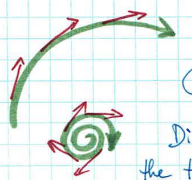


Unit tangent and curvature

(4/1)



Question: How quickly does the curve change direction?

Direction is represented by the tangent vector, so a change of direction should be represented by the derivative of the tangent vector. We will consider the unit tangent vector, since we are only interested in its direction, not magnitude.

We want this to have a geometric meaning independent of parametrization (depending only on the shape of the curve). Therefore we will use arclength parametrization (i.e. motion along the curve with constant speed 1.)

Note on arclength parametrization: we can use it to define concepts even when it would be difficult or impossible to find the arclength parametrization explicitly. Eventually, we will have formulas for the same concepts that do not require us to know an explicit formula for arclength.

Assume $\vec{r}(t): [a, b] \rightarrow \mathbb{R}^3$ is differentiable, $\vec{r}'(t)$ is continuous, $\vec{r}'(t) \neq 0$ on (a, b) .

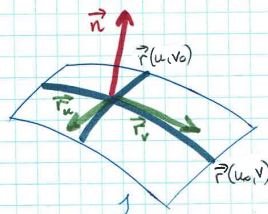
We want formulas for:

(3/2)

(1) Normal vector / tangent plane. If the surface is given in terms of parametrization $\vec{r}(u, v)$

(2) Surface area element, for the purpose of integration.

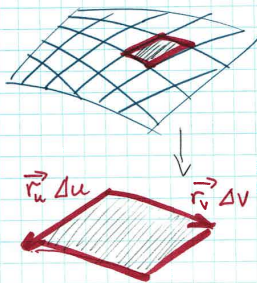
(1) Normal vector at $P = \vec{r}(u_0, v_0)$: the curves $\vec{r}(u, v_0)$ and $\vec{r}(u_0, v)$ pass through P and have tangent vectors $\vec{r}_u(u_0, v_0)$, $\vec{r}_v(u_0, v_0)$.



Assuming that $\vec{r}_u$ and $\vec{r}_v$ are not parallel, we can take

$$\vec{n} = \vec{r}_u \times \vec{r}_v$$

(2) Area element: for the purpose of integration, subdivide S using the (not necessarily rectangular) grid consisting of curves $u = \text{const}$, $v = \text{const}$. This divides S into approximate parallelograms spanned by vectors $\vec{r}_u \Delta u$ and $\vec{r}_v \Delta v$.



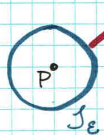
The area of such a parallelogram is

$$\begin{aligned} dS &= |(\vec{r}_u \Delta u) \times (\vec{r}_v \Delta v)| \\ &= |\vec{r}_u \times \vec{r}_v| \Delta u \Delta v \\ &= |\vec{n}| \Delta u \Delta v \end{aligned}$$

with $\vec{n}$ as above

Theorem (Divergence as flux density)

(4/4)



If S_E is the sphere of radius E centered at P , $\vec{F}$ smooth, $\vec{N}$ points outward, then

$$\text{div } \vec{F}(P) = \lim_{E \rightarrow 0} \frac{1}{\frac{4}{3}\pi E^3} \int_{S_E} \vec{F} \cdot \vec{N} dS$$

volume of the ball enclosed by S_E

Approximately,

$$\int_{S_E} \vec{F} \cdot \vec{N} dS \approx \left(\frac{4}{3}\pi E^3\right) \cdot \text{div } \vec{F}(P)$$

Proof: Assume $P=0$, let $\vec{F} = F_1 \vec{i} + F_2 \vec{j} + F_3 \vec{k}$, expand F_1, F_2, F_3 in Taylor series, evaluate F_1 contribution:

$$F_1(\vec{r}) = F_1(\vec{0}) + F_{1,x}(\vec{0})x + F_{1,y}(\vec{0})y + F_{1,z}(\vec{0})z + O(E^2)$$

differential approximation error estimate

$$\vec{N} = \frac{1}{E}(x\vec{i} + y\vec{j} + z\vec{k}) \text{ on } S_E \text{ (using } |\vec{r}|=E\text{)}$$

$$F_1 \vec{i} \cdot \vec{N} = F_1 \frac{x}{E} = \frac{1}{E} [F_1(\vec{0})x + F_{1,x}(\vec{0})x^2 + F_{1,y}(\vec{0})xy + F_{1,z}(\vec{0})xz] + \frac{1}{E} O(E^2) \cdot x$$

By symmetry and cancellation,

$O(E^2)$ since $|x| \leq E$

$$\int_{S_E} x dS = \int_{S_E} xy dS = \int_{S_E} xz dS = 0$$

Stokes's Theorem

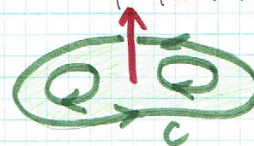
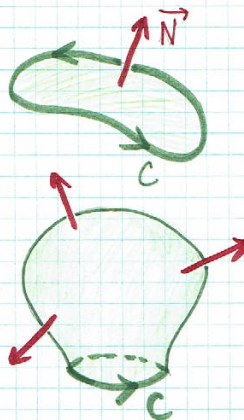
(4/3)

Theorem: Let S be a piecewise smooth, oriented surface in $\mathbb{R}^3$, with normal $\vec{N}$, and suppose that the boundary of S consists of one or more closed, piecewise smooth, simple curves with orientation inherited from S . Let $\vec{F}$ be a smooth vector field on an open set containing S .

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_S \text{curl } \vec{F} \cdot \vec{N} dS$$

Inherited orientation: counterclockwise when viewed from the direction of $\vec{N}$

(reverse if the surface has "holes" - same as for Green's Theorem)



Green's Theorem is a special case of Stokes's Theorem, with $S = D$ (a planar region) and $\vec{N} = \vec{k}$.