frac{\cos(x^2+y^2+z^2)}{\sqrt{x^2+y^2+z^2}}\,dV$, where D is the region that lies above the cone $z = \sqrt{x^2+y^2}$ and inside the sphere $x^2+y^2+z^2 = 4$.*

We use spherical coordinates:

$$\begin{aligned}
\iiint_D \frac{\cos(x^2+y^2+z^2)}{\sqrt{x^2+y^2+z^2}}\,dV &= \int_0^{2\pi} d\theta \int_0^{\pi/4} d\phi \int_0^2 \frac{\cos(R^2)R}{R}^2 \sin\phi\,dR \\
&= \int_0^{2\pi} d\theta \int_0^{\pi/4} \sin\phi\,d\phi \int_0^2 \cos(R^2)R\,dR \\
&= 2\pi\left[-\cos\phi\right]_0^{\pi/4}\left[\frac{\sin(R^2)}{2}\right]_0^2 \\
&= 2\pi\left(1 - \frac{\sqrt{2}}{2}\right)\left(\frac{\sin 4}{2}\right) = \pi(\sin 4)\left(1 - \frac{\sqrt{2}}{2}\right)
\end{aligned}$$

4. *Write the following integrals as iterated integrals in your choice of coordinates (rectangular (x,y,z), cylindrical (r,θ,z), or spherical (R,θ,ϕ)). You do not need to evaluate the integrals.*

(a) $\iiint_D (x+y)\,dV$, where D is the region bounded by the planes $y = x$, $y = 4x$, $x = 2$, $z = 0$, and $z = x$.

In rectangular coordinates:

$$\int_0^2 \int_x^{4x} \int_0^x (x+y)dz\,dy\,dx$$

(b) $\iiint_D z^2\,dV$*, where* D *is the spatial region outside the cone* $z = \sqrt{x^2+y^2}$*, inside the cylinder* $x^2+y^2=4$*, and above the* xy*-plane.*

We use cylindrical coordinates:

$$\int_0^2 \int_0^{2\pi} \int_0^r z^2 r\,dz\,d\theta\,dr$$

5. *Write out the integral* $\iiint_W x^2\,dV$ *as an iterated integral in cylindrical coordinates, if* W *is the three-dimensional solid inside the cylinder* $x^2+y^2=1$*, bounded from below by the cone* $z=\sqrt{x^2+y^2}$ *and from above by the sphere* $x^2+y^2+z^2=4$*. (You do not need to evaluate the integral.)*

   In cylindrical coordinates, the cylinder, cone and sphere have equations $r=1$, $z=r$, $r^2+z^2=1$. The inequalities describing the solid are $0 \le r \le 1$, $0 \le \theta \le 2\pi$, $r \le z \le \sqrt{4-r^2}$. The Jacobian for cylindrical coordinates is r. Also, $x^2 = r^2\cos^2\theta$. Hence the integral is

$$\int_0^{2\pi} \int_0^1 \int_r^{\sqrt{4-r^2}} r^3\cos^2\theta\,dz\,dr\,d\theta.$$

6. *Find the mass of the solid bounded from below by the cone* $\phi=\pi/3$ *and from above by the sphere* $R=2$*, if the density at a point with spherical coordinates* (R,θ,ϕ) *is* R*. (Evaluate the integral.)*

   The solid (shaped like an ice-cream cone) is described in spherical coordinates by the inequalities $0 \le R \le 2$, $0 \le \theta \le 2\pi$, $0 \le \phi \le \pi/3$. The Jacobian is $R^2\sin\phi$. Hence the mass is

$$\int_0^2 \int_0^{\pi/3} \int_0^{2\pi} R\cdot R^2\sin\phi\,d\theta\,d\phi\,d\rho = \int_0^2 \int_0^{\pi/3} 2\pi R^3\sin\phi\,d\phi\,dR$$

$$= \int_0^2 2\pi R^3 \left(-\cos\phi\Big|_0^{\pi/3}\right) dR = \int_0^2 2\pi R^3(-\frac{1}{2}+1)\,dR$$

$$= \int_0^2 \pi R^3\,dR = \frac{\pi R^4}{4}\Big|_0^2 = \frac{16\pi}{4} = 4\pi.$$

7. *Use the change of variables $x = u^2$, $y = v^2$, $z = w^2$ to evaluate the volume of the region bounded by the coordinate planes and the surface $\sqrt{x} + \sqrt{y} + \sqrt{z} = 1$.*

We first change variables as indicated above. The Jacobian is $\frac{\partial(x,y,z)}{\partial(u,v,w)} = 8uvw$, and our region in the new coordinates is given by the inequalities $u \geq 0$, $v \geq 0$, $w \geq 0$, $u + v + w \leq 1$. We compute its volume:

$$V = \int_0^1 \int_0^{1-u} \int_0^{1-u-v} 8uvw \, dwdvdu = \int_0^1 \int_0^{1-u} 4uvw^2 \Big|_{w=0}^{1-u-v} dvdu$$

$$= \int_0^1 \int_0^{1-u} 4uv(1-u-v)^2 dvdu.$$

This could be expanded and integrated in a straightforward way, but the calculations are a little bit shorter if we change variables again, from (u, v) to (u, t) with $t = 1 - u - v$:

$$V = \int_0^1 \int_0^{1-u} 4u(1-u-t)t^2 dtdu = \int_0^1 \int_0^{1-u} (4u(1-u)t^2 - 4ut^3) dtdu$$

$$= \int_0^1 \Big(\frac{4u(1-u)t^3}{3} - ut^4\Big)\Big|_{t=0}^{1-u} du$$

$$= \int_0^1 \Big(\frac{4u(1-u)^4}{3} - u(1-u)^4\Big) du = \int_0^1 \frac{u(1-u)^4}{3} du.$$

Again, we change variables for convenience, letting $s = 1 - u$:

$$V = \int_0^1 \frac{(1-s)s^4}{3} ds = \int_0^1 \frac{s^4 - s^5}{3} ds = \frac{1}{3}\Big(\frac{s^5}{5} - \frac{s^6}{6}\Big)\Big|_0^1 = \frac{1}{90}.$$