

Notes on Multivariable Calculus II

SOLUTIONS MANUAL

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Chapter 1

Parametrized Curves

1.1 Curves and Vector-Valued Functions

1. Let C be the curve in the xy -plane that consists of the half-line $y = -x$, $x \leq 0$, and the half-line $y = x$, $x \geq 0$. Give an example of a vector-valued function $\mathbf{r} : \mathbb{R} \rightarrow \mathbb{R}^2$ which parametrizes C and such that $\mathbf{r}(t)$ is differentiable for all t .

We are looking for a differentiable parametrization of the absolute value curve. We claim that $\mathbf{r}(t) = (t^3, |t^3|)$ works. This parametrization is clearly a bijection from $\mathbb{R}$ to C , the component $x(t) = t^3$ is differentiable for all t , and $y(t) = |t^3|$ is differentiable for all $t \neq 0$. It remains to check that $y(t)$ is differentiable at $t = 0$:

$$\begin{aligned}\lim_{x \rightarrow 0^+} |t^3|/t &= \lim_{x \rightarrow 0^+} t^3/t = \lim_{x \rightarrow 0^+} t^2 = 0, \\ \lim_{x \rightarrow 0^-} |t^3|/t &= \lim_{x \rightarrow 0^-} -t^3/t = \lim_{x \rightarrow 0^+} -t^2 = 0.\end{aligned}$$

Since the one-sided derivatives exist and are equal, $y(t)$ is differentiable at $t = 0$.

2. Suppose that $\mathbf{r} : [a, b] \rightarrow \mathbb{R}^3$ is a parametrized curve such that $\mathbf{r}$ and $\mathbf{r}'$ are continuous on $[a, b]$ and $\mathbf{r}'(t) \neq 0$. Does there have to exist a $t_0 \in (a, b)$ such that $\mathbf{r}(b) - \mathbf{r}(a) = (b - a)\mathbf{r}'(t_0)$? If yes, prove it. If no, give a counterexample.

The answer is no. Let $\mathbf{r} = \cos t \mathbf{i} + \sin t \mathbf{j} + t \mathbf{k}$ and $[a, b] = [0, 2\pi]$, so that $\mathbf{r}(b) - \mathbf{r}(a) = \mathbf{i} + 2\pi\mathbf{k} - \mathbf{i} = 2\pi\mathbf{k}$. But $\mathbf{r}'(t) = -\sin t \mathbf{i} + \cos t \mathbf{j} + \mathbf{k}$ is never parallel to the z axis.

3. Let $A = (a_1, a_2)$ and $B = (b_1, b_2)$ be two points in $\mathbb{R}^2$ such that $|AB| = 1$. Let also $\mathbf{a} = \overrightarrow{OA} = a_1\mathbf{i} + a_2\mathbf{j}$.

(a) Find an orthogonal 2×2 matrix Q such that

- $\det(Q) = 1$,

- the mapping $L : \mathbb{R}^2 \rightarrow \mathbb{R}^2$, defined by

$$L \begin{pmatrix} x \\ y \end{pmatrix} = \mathbf{a} + Q \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\text{satisfies } L \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \text{ and } L \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}.$$

We need to find $Q = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}$ so that Q is orthogonal and the above equations hold. The first equation

$$L \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \mathbf{a} + Q\mathbf{0} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$

is true for any Q . Let $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j}$, where $v_1 = b_1 - a_1$ and $v_2 = b_2 - a_2$. Then the second equation is

$$\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = L \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} + \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} + \begin{pmatrix} c_{11} \\ c_{21} \end{pmatrix}$$

so that $c_{11} = b_1 - a_1 = v_1$ and $c_{21} = b_2 - a_2 = v_2$.

For Q to be orthogonal, the second column must be orthogonal to the first one, therefore a multiple of $\begin{pmatrix} -v_2 \\ v_1 \end{pmatrix}$. We also have the condition $\det(Q) = 1$:

$$1 = \det(Q) = \begin{vmatrix} v_1 & -\lambda v_2 \\ v_2 & \lambda v_1 \end{vmatrix} = \lambda(v_1^2 + v_2^2).$$

But $v_1^2 + v_2^2 = |AB|^2 = 1$ by assumption, so that $\lambda = 1$ and $Q = \begin{pmatrix} v_1 & -v_2 \\ v_2 & v_1 \end{pmatrix}$.

(b) Let $\mathbf{r} : [0, 1] \rightarrow \mathbb{R}^2$ be given by $\mathbf{r}(t) = t\mathbf{i} + g(t)\mathbf{j}$, where $g : [0, 1] \rightarrow \mathbb{R}$ is continuous on $[0, 1]$ and satisfies $g(0) = g(1) = 0$. This parametrizes a curve $\mathcal{C}$.

Let L be as in (a). Then $L(\mathbf{r}(t))$ parametrizes the curve obtained from $\mathcal{C}$ by translating and rotating it so that the initial point is moved to A and the terminal point is moved to B . (This is a fact from linear algebra that you do not have to prove.)

Evaluate $L(\mathbf{r}(t))$ in terms of components. Prove that we may choose a parametrization $\mathbf{w} : [0, 1] \rightarrow \mathbb{R}^2$ of the line segment $\overline{AB}$, and a unit vector $\mathbf{n}$ perpendicular to $\overrightarrow{AB}$, so that

$$L(\mathbf{r}(t)) = \mathbf{w}(t) + g(t)\mathbf{n}.$$

We compute

$$\begin{aligned} L(\mathbf{r}(t)) &= \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} + \begin{pmatrix} v_1 & -v_2 \\ v_2 & v_1 \end{pmatrix} \begin{pmatrix} t \\ g(t) \end{pmatrix} \\ &= \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} + \begin{pmatrix} v_1 t - v_2 g(t) \\ v_2 t + v_1 g(t) \end{pmatrix} \\ &= \begin{pmatrix} a_1 + tv_1 - v_2 g(t) \\ a_2 + tv_2 + v_1 g(t) \end{pmatrix} = \mathbf{w}(t) + g(t)\mathbf{n}, \end{aligned}$$

where $\mathbf{w}(t) = \mathbf{a} + t\mathbf{v}$ for $t \in [0, 1]$ is a parametrization of the line segment from A to B , and $\mathbf{n} = -v_2\mathbf{i} + v_1\mathbf{j}$ is orthogonal to $\mathbf{v}$. We also have $|\mathbf{n}| = 1$, using again that $|AB| = 1$.

(c) Let $\mathcal{C}$ be the curve in $\mathbb{R}^2$ parametrized by $\mathbf{r}(t) = t\mathbf{i} + \sin(\pi t)\mathbf{j}$, $t \in [0, 1]$. Let $\mathcal{C}_1$ be the curve in $\mathbb{R}^2$ obtained from $\mathcal{C}$ by translating and rotating it in the plane so that the initial point is moved to $(3, 4)$ and the terminal point is moved to $(\frac{7}{2}, 4 + \frac{\sqrt{3}}{2})$. Parametrize $\mathcal{C}_1$.

Let $A = (3, 4)$ and $B = (\frac{7}{2}, 4 + \frac{\sqrt{3}}{2})$, and note that

$$\mathbf{v} = \overrightarrow{AB} = \frac{1}{2}\mathbf{i} + \frac{\sqrt{3}}{2}\mathbf{j}$$

is a unit vector. We use (b) with $\mathbf{n} = -v_2\mathbf{i} + v_1\mathbf{j} = -\frac{\sqrt{3}}{2}\mathbf{i} + \frac{1}{2}\mathbf{j}$, and with the line segment $\overline{AB}$ parametrized by

$$\mathbf{w}(t) = \mathbf{a} + t\mathbf{v} = (3\mathbf{i} + 4\mathbf{j}) + t\left(\frac{1}{2}\mathbf{i} + \frac{\sqrt{3}}{2}\mathbf{j}\right).$$

This gets us the parametrization

$$\begin{aligned} L(\mathbf{r}(t)) &= \mathbf{w}(t) + \sin(t)\mathbf{n} = (3\mathbf{i} + 4\mathbf{j}) + t\left(\frac{1}{2}\mathbf{i} + \frac{\sqrt{3}}{2}\mathbf{j}\right) + \sin(t)\left(-\frac{\sqrt{3}}{2}\mathbf{i} + \frac{1}{2}\mathbf{j}\right) \\ &= \left(3 + \frac{t}{2} - \frac{\sqrt{3}\sin(t)}{2}\right)\mathbf{i} + \left(4 + \frac{\sqrt{3}t}{2} + \frac{\sin(t)}{2}\right)\mathbf{j} \end{aligned}$$

1.2 Arc Length for Curves

1. Let C be the curve $x = e^{2t}$, $y = 2e^t$, $z = t$ for $t \in \mathbb{R}$. Find the length of the curve from $t = 0$ to $t = 1$.

We have $\mathbf{r}'(t) = 2e^{2t}\mathbf{i} + 2e^t\mathbf{j} + \mathbf{k}$, so that

$$|\mathbf{r}'(t)| = \sqrt{4e^{4t} + 4e^{2t} + 1} = 2e^{2t} + 1.$$

The length is $\int_0^1 (2e^{2t} + 1)dt = e^{2t} + t \Big|_0^1 = e^2 + 1 - 1 = e^2$.

2. Let $\mathbf{r}(t) = e^t \cos(2t)\mathbf{i} + e^t \sin(2t)\mathbf{j}$ be a parametrization of a curve C in $\mathbb{R}^2$.

(a) Let $s(b)$ be the length of the curve $\mathbf{r}(t)$ from $t = -\infty$ to $t = b$. (This is defined as the limit as $a \rightarrow -\infty$ of the length of $\mathbf{r}(t)$ from $t = a$ to $t = b$.) Find $s(b)$ as a function of b .

We have

$$\begin{aligned} x' &= e^t \cos(2t) - 2e^t \sin(2t) \\ y' &= e^t \sin(2t) + 2e^t \cos(2t) \\ |\mathbf{r}'|^2 &= e^{2t} \left[\cos^2(2t) - 4 \sin(2t) \cos(2t) + 4 \sin^2(2t) \right. \\ &\quad \left. + \sin^2(2t) + 4 \sin(2t) \cos(2t) + 4 \cos^2(2t) \right] \\ &= 5e^{2t} \\ |\mathbf{r}'| &= \sqrt{5}e^t \end{aligned}$$

so that

$$s(b) = \int_{-\infty}^b \sqrt{5}e^t dt = \sqrt{5}e^b.$$

(b) Use part (a) to give an arclength parametrization of C .

Let $s(t) = \sqrt{5}e^t$ from (a), then $e^t = s/\sqrt{5}$, so that $t = \ln(s/\sqrt{5})$. Therefore the arclength parametrization of C in terms of s is

$$\mathbf{r}(s) = \frac{s}{\sqrt{5}} \cos \left(2 \ln \frac{s}{\sqrt{5}} \right) \mathbf{i} + \frac{s}{\sqrt{5}} \sin \left(2 \ln \frac{s}{\sqrt{5}} \right) \mathbf{j}.$$

3. The paraboloid $z = x^2 + y^2 + 1$ intersects the surface $x^2 - 2x + y^2 = 0$ in a curve. Prove that the length of this curve is less than 15.

Along the intersection curve, we have $z = x^2 + y^2 + 1$ and $x^2 + y^2 = 2x$, so that $z = 2x + 1$. Now, notice that the equation of the second surface can be rewritten as

$$(x - 1)^2 + y^2 = 1,$$

which is a right circular cylinder with the symmetry axis given by $x = 1, y = 0$. This gives the parametrization

$$x = 1 + \cos t, \quad y = \sin t, \quad z = 2(1 + \cos t) + 1 = 2 \cos t + 3,$$

with $0 \leq t \leq 2\pi$. The arc length integral is

$$L = \int_0^{2\pi} \sqrt{(-\sin t)^2 + (\cos t)^2 + (-2 \sin t)^2} dt = \int_0^{2\pi} \sqrt{1 + 4 \sin^2 t} dt.$$

Since $|\cos t| \leq 1$, we get

$$L \leq \int_0^{2\pi} \sqrt{5} dt = 2\pi\sqrt{5} \approx 14.05 < 15.$$

1.3 Unit Tangent and Curvature

1. Let $\mathcal{C}$ be the curve $x = \frac{t^3}{3} - 2t$, $y = \frac{t^3}{3} + 2t$, $z = \sqrt{2}t^2$ for $t \in \mathbb{R}$.

(a) Find the arclength of the curve from $t = 0$ to $t = 1$.

We have $\mathbf{r}'(t) = (t^2 - 2)\mathbf{i} + (t^2 + 2)\mathbf{j} + 2\sqrt{2}t\mathbf{k}$, so that

$$\begin{aligned} |\mathbf{r}'(t)|^2 &= (t^2 - 2)^2 + (t^2 + 2)^2 + 8t^2 = t^4 - 4t^2 + 4 + t^4 + 4t^2 + 4 + 8t^2 \\ &= 2t^4 + 8t^2 + 8 = 2(t^2 + 2)^2, \\ |\mathbf{r}'(t)| &= \sqrt{2}(t^2 + 2). \end{aligned}$$

The arclength is

$$\int_0^1 \sqrt{2}(t^2 + 2) dt = \sqrt{2} \left(\frac{t^3}{3} + 2t \right) \Big|_0^1 = \frac{7\sqrt{2}}{3}.$$

(b) Find the curvature at the point where $t = 0$.

We have $\mathbf{r}'(t) = (t^2 - 2)\mathbf{i} + (t^2 + 2)\mathbf{j} + 2\sqrt{2}t\mathbf{k}$ as above, so that $\mathbf{r}''(t) = 2t\mathbf{i} + 2t\mathbf{j} + 2\sqrt{2}\mathbf{k}$.

At $t = 0$, we have $|\mathbf{r}'| = 2\sqrt{2}$ (from part (a)), and

$$\mathbf{r}' \times \mathbf{r}'' = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & 2 & 0 \\ 0 & 0 & 2\sqrt{2} \end{vmatrix} = 4\sqrt{2}\mathbf{i} + 4\sqrt{2}\mathbf{j},$$

$$|\mathbf{r}' \times \mathbf{r}''| = \sqrt{(4\sqrt{2})^2 + (4\sqrt{2})^2} = \sqrt{32 + 32} = \sqrt{64} = 8.$$

For the curvature, we have

$$\kappa = \frac{|\mathbf{r}' \times \mathbf{r}''|}{|\mathbf{r}'|^3} = \frac{8}{(2\sqrt{2})^3} = \frac{1}{\sqrt{8}}.$$

2. Let $\mathbf{r}(t) = t\mathbf{i} + \frac{t^2}{2}\mathbf{j}$, $t \in \mathbb{R}$ (a parabola in $\mathbb{R}^2$), and let $\mathbf{T}(t)$ be the unit tangent vector to $\mathbf{r}(t)$. Suppose that $\mathbf{r}(u)$, $u \in I$ (for some interval I), is a smooth reparametrization of the parabola such that

$$\left| \frac{d\mathbf{T}}{du} \right| = 1.$$

Find $\frac{du}{dt}$ as a function of t . Prove that the interval I is finite, and find the length of it.

We have $\mathbf{r}(t) = (t, t^2/2)$, and $\mathbf{r}'(t) = (1, t)$. Therefore,

$$\mathbf{T}(t) = \left\langle \frac{1}{\sqrt{1+t^2}}, \frac{t}{\sqrt{1+t^2}} \right\rangle, \quad d\mathbf{T}/dt = \left\langle \frac{-t}{(1+t^2)^{3/2}}, \frac{1}{(1+t^2)^{3/2}} \right\rangle.$$

By a short calculation (you should do it!), we obtain $|d\mathbf{T}/dt| = \frac{1}{1+t^2}$. Applying the Chain Rule, we get

$$\left| \frac{d\mathbf{T}}{du} \right| = \left| \frac{d\mathbf{T}}{dt} \frac{dt}{du} \right| = 1$$

Therefore

$$\left| \frac{dt}{du} \right| = \frac{1}{\left| \frac{d\mathbf{T}}{dt} \right|} = \frac{1}{1/(1+t^2)},$$

so that $\left| \frac{du}{dt} \right| = \left| \frac{d\mathbf{T}}{dt} \right| = \frac{1}{1+t^2}$. This is always positive, so $\frac{du}{dt} = \frac{1}{1+t^2}$.

To find the length of the interval I , we note that I is the image of $t \mapsto u(t)$ for $t \in \mathbb{R}$. We have that $\frac{du}{dt} = \frac{1}{1+t^2}$, and so we can solve for $u = \arctan(t) + c$ for some constant c . The range of $\arctan(t)$ for $t \in \mathbb{R}$ is the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$, so that u has range $I = (-\frac{\pi}{2} + c, \frac{\pi}{2} + c)$, a finite interval of length π .

3. Let $\mathbf{r} : \mathbb{R} \rightarrow \mathbb{R}^3$ be given by

$$\mathbf{r}(t) = \cos(t)\mathbf{i} + \sin(t)\mathbf{j} + f(t)\mathbf{k},$$

where $f : \mathbb{R} \rightarrow \mathbb{R}$ satisfies $f(t) = f(t+2\pi)$. Assume that f and all its derivatives are continuous. Thus $\mathbf{r}(t)$ parametrizes a closed smooth curve that lies on the cylinder $x^2 + y^2 = 1$, with each t -interval of length 2π covering the entire curve. Prove that there is a point $\mathbf{r}(t_0)$ on the curve where $\kappa \leq 1$.

With $\mathbf{r}(t)$ as above, we have

$$\mathbf{r}'(t) = -\sin(t)\mathbf{i} + \cos(t)\mathbf{j} + f'(t)\mathbf{k}, \quad \mathbf{r}''(t) = -\cos(t)\mathbf{i} - \sin(t)\mathbf{j} + f''(t)\mathbf{k},$$

so that

$$|\mathbf{r}'|^2 = \sin^2 t + \cos^2 t + (f'(t))^2 = 1 + (f'(t))^2,$$

$$\begin{aligned}\mathbf{r}' \times \mathbf{r}'' &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\sin t & \cos t & f'(t) \\ -\cos t & -\sin t & f''(t) \end{vmatrix} \\ &= (\cos(t)f'' + \sin(t)f')\mathbf{i} + (\sin(t)f'' - \cos(t)f')\mathbf{j} + \mathbf{k},\end{aligned}$$

$$\begin{aligned}|\mathbf{r}' \times \mathbf{r}''|^2 &= (\cos(t)f'' + \sin(t)f')^2 + (\sin(t)f'' - \cos(t)f')^2 + 1 \\ &= (f'')^2 + (f')^2 + 1.\end{aligned}$$

This gives us a formula for curvature,

$$\kappa = \frac{((f'')^2 + (f')^2 + 1)^{1/2}}{(1 + (f')^2)^{3/2}}.$$

We need to prove that there is a point on the curve where $\kappa \leq 1$. This is equivalent to $\kappa^2 \leq 1$, or

$$(f'')^2 + (f')^2 + 1 \leq (1 + (f')^2)^3 = 1 + 3(f')^2 + 3(f')^4 + (f')^6.$$

If we can find a point t_0 where $f''(t_0) = 0$, then the inequality holds at that point. To find such t_0 , we observe that since f is periodic with $f(t + 2\pi) = f(t)$, we also have that f' is periodic with $f'(t) = f'(t + 2\pi)$. In particular, $f'(0) = f'(2\pi)$. By the Mean Value Theorem, there is a $t_0 \in (0, 2\pi)$ such that $f''(t_0) = 0$. That gives us the point we want.

4. Let $\mathbf{r}(t)$ be a smooth vector-valued function, and suppose that there is a point P that lies on every normal plane to $\mathbf{r}(t)$. Prove that the curve lies on the surface of a sphere.

We may assume that $P = (0, 0, 0)$. The equation of the normal plane to the curve at $\mathbf{r}(t_0) = (x_0, y_0, z_0)$ is

$$\mathbf{r}'(t_0) \cdot (x - x_0, y - y_0, z - z_0) = 0.$$

If P lies on every such plane, it means that for all t_0 ,

$$\mathbf{r}'(t_0) \cdot \mathbf{r}(t_0) = 0.$$

But this characterizes curves that lie on the surface of a sphere $|\mathbf{r}(t)| = c$, as proved in Worked Problems, Section 1.1.

1.4 The Principal Normal and The Osculating Plane

1. Let

$$\mathbf{r}(t) = \langle \sqrt{2}t^{3/2}, (t+2)^{3/2}, (t-2)^{3/2} \rangle, \text{ for } t \in (2, \infty).$$

Find the unit tangent vector $\mathbf{T}(t)$, the principal normal vector $\mathbf{N}(t)$, and the curvature $\kappa(t)$. Find the limit of $\mathbf{N}(t)$ as $t \rightarrow 2^+$. (No need for an $\epsilon - \delta$ proof here, it suffices to use the general rules for limits.)

We calculate $\mathbf{T}$ and $\mathbf{N}$ as follows:

$$\begin{aligned} \mathbf{r}'(t) &= \left\langle \frac{3\sqrt{2}}{2}t^{1/2}, \frac{3}{2}(t+2)^{1/2}, \frac{3}{2}(t-2)^{1/2} \right\rangle, \\ |\mathbf{r}'(t)| &= \frac{3}{2} \left(2t + (t+2) + (t-2) \right)^{1/2} = \frac{3}{2} \sqrt{4t} = 3\sqrt{t}, \\ \mathbf{T}(t) &= \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \frac{3}{2 \cdot 3\sqrt{t}} \left\langle \sqrt{2t}, \sqrt{t+2}, \sqrt{t-2} \right\rangle = \frac{1}{2} \left\langle \sqrt{2}, \sqrt{1+\frac{2}{t}}, \sqrt{1-\frac{2}{t}} \right\rangle, \\ \mathbf{T}'(t) &= \frac{1}{2} \left\langle 0, \frac{1}{2\sqrt{1+\frac{2}{t}}} \cdot \frac{-2}{t^2}, \frac{1}{2\sqrt{1-\frac{2}{t}}} \cdot \frac{2}{t^2} \right\rangle = \frac{1}{2t^2} \left\langle 0, -\sqrt{\frac{t}{t+2}}, \sqrt{\frac{t}{t-2}} \right\rangle, \\ |\mathbf{T}'(t)| &= \frac{1}{2t^2} \left(\frac{t}{t+2} + \frac{t}{t-2} \right)^{1/2} = \frac{1}{2t^2} \left(\frac{t^2 - 2t + t^2 + 2t}{t^2 - 4} \right)^{1/2} \\ &= \frac{1}{2t^2} \left(\frac{2t^2}{t^2 - 4} \right)^{1/2} = \frac{1}{t\sqrt{2(t^2 - 4)}}, \\ \mathbf{N}(t) &= \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|} = \frac{t\sqrt{2(t^2 - 4)}}{2t^2} \cdot \left\langle 0, \sqrt{\frac{t}{t+2}}, -\sqrt{\frac{t}{t-2}} \right\rangle \\ &= \sqrt{\frac{(t+2)(t-2)}{2t^2}} \left\langle 0, -\sqrt{\frac{t}{t+2}}, \sqrt{\frac{t}{t-2}} \right\rangle = \left\langle 0, -\sqrt{\frac{t-2}{2t}}, \sqrt{\frac{t+2}{2t}} \right\rangle. \end{aligned}$$

From the last formula, we have

$$\lim_{t \rightarrow 2^+} \mathbf{N}(t) = (0, 0, 1).$$

With the work done so far, the easiest way to compute the curvature is

$$\kappa(t) = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} = \frac{1}{t\sqrt{2(t^2 - 4)}} \cdot \frac{1}{3\sqrt{t}} = \frac{1}{3t^{3/2}\sqrt{2(t^2 - 4)}}.$$

2. Let $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}$ for $t \in \mathbb{R}$.

(a) Find the equation of the osculating plane at the point $t = 0$.

We have

$$\mathbf{r}' = \mathbf{i} + 2t\mathbf{j} + 3t^2\mathbf{k}, \quad \mathbf{r}'' = 2\mathbf{j} + 6t\mathbf{k}.$$

At $t = 0$, we have $\mathbf{r}(0) = \mathbf{0}$, $\mathbf{r}'(0) = \mathbf{i}$, $\mathbf{r}''(0) = 2\mathbf{j}$. The osculating plane is parallel to both $\mathbf{r}'(0)$ and $\mathbf{r}''(0)$, therefore to $\mathbf{i}$ and $\mathbf{j}$. It also passes through $(0, 0, 0)$. Therefore the osculating plane is the plane $z = 0$.

(b) For $s, t \in \mathbb{R}$ such that $s, t, 0$ are all distinct, find the unit vector $\mathbf{n}(t, s)$ normal to the plane that passes through the points $\mathbf{r}(t)$, $\mathbf{r}(s)$, $\mathbf{r}(0)$. (There are two such vectors, pointing in opposite directions; choose the one that has a positive $\mathbf{k}$ -component.) Prove that

$$\lim_{s, t \rightarrow 0, s \neq t} \mathbf{n}(t, s)$$

is perpendicular to the osculating plane found in (a).

The vector $\mathbf{n}$ is perpendicular to $t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}$ and $s\mathbf{i} + s^2\mathbf{j} + s^3\mathbf{k}$, therefore to $\mathbf{i} + t\mathbf{j} + t^2\mathbf{k}$ and $\mathbf{i} + s\mathbf{j} + s^2\mathbf{k}$. It must therefore be parallel to their cross product:

$$\begin{aligned} \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & t & t^2 \\ 1 & s & s^2 \end{vmatrix} &= (ts^2 - st^2)\mathbf{i} - (s^2 - t^2)\mathbf{j} + (s - t)\mathbf{k} \\ &= (s - t)\left(st\mathbf{i} - (s + t)\mathbf{j} + \mathbf{k}\right), \end{aligned}$$

so that

$$\mathbf{n}(s, t) = \frac{1}{g(s, t)} \left(st\mathbf{i} - (s + t)\mathbf{j} + \mathbf{k} \right), \text{ where } g(s, t) = \sqrt{(st)^2 + (s + t)^2 + 1}.$$

As $(s, t) \rightarrow (0, 0)$, we have $g(s, t) \rightarrow 1$, $st \rightarrow 0$, and $s + t \rightarrow 0$, so that $\mathbf{n}(s, t) \rightarrow \mathbf{k}$. This is perpendicular to the osculating plane from (a) as claimed.

3. (12 marks) Let $\mathbf{r}(t) = t\mathbf{i} + at^2\mathbf{j}$ for some $a > 0$.

(a) Find the center Q and radius ρ of the osculating circle at the point $(0, 0)$.

We have

$$\begin{aligned} \mathbf{r}' &= \mathbf{i} + 2at\mathbf{j}, \\ |\mathbf{r}'| &= \sqrt{1 + 4a^2t^2}, \\ \mathbf{r}'' &= 2a\mathbf{j}, \end{aligned}$$

so that

$$\mathbf{r}' \times \mathbf{r}'' = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2at & 0 \\ 0 & 2a & 0 \end{vmatrix} = 2a\mathbf{k},$$

$$\kappa(t) = \frac{|\mathbf{r}' \times \mathbf{r}''|}{|\mathbf{r}'|^3} = \frac{2a}{(1 + 4a^2t^2)^{3/2}}.$$

At $t = 0$, we have $\kappa = 2a$, so that $\rho = (2a)^{-1}$. The center Q of the osculating circle lies on the positive y half-axis at distance ρ from $(0, 0)$, so that $Q = (0, (2a)^{-1})$.

(Alternatively, we could solve this using the calculations from Worked Problem 3. You should think about how to do this!)

(b) For $t > 0$, find the center $Q(t)$ and radius $R(t)$ of the circle in the xy -plane that passes through the points $\mathbf{r}(t)$, $\mathbf{r}(-t)$, $\mathbf{r}(0)$. Prove that $Q(t) \rightarrow Q$ and $R(t) \rightarrow \rho$ as $t \rightarrow 0$, where Q and ρ are the answers in (a).)

By symmetry, $Q(t)$ lies on the positive y half-axis, so that $Q(t) = (0, b)$ for some $b = b(t)$. The distance from $Q(t)$ to all of the points $\mathbf{r}(t)$, $\mathbf{r}(-t)$, $\mathbf{r}(0)$ has to be the same (and equal to $R(t)$). Let $O = (0, 0)$, $P = (t, at^2)$, and $P' = (-t, at^2)$, then

$$\begin{aligned} |OQ(t)|^2 &= b^2, \\ |PQ(t)|^2 &= |P'Q(t)|^2 = (0 - t)^2 + (b - at^2)^2 \\ &= t^2 + b^2 - 2abt^2 + a^2t^4. \end{aligned}$$

For these to be equal, we solve for b as a function of t with :

$$\begin{aligned} b^2 &= t^2 + b^2 - 2abt^2 + a^2t^4 \\ 0 &= t^2 - 2abt^2 + a^2t^4 = t^2(1 - 2ab + a^2t^2). \end{aligned}$$

For $t \neq 0$, this implies that $1 - 2ab + a^2t^2 = 0$, so that $b = \frac{a^2t^2+1}{2a}$ and

$$Q(t) = \left(0, \frac{a^2t^2+1}{2a}\right), \quad R(t) = \frac{a^2t^2+1}{2a}.$$

As $t \rightarrow 0$, we have $Q(t) \rightarrow (0, (2a)^{-1}) = Q$ and $R(t) \rightarrow (2a)^{-1} = \rho$.

4. Let $\mathbf{r}$ be a parametrized curve in $\mathbb{R}^3$ such that $\mathbf{r}$ and its derivatives of all orders are continuous, $\mathbf{r}'(t) \neq 0$ for t in some interval I , and $|\mathbf{r}(t)| = r$ for all $t \in I$ (i.e. the curve lies on the sphere S of radius r centered at the origin O). Let $P = \mathbf{r}(t)$ be a point on the curve.

(a) Prove that $\rho = r \cos \theta$, where θ is the angle between $\vec{PO}$ and $\mathbf{N}$. Use this to prove that $\kappa \geq 1/r$.

(Hint: Use arclength parametrization. We have $\mathbf{r}(s) \cdot \frac{d\mathbf{r}}{ds} = 0$ (why?). If we differentiate this in s and use that $|\frac{d\mathbf{r}}{ds}| = 1$, what does it say about the dot product $\vec{PO} \cdot \mathbf{N}$?)

Let s be the arclength parameter. Since $|\mathbf{r}| = r$ is constant, we know from class that $0 = \mathbf{r}(s) \cdot \mathbf{r}'(s)$. Differentiating this in s , and then using that $|\mathbf{r}'(s)| = 1$, we get

$$0 = \mathbf{r}'(s) \cdot \mathbf{r}'(s) + \mathbf{r}(s) \cdot \mathbf{r}''(s) = 1 + \mathbf{r}(s) \cdot \mathbf{r}''(s). \quad (1)$$

Hence $\mathbf{r}(s) \cdot \mathbf{r}''(s) = -1$. But $\mathbf{r}'(s) = \mathbf{T}$, so $\mathbf{r}''(s) = \frac{d\mathbf{T}}{ds} = \kappa\mathbf{N}$, therefore we get from (1) that

$$\mathbf{r} \cdot \kappa\mathbf{N} = -1. \quad (2)$$

Therefore $\kappa \neq 0$. Let θ be the angle between $\vec{PO} = -\mathbf{r}$ and $\mathbf{N}$, then, on one hand, from (2) we have

$$\vec{PO} \cdot \mathbf{N} = -\mathbf{r} \cdot \mathbf{N} = \frac{1}{\kappa} = \rho.$$

On the other hand, $\vec{PO} \cdot \mathbf{N} = |\vec{PO}| |\mathbf{N}| \cos \theta = r \cos \theta$, so that $\rho = r \cos \theta$. (This in particular implies that $\cos \theta > 0$.) Since $0 < \cos \theta \leq 1$, we get that

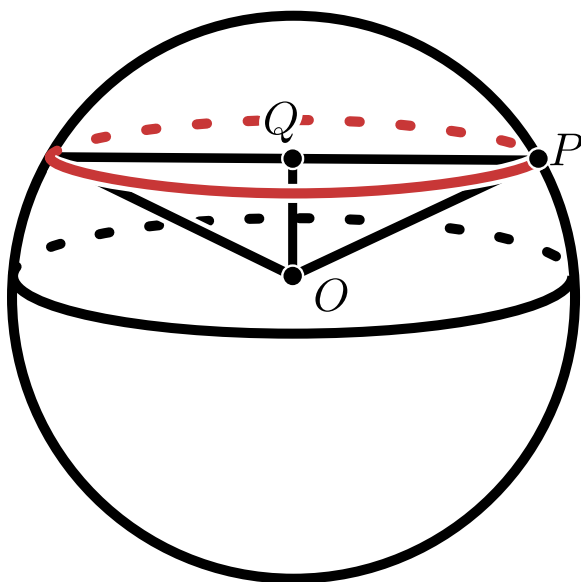
$$\kappa = \frac{1}{r \cos \theta} \geq \frac{1}{r}.$$

(b) Prove that for any point $P = \mathbf{r}(t)$ on the curve, the osculating circle for the curve at that point lies on the sphere S .

The osculating circle at P is the unique circle that lies in the osculating plane at P , has radius $\rho = 1/\kappa$, and its center Q satisfies $\vec{PQ} = \rho\mathbf{N}$. If Q coincides with the origin O , then the osculating circle is centered at O and has radius $\rho = |\vec{OP}| = r$, so it lies on the sphere. Otherwise, we claim that the vector $\vec{OQ}$ is perpendicular to both $\mathbf{N}$ and $\mathbf{T}$, therefore to the osculating plane. To prove this:

- Consider the triangle OPQ . We have $|\vec{OP}| = r$, $|\vec{PQ}| = \rho$, and we proved in (a) that $\rho = r \cos \theta$, where θ is the angle at P . Therefore OPQ is a right triangle with a right angle at Q , so that $\vec{OQ} \perp \vec{PQ}$. Since $\vec{PQ} = \rho\mathbf{N}$, $\vec{OQ}$ is perpendicular to $\mathbf{N}$.
- We also have $\vec{OQ} \cdot \mathbf{T} = (\vec{OP} + \vec{PQ}) \cdot \mathbf{T} = (\mathbf{r} + \rho\mathbf{N}) \cdot \mathbf{T} = \mathbf{r} \cdot \mathbf{T} + \rho\mathbf{N} \cdot \mathbf{T} = 0 + 0 = 0$. At the last step, we used that $\mathbf{r} \cdot \mathbf{r}' = 0$, so that $\mathbf{r} \cdot \mathbf{T} = 0$.

It follows that the osculating circle is centered at Q , lies in the plane perpendicular to $\vec{OQ}$, and passes through P . Therefore the entire circle lies on the sphere.



5. Let $\mathbf{r}$ be a closed, simple parametrized curve in $\mathbb{R}^3$. We use a “wrap-around” parametrization: if $\mathbf{r}(t)$ for $t \in [0, a]$ parametrizes one full loop around C returning to the starting point, we extend $\mathbf{r}$ to a periodic function $\mathbf{r} : \mathbb{R} \rightarrow \mathbb{R}^3$ so that $\mathbf{r}(t + a) = \mathbf{r}(t)$ for all t . (This is similar to parametrizing the unit circle by $\mathbf{r}(t) = \cos(t)\mathbf{i} + \sin(t)\mathbf{j}$, with period 2π .)

Assume that $\mathbf{r}$ and its derivatives of all orders are continuous, $\mathbf{r}'(t) \neq 0$, and $|\mathbf{r}(t)| \leq 1$ for all t (i.e. the curve lies in the closed ball of radius 1 centered at the origin O). Prove that there is a point $\mathbf{r}(t_0)$ on the curve where $\kappa(t_0) \geq 1$.

(Hint: Let $f(t) = |\mathbf{r}(t)|^2$, then $f(t)$ attains its maximum at some point $t = t_0$. What does this say about $f'(t_0)$ and $f''(t_0)$? It may be helpful to use arclength parametrization, starting from $t = t_0$ so that $s(t_0) = 0$.)

Let s be the arclength parameter, measured from $\mathbf{r}(t_0)$ so that $s(t_0) = 0$. Let $f(s) = |\mathbf{r}(s)|^2$, then

$$\begin{aligned} f'(s) &= 2\mathbf{r}(s) \cdot \mathbf{r}'(s) = 2\mathbf{r}(s) \cdot \mathbf{T}(s), \\ f''(s) &= 2\mathbf{r}'(s) \cdot \mathbf{T}(s) + 2\mathbf{r}(s) \cdot \mathbf{T}'(s) = 2\mathbf{T}(s) \cdot \mathbf{T}(s) + 2\mathbf{r}(s) \cdot \kappa\mathbf{N}(s) \\ &= 2 + 2\kappa\mathbf{r}(s) \cdot \mathbf{N}(s) \end{aligned}$$

Since f has a local maximum at $s = 0$, we have $f'(0) = 0$ and $f''(0) \leq 0$. This means that

$$\begin{aligned} \mathbf{r}(0) \cdot \mathbf{T}(0) &= 0, \\ 2 + 2\kappa\mathbf{r}(0) \cdot \mathbf{N}(0) &\leq 0. \end{aligned}$$

Since $|\mathbf{N}(0)| = 1$, we have $\mathbf{r}(0) \cdot \mathbf{N}(0) = |\mathbf{r}(0)| \cos \theta$, where θ is the angle between $\mathbf{r}(0)$ and $\mathbf{N}(0)$. Therefore

$$2 + 2\kappa|\mathbf{r}(0)| \cos \theta \leq 0,$$

so that $\kappa|\mathbf{r}(0)|\cos\theta \leq -1$. Since $|\mathbf{r}(0)| \leq 1$ and $|\cos\theta| \leq 1$, it follows that $\kappa(0) \geq 1$.

1.5 Torsion

1. Let C be the curve $x = e^{2t}$, $y = 2e^t$, $z = t$ for $t \in \mathbb{R}$. Find all points on the curve where the osculating plane is parallel to the plane $x - 6y + 18z = 0$. (If there are no such points, prove it.)

We have $\mathbf{r}'(t) = 2e^{2t}\mathbf{i} + 2e^t\mathbf{j} + \mathbf{k}$, so that $\mathbf{r}''(t) = 4e^{2t}\mathbf{i} + 2e^t\mathbf{j}$. The vector normal to the osculating plane is $\mathbf{B}$, which is parallel to

$$\mathbf{r}' \times \mathbf{r}'' = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2e^{2t} & 2e^t & 1 \\ 4e^{2t} & 2e^t & 0 \end{vmatrix} = -2e^t\mathbf{i} + 4e^{2t}\mathbf{j} - 4e^{3t}\mathbf{k} = -2e^t(\mathbf{i} - 2e^t\mathbf{j} + 2e^{2t}\mathbf{k}).$$

For the osculating plane to be parallel to $x - 6y + 18z = 0$, this vector needs to be parallel to $\mathbf{i} - 6\mathbf{j} + 18\mathbf{k}$, so that we need to have

$$2e^t = 6, \quad 2e^{2t} = 18.$$

From the first equation we get $e^t = 3$, $t = \ln 3$. Then $2e^{2t} = 2 \cdot 9 = 18$, so that the second equation is also satisfied. The point where $t = \ln 3$ has coordinates $(x, y, z) = (9, 6, \ln 3)$.

2. Prove that if $\mathbf{r}(t)$ is a smooth curve such that $\kappa \neq 0$ and $\mathbf{r} + \kappa^{-1}\mathbf{N}$ (the center of the osculating circle) is constant, then $\mathbf{r}(t)$ is a circle. (Hint: By the uniqueness theorem for curves, it suffices to prove that κ is constant and the torsion τ is 0.)

We use the arclength parametrization. By the Frenet-Serret formulas, we have

$$\frac{d\mathbf{N}}{ds} = \tau\mathbf{B} - \kappa\mathbf{T}.$$

Therefore

$$\begin{aligned} \frac{d}{ds}(\mathbf{r} + \kappa^{-1}\mathbf{N}) &= \frac{d\mathbf{r}}{ds} + \left(\frac{d}{ds}\kappa^{-1}\right)\mathbf{N} + \kappa^{-1}\frac{d\mathbf{N}}{ds} \\ &= \mathbf{T} + \left(\frac{d}{ds}\kappa^{-1}\right)\mathbf{N} + \kappa^{-1}(-\kappa\mathbf{T} + \tau\mathbf{B}) \\ &= \left(\frac{d}{ds}\kappa^{-1}\right)\mathbf{N} + \kappa^{-1}\tau\mathbf{B}. \end{aligned}$$

By assumption, this is equal to $\mathbf{0}$ for all s . Since $\mathbf{N}$ and $\mathbf{B}$ are linearly independent, it follows that both coefficients must be zero. Thus, $\frac{d}{ds}\kappa^{-1} \equiv 0$, so that κ^{-1} is constant and therefore κ is constant. Also, $\kappa^{-1}\tau = 0$, so that $\tau \equiv 0$. By the uniqueness theorem for curves, $\mathbf{r}(t)$ is a circle.

3. Let C be a smooth curve parametrized by $\mathbf{r}(t) : \mathbb{R} \rightarrow \mathbb{R}^3$. Assume that $(0,0,0)$ belongs to the osculating plane at some point $\mathbf{r}(t_0)$. Prove that at $t = t_0$,

$$(\mathbf{r} \times \mathbf{r}') \cdot \mathbf{r}'' = 0.$$

The normal vector to the osculating plane is $\mathbf{B} = \frac{\mathbf{r}' \times \mathbf{r}''}{|\mathbf{r}' \times \mathbf{r}''|}$, parallel to $\mathbf{r}' \times \mathbf{r}''$. Therefore the equation of the osculating plane at $t = t_0$ is

$$(\langle x, y, z \rangle - \mathbf{r}(t_0)) \cdot (\mathbf{r}'(t_0) \times \mathbf{r}''(t_0)) = 0.$$

If $(x, y, z) = (0, 0, 0)$ satisfies this equation, we have

$$\mathbf{r}(t_0) \cdot (\mathbf{r}'(t_0) \times \mathbf{r}''(t_0)) = 0.$$

But this is the triple product:

$$\mathbf{r}(t) \cdot (\mathbf{r}'(t) \times \mathbf{r}''(t)) = \begin{vmatrix} x(t) & y(t) & z(t) \\ x'(t) & y'(t) & z'(t) \\ x''(t) & y''(t) & z''(t) \end{vmatrix} = (\mathbf{r}(t) \times \mathbf{r}'(t)) \cdot \mathbf{r}''(t),$$

so the last expression must be 0 at $t = t_0$.

1.6 Velocity and Acceleration

1. The position of an object is given by a smooth vector-valued function $\mathbf{r}(t)$. Let $\mathbf{v}(t) = \mathbf{r}'(t)$, and $\mathbf{a}(t) = \mathbf{r}''(t)$. Assume that $\mathbf{a}(t) = \mathbf{k} \times \mathbf{v}(t)$ for all t .

(a) Prove that the speed v is constant.

(b) Assuming that v is nonzero, prove that the angle $\theta(t)$ between $\mathbf{v}(t)$ and $\mathbf{k}$ is constant.

(a) We have

$$\frac{d}{dt}(\mathbf{v} \cdot \mathbf{v}) = \mathbf{a} \cdot \mathbf{v} + \mathbf{v} \cdot \mathbf{a} = 0,$$

since $\mathbf{a}$ is perpendicular to $\mathbf{v}$. Therefore $\mathbf{v} \cdot \mathbf{v} = |\mathbf{v}|^2 = v^2$ is constant, so that v is constant.

(b) Since $\mathbf{a}$ is perpendicular to $\mathbf{u}$, we have

$$\frac{d}{dt}(\mathbf{v} \cdot \mathbf{k}) = \mathbf{a} \cdot \mathbf{k} + \mathbf{v} \cdot \mathbf{0} = 0.$$

Therefore $\mathbf{v} \cdot \mathbf{k}$ is constant. We also have $|\mathbf{k}| = 1$, and $|\mathbf{v}| = v$ is constant by (a) and nonzero by assumption. It follows that

$$\cos \theta = \frac{\mathbf{v} \cdot \mathbf{k}}{v}$$

is constant, as required.

2. Assume that $\mathbf{a}(t) = c\mathbf{r}(t)$ for some constant c . (This describes the motion of an object under a central force.)

(a) Prove that $\mathbf{w} := \mathbf{r} \times \mathbf{v}$ is constant.

(b) Assuming that $\mathbf{w} \neq \mathbf{0}$, prove that the curve $\mathbf{r}(t)$ lies in the plane passing through the origin and perpendicular to $\mathbf{w}$.

(a) We first prove that $\mathbf{w}$ is constant:

$$\mathbf{w}'(t) = \frac{d}{dt}(\mathbf{r} \times \mathbf{v}) = \mathbf{v} \times \mathbf{v} + \mathbf{r} \times \mathbf{a} = \mathbf{0} + \mathbf{r} \times (c\mathbf{r}) = \mathbf{0}.$$

(b) It follows that $\mathbf{r}$ is always perpendicular to the fixed vector $\mathbf{w}$, which we assume to be nonzero. Therefore the curve lies in the plane $\mathbf{r} \cdot \mathbf{w} = 0$.

3. Suppose that a curve $\mathcal{C}$ in $\mathbb{R}^3$ has a smooth parametrization $\mathbf{r}(t)$ for all $t \in \mathbb{R}$, the curvature κ is a positive constant, and the torsion τ is zero. Prove, without using the Fundamental Theorem of Calculus for curves, that $\mathcal{C}$ is a circle. (Hint: how would you find the center of the circle? You may also want to use the Frenet-Serret formulas.)

Reverse-engineering from the conclusion, if $\mathcal{C}$ were a circle, then $\mathcal{C}$ would be its own osculating circle at all points. In particular, the center of the osculating circle should remain fixed as we travel the curve.

Now, we do not actually know that $\mathcal{C}$ is a circle – we need to prove this. As a first step, we will try to prove that the center Q of the osculating circle remains at a fixed point. Let $\mathbf{w} = \vec{OQ} = \mathbf{r} + \rho\mathbf{N}$, where $\rho = 1/\kappa$. Since κ is constant, so is ρ . We want Q to be independent of t . To prove this, we check the derivative with respect to s (this is easier here than differentiating in t):

$$\begin{aligned} \frac{d}{ds}(\mathbf{r} + \rho\mathbf{N}) &= \mathbf{T} + \frac{d}{ds}(\rho\mathbf{N}) \\ &= \mathbf{T} + \kappa^{-1} \frac{d}{ds}\mathbf{N} \\ &= \mathbf{T} + \kappa^{-1}(-\kappa\mathbf{T}) = \mathbf{0}, \end{aligned}$$

where at the last step we used Frenet-Serret. So Q is constant as claimed.

Based on this, we need to prove that $\mathcal{C}$ is a circle centered at Q . In Section 1.5 of this manual, this was done using the Fundamental Theorem of Calculus for curves. Here, you are asked to find an alternative proof that does not rely on the theorem. It suffices to prove the following two facts:

- The distance from Q to $\mathbf{r}(t)$ is constant: this follows from

$$|\mathbf{r} - \vec{OQ}| = |\mathbf{r} - (\mathbf{r} + \rho\mathbf{N})| = |\rho\mathbf{N}| = \rho.$$

- The curve lies in a fixed 2-dimensional plane through Q . Indeed, we claim that the binormal vector $\mathbf{B}$ is constant, and that the curve lies in the plane $(\mathbf{r} - \vec{OQ}) \cdot \mathbf{B} = 0$. To see this, we first verify that $\mathbf{B}$ is constant:

$$\frac{d}{ds}\mathbf{B} = -\tau\mathbf{N} = \mathbf{0},$$

using Frenet-Serret again. Finally, we check the equation of the plane:

$$(\mathbf{r} - \vec{OQ}) \cdot \mathbf{B} = (-\rho\mathbf{N}) \cdot \mathbf{B} = 0.$$

The two bullet points together imply that the curve is a circle.

4. Let C be a curve on the sphere $\mathcal{S}$ of radius 1 centered at $\mathbf{0}$. Suppose that C has a smooth parametrization $\mathbf{r}(t)$ such that the velocity $\mathbf{r}'(t)$ is never zero and the acceleration $\mathbf{r}''(t)$ is always perpendicular to $\mathcal{S}$ at $\mathbf{r}(t)$. Prove that the curve is contained in a great circle of $\mathcal{S}$ (that is, a circle where $\mathcal{S}$ intersects a plane through $\mathbf{0}$).

Let $\mathbf{b} = \mathbf{r}' \times \mathbf{r}$. Then:

- $\mathbf{b}$ is constant since

$$\frac{d}{dt}(\mathbf{r}' \times \mathbf{r}) = \mathbf{r}'' \times \mathbf{r} + \mathbf{r}' \times \mathbf{r}' = \mathbf{r}'' \times \mathbf{r}.$$

Since $\mathbf{r}''$ is perpendicular to $\mathcal{S}$, it is parallel to the normal vector to $\mathcal{S}$, therefore parallel to $\mathbf{r}$. It follows that the last cross-product is zero, and so $\mathbf{b}$ is constant.

- $\mathbf{b}$ is nonzero: since $|\mathbf{r}| = 1$ for all t , $\mathbf{r}'$ is perpendicular to $\mathbf{r}$, so that $|\mathbf{b}| = |\mathbf{r}' \times \mathbf{r}| = |\mathbf{r}'||\mathbf{r}| = |\mathbf{r}'| \neq 0$ by assumption.
- $\mathbf{r} \cdot \mathbf{b} = 0$, since $\mathbf{b} = \mathbf{r}' \times \mathbf{r}$ is perpendicular to $\mathbf{r}$.

Therefore $\mathbf{r}$ lies in the plane through the origin perpendicular to $\mathbf{b}$, as claimed.

5. Let $\mathbf{r}(s) : \mathbb{R} \rightarrow \mathbb{R}^3$ be a smooth curve, parametrized by arc length. Assume that $\frac{d^2\mathbf{r}}{ds^2} = \mathbf{r} \times \mathbf{w}$ for all s , where $\mathbf{w}$ is a fixed nonzero vector. Prove that this is only possible when $\frac{d^2\mathbf{r}}{ds^2} = \mathbf{r} \times \mathbf{w} = \mathbf{0}$ for all s .

(Hint: suppose that there is a value of s where $\mathbf{r} \times \mathbf{w} \neq \mathbf{0}$. Prove first that there are scalar functions $\lambda(s), \mu(s)$ such that $\frac{d\mathbf{r}}{ds} = \lambda\mathbf{r} + \mu\mathbf{w}$ near that point. What does this imply about $\frac{d^2\mathbf{r}}{ds^2}$?)

As per hint, assume towards contradiction that $\mathbf{r} \times \mathbf{w} \neq \mathbf{0}$ for some s . By continuity, this is also true on some neighbourhood of that point. In particular, we may find a part of the curve where $\mathbf{r} \times \mathbf{w} \neq \mathbf{0}$.

We always have $|\frac{d\mathbf{r}}{ds}| = |\mathbf{T}| = 1$, so that $\frac{d^2\mathbf{r}}{ds^2}$ is perpendicular to $\frac{d\mathbf{r}}{ds}$. But we know that $\frac{d^2\mathbf{r}}{ds^2}$ is also perpendicular to $\mathbf{r}$ and $\mathbf{w}$, and these two vectors are nonzero and not parallel to each other. Therefore $\frac{d\mathbf{r}}{ds}$ must belong to the span of these two vectors, so that $\frac{d\mathbf{r}}{ds} = \lambda\mathbf{r} + \mu\mathbf{w}$ for some scalar functions $\lambda(s), \mu(s)$.

Next, differentiating this in s and using that $\mathbf{w}$ is constant, we get

$$\begin{aligned}\frac{d^2\mathbf{r}}{ds^2} &= \frac{d}{ds}(\lambda\mathbf{r} + \mu\mathbf{w}) \\ &= \frac{d\lambda}{ds}\mathbf{r} + \lambda\frac{d\mathbf{r}}{ds} + \frac{d\mu}{ds}\mathbf{w} \\ &= \frac{d\lambda}{ds}\mathbf{r} + \lambda(\lambda\mathbf{r} + \mu\mathbf{w}) + \frac{d\mu}{ds}\mathbf{w}.\end{aligned}$$

This is a linear combination of $\mathbf{r}$ and $\mathbf{w}$. But $\frac{d^2\mathbf{r}}{ds^2}$ was also supposed to be perpendicular to both of these vectors. Therefore it is also perpendicular to any linear combination of its vectors, including itself. Hence

$$\frac{d^2\mathbf{r}}{ds^2} = \mathbf{r} \times \mathbf{w} = \mathbf{0},$$

contradicting our assumption.

Chapter 2

Vector Fields

2.1 Introduction to Vector Fields

1. Find and sketch the field lines of the vector field $\mathbf{F}(x, y) = -x^3\mathbf{i} + y\mathbf{j}$ in $\mathbb{R}^2$. Indicate the flow direction in your sketch.

For a field line $\mathbf{r}(t)$, we have $\mathbf{r}'(t) = c\mathbf{F}(\mathbf{r}(t))$ with $c > 0$, so that

$$\frac{dx}{dt} = -cx^3, \quad \frac{dy}{dt} = cy.$$

If $x, y \neq 0$, we have $-\frac{dx}{x^3} = cdt = \frac{dy}{y}$. Integrating this, we get

$$\ln |y| = \frac{1}{2x^2} + C, \quad |y| = e^C e^{1/2x^2}, \quad y = C_1 e^{1/2x^2},$$

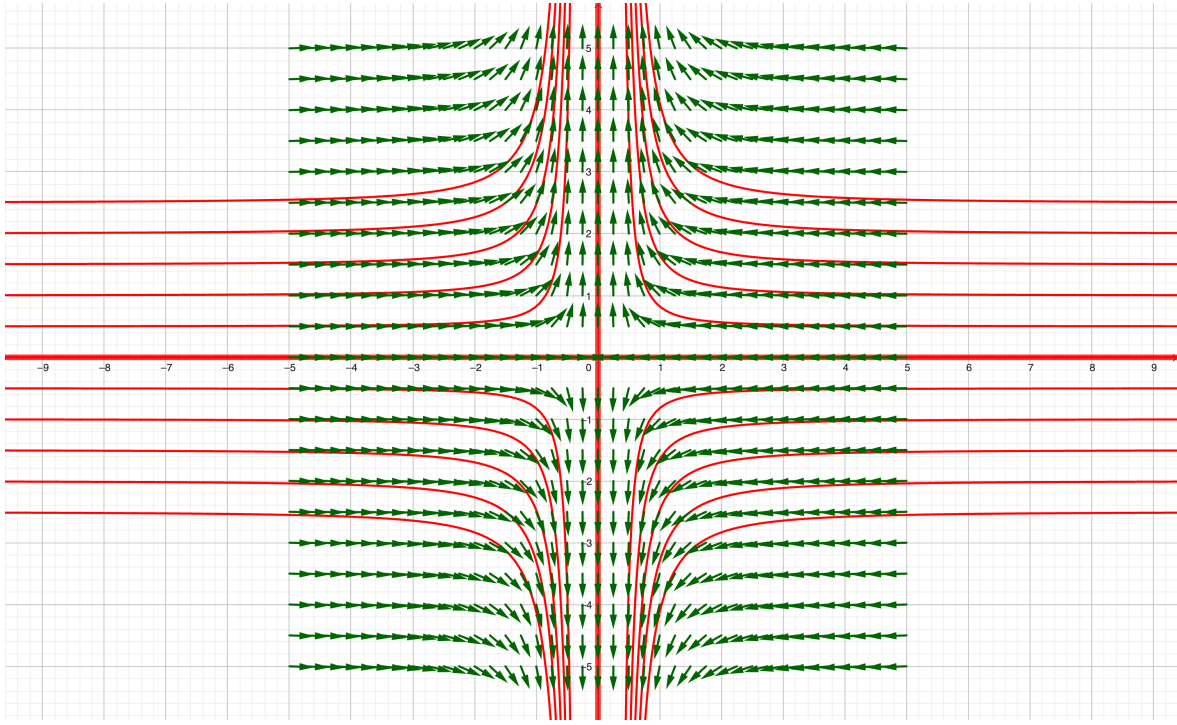
where $C_1 = \pm e^C$ is a constant. If $x = 0$, we get

$$\frac{dx}{dt} = 0, \quad \frac{dy}{dt} = cy,$$

which corresponds to the two half-lines on the y -axis, oriented away from the origin. Similarly, if $y = 0$, we get

$$\frac{dx}{dt} = -cx^3, \quad \frac{dy}{dt} = 0,$$

which corresponds to the two half-lines on the x -axis, oriented towards the origin. At $(0, 0)$, we have $\mathbf{F} = \mathbf{0}$, so that this is a stationary point with no field lines through it.



2. Let $f(x, y) = x^2 + 4y^4$. Find the field line of the vector field $\mathbf{F} = \nabla f$ which passes through the point $(2, 6)$.

We are looking for a field line that satisfies

$$(x'(t), y'(t)) = \nabla f(x, y) = (2x, 16y^4)$$

and that passes through $(2, 6)$. We will assume that $x(0) = 2, y(0) = 6$ (for a field line that passes through $(2, 6)$ at time t_0 , replace t by $t - t_0$). This means

$$\begin{aligned} x' &= 2x, \quad x(0) = 2, \\ y' &= 16y^3, \quad y(0) = 6. \end{aligned}$$

We solve for $x(t)$ and $y(t)$:

$$\begin{aligned} x(t) &= C_1 e^{2t}, \quad 2 = x(0) = C_1, \quad x = 2e^{2t}, \\ \frac{y'}{y^3} &= 16, \quad \int \frac{dy}{y^3} = \int 16dt, \quad -\frac{1}{2y^2} = 16t + C_2, \quad C_2 = -\frac{1}{2 \cdot 6^2} = -\frac{1}{72}, \\ y^2 &= -\left(2\left(16t - \frac{1}{72}\right)\right)^{-1} = \left(\frac{1}{36} - 32t\right)^{-1}, \\ y &= \left(\frac{1}{36} - 32t\right)^{-1/2} \end{aligned}$$

(we took the positive square root because $y(0) = 6 > 0$).

3. Find all field lines of the vector field $\mathbf{F} = \frac{1-x^2}{x}\mathbf{i} + y^2\mathbf{j}$, using the following procedure.

(a) Assuming that x , $1-x^2$, and y are all non-zero, solve the corresponding ordinary differential equation using the Method of Separation of Variables. (You may have to consider multiple cases.)

We have the differential equation

$$\frac{dy}{y^2} = \frac{xdx}{1-x^2} \quad (2.1.1)$$

If we integrate both sides, we get

$$-\frac{1}{y} + C = \int \frac{x}{1-x^2} dx$$

We substitute $u = x^2$ to get

$$\frac{1}{y} + C = \frac{1}{2} \int \frac{1}{u-1} du = \frac{1}{2} \ln |u-1| = \frac{1}{2} \ln |x^2-1|,$$

which gives us

$$\frac{1}{y} = \frac{1}{2} \ln |x^2-1| + C, \quad y = \frac{2}{\ln |x^2-1| + 2C},$$

where C is a constant. Alternatively, we can let $D = e^{2C}$, so that $2C = \ln D$. Then the field line equation can be rewritten as $y = \frac{2}{\ln(D|x^2-1|)}$, where D is a positive constant.

(b) What happens when $x = 0$, $1-x^2 = 0$, or $y = 0$? Are there any field lines corresponding to these cases?

- The vector field $\mathbf{F}$ is undefined at $x = 0$, so there are no field lines there.
- If $1-x^2 = 0$, then $\mathbf{F} = y^2\mathbf{j}$. This gives two vertical lines at $x = 1$ and $x = -1$, oriented upwards.
- If $y = 0$, we have $\mathbf{F} = \frac{1-x^2}{x}\mathbf{i}$. The corresponding field lines lie on the x -axis, with orientation depending on the sign of $\frac{1-x^2}{x}$.

(c) Draw a picture of the field lines you found. Make sure to indicate their orientation. (You can check that by evaluating $\mathbf{F}$ at sample points.) You should have exactly one field line through every point where $\mathbf{F} \neq \mathbf{0}$.

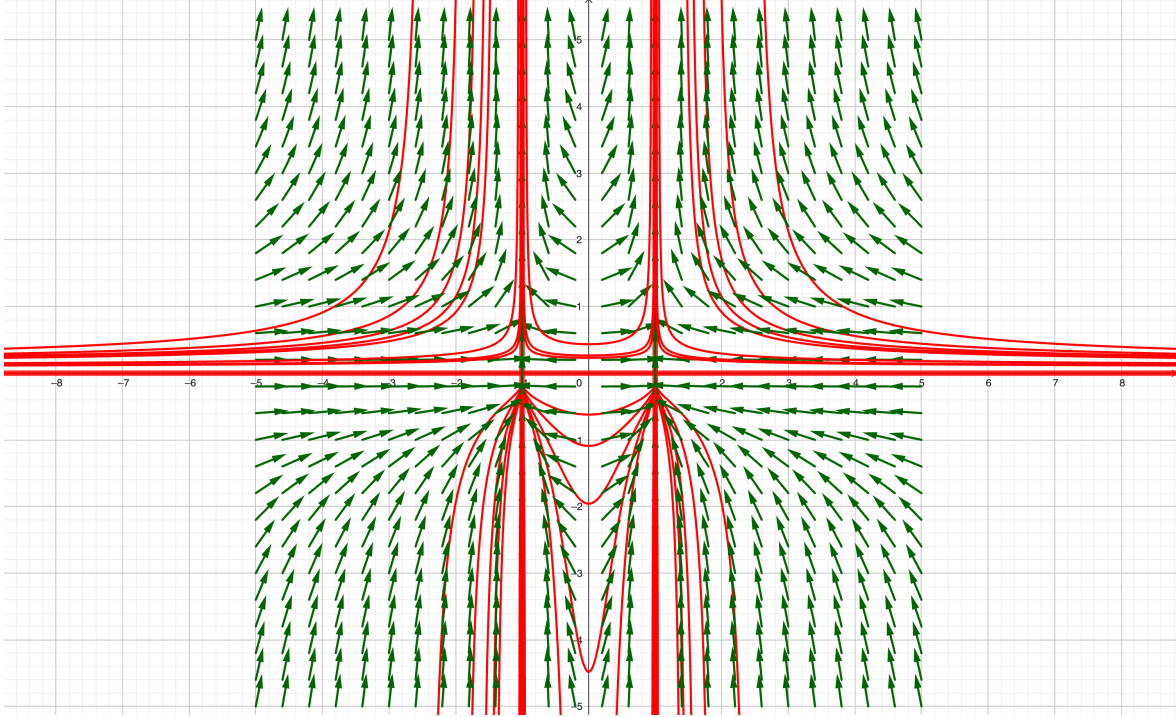


Figure 2.1: The field lines of $\mathbf{F}(x, y) = \frac{1-x^2}{x}\mathbf{i} + y^2\mathbf{j}$ overlain on the vector field. The solutions corresponding to $x = -1$, $x = 1$ and $y = 0$ are the heavy red lines.

2.2 Conservative Vector Fields

1. Let $\mathbf{F}(x, y, z) = e^y\mathbf{i} + (xe^y - \cos z)\mathbf{j} + y\sin z$ in $\mathbb{R}^3$. Prove that $\mathbf{F}$ conservative, and find a potential function f .

We look for a potential function such that $\mathbf{F} = \nabla f$:

$$\frac{\partial}{\partial x}f(x, y, z) = F_1 = e^y, \quad f(x, y, z) = xe^y + g(y, z),$$

$$\frac{\partial}{\partial y}f(x, y, z) = xe^y + \frac{\partial}{\partial y}g(y, z) = xe^y - \cos z,$$

$$g(y, z) = -y \cos z + h(z),$$

$$\frac{\partial}{\partial z}f(x, y, z) = y \sin z + h'(z) = F_3 = y \sin z, \quad h(z) = c.$$

The field is conservative, with $f(x, y, z) = xe^y - y \cos z + c$ for any constant c .

2. Consider the vector field

$$\mathbf{G} = \frac{F_1(x)}{F_3(z)}\mathbf{i} + \frac{F_2(y)}{F_3(z)}\mathbf{j} + \mathbf{k},$$

where F_1, F_2, F_3 are smooth functions on a domain $D \subset \mathbb{R}^3$ and F_3 is never zero on D . Does $\mathbf{G}$ have to be conservative on D ? If yes, prove it. If no, give a counterexample.

The answer is no. Let $F_1(x) = 1$, $F_2(y) = 1$, $F_3(z) = z$. Then

$$\mathbf{G} = G_1\mathbf{i} + G_2\mathbf{j} + G_3\mathbf{k} = \frac{1}{z}\mathbf{i} + \frac{1}{z}\mathbf{j} + \mathbf{k},$$

is defined on the domain $z > 0$, but $\mathbf{G}$ does not satisfy the necessary condition of Proposition 1.5 for a conservative vector field in $\mathbb{R}^3$. Namely, we have:

$$\frac{\partial G_1}{\partial z} = -\frac{1}{z^2} \neq 0 = \frac{\partial G_3}{\partial x},$$

contradicting Proposition 2.2.5 in Section 2.2.

3. Give an example of two smooth vector fields $\mathbf{F}$ and $\mathbf{G}$ on $\mathbb{R}^2$ such that $\mathbf{F}$ and $\mathbf{G}$ have the same field lines, $\mathbf{F}$ is conservative, and $\mathbf{G}$ is not conservative.

(Hint: Worked Problem 1 of this Section might be helpful in constructing a simple conservative field $\mathbf{F}$. Then, try modifying it to make a $\mathbf{G}$ that has the same field lines but is non-conservative.)

Take $\mathbf{F} = e^x\mathbf{i}$ and $\mathbf{G} = e^{x+y}\mathbf{i}$. For both fields, the vectors are always non-zero and pointing in the direction of $\mathbf{i}$, so the field lines are lines $y = C$, oriented in the direction of increasing x . The field $\mathbf{F}$ is conservative, as a special case of Practice Problem 1. On the other hand, $\mathbf{G}$ is not conservative, since $\frac{\partial G_1}{\partial y} = \frac{\partial e^{x+y}}{\partial y} = e^{x+y}$ and $\frac{\partial G_2}{\partial x} = 0$.

4. Find all functions $f(y)$ and $g(x)$, where $f(y)$ depends only on y and $g(x)$ depends only on x , such that the vector field $\mathbf{F}(x, y) = f(y)\mathbf{i} + g(x)\mathbf{j}$ is conservative on $\mathbb{R}^2$. For such f and g , find the potential function for $\mathbf{F}$.

For $\mathbf{F}$ to be conservative, we must have

$$\frac{\partial}{\partial y}f(y) = \frac{\partial}{\partial x}g(x).$$

But since the functions f and g depend only on one variable each, so do the functions $\frac{\partial}{\partial y}f(y) = f'(y)$ and $\frac{\partial}{\partial x}g(x) = g'(x)$. Therefore $f'(y) = g'(x) = A$ has to be constant.

This means that $f(y) = Ay + B$ and $g(y) = Ax + C$, where B and C are also constants. In that case, integrating in x and y we get

$$\phi(x, y) = Axy + Bx + Cy + D,$$

where A, B, C, D are constants, so that $\mathbf{F}$ is indeed a conservative field. (The constant D is optional, since the potential function is defined only up to a constant.)

2.3 Line Integrals

1. Evaluate the integral $\int_C \sqrt{x^2 + y^2} ds$, where C is the parametrized curve $\mathbf{r}(t) = (t \cos t, t \sin t)$, $0 \leq t \leq 2\pi$.

We have $\sqrt{x(t)^2 + y(t)^2} = \sqrt{t^2 \cos^2 t + t^2 \sin^2 t} = t$ and

$$\begin{aligned} ds &= |\mathbf{r}'(t)| = \sqrt{(-t \sin t + \cos t)^2 + (t \cos t + \sin t)^2} dt \\ &= (t^2 \sin^2 t - 2t \sin t \cos t + \cos^2 t + t^2 \cos^2 t + 2t \cos t \sin t + \sin^2 t)^{1/2} dt \\ &= \sqrt{t^2 + 1} dt. \end{aligned}$$

Hence the integral is

$$\int_0^{2\pi} t \sqrt{t^2 + 1} dt = \frac{1}{2} \frac{(t^2 + 1)^{3/2}}{3/2} \Big|_0^{2\pi} = \frac{1}{3} \left((4\pi^2 + 1)^{3/2} - 1 \right).$$

2. Evaluate the line integral $\int_C f ds$, where $f(x, y, z) = z - x$ and $\mathbf{r}(t) = e^{-t}\mathbf{i} + \sqrt{2}t\mathbf{j} + e^t\mathbf{k}$, $0 \leq t \leq 1$.

We have $\mathbf{r}'(t) = -e^{-t}\mathbf{i} + \sqrt{2}\mathbf{j} + e^t\mathbf{k}$, so that

$$\begin{aligned} |\mathbf{r}'| &= \sqrt{e^{-2t} + 2 + e^{2t}} = e^t + e^{-t}, \\ \int_C f ds &= \int_0^1 (e^t - e^{-t})(e^t + e^{-t}) dt = \int_0^1 (e^{2t} - e^{-2t}) dt \\ &= \frac{e^{2t} + e^{-2t}}{2} \Big|_{t=0}^1 = \frac{e^2 + e^{-2}}{2} - 1. \end{aligned}$$

3. Evaluate the line integral $\int_C 2yz ds$, if C is parametrized by $\mathbf{r}(t) = t\mathbf{i} + 2e^t\mathbf{j} + e^{2t}\mathbf{k}$ for $0 \leq t \leq 1$.

We have $\mathbf{r}'(t) = \mathbf{i} + 2e^t\mathbf{j} + 2e^{2t}\mathbf{k}$, so that

$$ds = |\mathbf{r}'(t)| dt = \sqrt{1 + 4e^{2t} + 4e^{4t}} dt = (1 + 2e^{2t}) dt,$$

$$\begin{aligned}
\int_C 2yz \, ds &= \int_0^1 4e^{3t}(1 + 2e^{2t})dt \\
&= \int_0^1 (4e^{3t} + 8e^{5t})dt \\
&= \left. \frac{4}{3}e^{3t} + \frac{8}{5}e^{5t} \right|_0^1 = \frac{4}{3}(e^3 - 1) + \frac{8}{5}(e^5 - 1).
\end{aligned}$$

4. Evaluate the line integral $\int_C xy \, dx - xy^2 \, dy$, where C is the curve with equation $x^2 = y^3$, from $(-1, 1)$ to $(1, 1)$.

Parametrize the curve as $x = t^3, y = t^2, -1 \leq t \leq 1$. Then

$$\begin{aligned}
\int_C xy \, dx - xy^2 \, dy &= \int_{-1}^1 (t^3 t^2 \cdot 3t^2 - t^3 t^4 \cdot 2t)dt = \int_{-1}^1 (3t^7 - 2t^8)dt \\
&= \left. \frac{3t^8}{8} - \frac{2t^9}{9} \right|_{t=-1}^1 = -\frac{4}{9}.
\end{aligned}$$

5. Evaluate the line integral $\int_C x \, dy - y \, dz$, where C is the oriented line segment from $(1, 0, 0)$ to $(3, 1, 6)$.

We parametrize the line segment as

$$\mathbf{r}(t) = (1 - t)(1, 0, 0) + t(3, 1, 6) = (1 + 2t, t, 6t), \quad 0 \leq t \leq 1.$$

Hence the integral is

$$\int_0^1 (1 + 2t)dt - t \cdot 6dt = \int_0^1 (1 - 4t)dt = \left. t - 2t^2 \right|_0^1 = 1 - 2 = -1.$$

6. Evaluate the integral $\int_C y \, dx + x \, dy$, where C is the arc of the circle $(x - 1)^2 + y^2 = 1$ from $(0, 0)$ to $(1, 1)$ in the first quadrant of the xy -plane.

Parametrize the arc by $x = 1 - \cos \theta, y = \sin \theta, 0 \leq \theta \leq \pi/2$. Then

$$\begin{aligned}
\int_C y \, dx + x \, dy &= \int_0^{\pi/2} (\sin \theta)(\sin \theta) + (1 - \cos \theta)(\cos \theta) d\theta \\
&= \int_0^{\pi/2} (\sin^2 \theta + \cos \theta - \cos^2 \theta) d\theta = \int_0^{\pi/2} (\cos \theta - \cos(2\theta)) d\theta \\
&= \left. \sin \theta - \frac{\sin(2\theta)}{2} \right|_0^{\pi/2} = 1 - 0 = 1.
\end{aligned}$$

7. Evaluate the integral $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F} = y\mathbf{i} - z\mathbf{j}$ and C is the part of the curve of intersection of the surfaces $x + y + z = 2$ and $y = z^2$ from $(0, 1, 1)$ to $(2, 0, 0)$.

If we use the parametrization $z = t$, $y = t^2$, $x = 2 - t - t^2$, $0 \leq t \leq 1$, then this parametrizes the curve C , but in the opposite direction (from $z = 0$ to $z = 1$, instead of the other way around). We are going to use this parametrization anyway and just add a minus sign. We have $\mathbf{r}'(t) = (-2t - 1)\mathbf{i} + 2t\mathbf{j} + \mathbf{k}$, so that

$$\begin{aligned} \int_C \mathbf{F} \cdot d\mathbf{r} &= - \int_0^1 \left[t^2(-2t - 1) - t(2t) \right] dt = \int_0^1 (2t^3 + 3t^2) dt \\ &= \frac{t^4}{2} + t^3 \Big|_{t=0}^1 = \frac{1}{2} + 1 = \frac{3}{2}. \end{aligned}$$

8. Evaluate the integral $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F} = y\mathbf{i} + z\mathbf{j} + x^2\mathbf{k}$ and C is the intersection curve of the surfaces $z = y^2$ and $x^2 + y^2 = 4$, oriented counterclockwise when viewed from the direction of the positive z -axis.

We use the parametrization $x = 2 \cos t$, $y = 2 \sin t$, $z = y^2 = 4 \sin^2 t$, $0 \leq t \leq 2\pi$. Then $\mathbf{r}'(t) = -2 \sin t \mathbf{i} + 2 \cos t \mathbf{j} + 8 \sin t \cos t \mathbf{k}$, so that

$$\begin{aligned} \int_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^{2\pi} \left[(2 \sin t)(-2 \sin t) + (4 \sin^2 t)(2 \cos t) + (4 \cos^2 t)(8 \sin t \cos t) \right] dt \\ &= \int_0^{2\pi} \left[-4 \sin^2 t + 8 \sin^2 t \cos t + 32 \cos^3 t \sin t \right] dt \end{aligned}$$

We have $\cos(2t) = 1 - 2 \sin^2 t$, so that

$$\int_0^{2\pi} (-4 \sin^2 t) dt = \int_0^{2\pi} (2 \cos(2t) - 2) dt = [\sin(2t) - 2t]_0^{2\pi} = -4\pi,$$

and also

$$\begin{aligned} \int_0^{2\pi} 8 \sin^2 t \cos t dt &= \left[\frac{8}{3} \sin^3 t \right]_0^{2\pi} = 0, \\ \int_0^{2\pi} 32 \cos^3 t \sin t dt &= [-8 \cos^4 t]_0^{2\pi} = 0. \end{aligned}$$

Therefore

$$\int_C \mathbf{F} \cdot d\mathbf{r} = -4\pi + 0 + 0 = -4\pi.$$

9. Consider the integral $\int_C f ds$, where $f(x, y, z) = e^{-(x^2+y^2+z^2)}$ and C is the line where the planes $x + y - z = 0$ and $x + 2y + z = 0$ intersect. Prove that this integral is finite.

(Hint: if you set up this integral using the “usual” parametrization of a line where the velocity is constant, you will get an improper integral. If you do not know how to evaluate it, it will suffice if you use inequalities to prove that the integral is finite.

Solution 1. We first parametrize the line. From the first plane equation we have $x = z - y$, which we plug into the second equation:

$$0 = x + 2y + z = (z - y) + 2y + z = y + 2z, \quad y = -2z, \quad x = z - y = z - (-2z) = 3z.$$

We get a parametrization $\mathbf{r}(t) = 3t\mathbf{i} - 2t\mathbf{j} + t\mathbf{k}$, with $|\mathbf{r}'| = \sqrt{14}$. The line integral is

$$\int_C f \, ds = \int_{-\infty}^{\infty} e^{-14t^2} \sqrt{14} \, dt.$$

One way to complete the proof is to use the fact that $\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$. (This can be done using polar coordinates in the plane – see if you can find it in your textbook!) Then by a change of variables $u = t\sqrt{14}$, $du = dt\sqrt{14}$,

$$\int_C f \, ds = \int_{-\infty}^{\infty} e^{-u^2} du = \sqrt{\pi}.$$

Solution 2. Start as above, but if you do not know or do not remember the last integral, there are alternative solutions using inequalities. For example, we can use that $e^{-14t^2} \leq 1$ for $|t| \leq 1$ and $e^{-14t^2} \leq e^{-|t|}$ for $|t| \geq 1$. Then

$$\begin{aligned} \int_C f \, ds &= 2 \int_0^{\infty} e^{-14t^2} \sqrt{14} \, dt \\ &\leq 2 \int_0^1 \sqrt{14} \, dt + 2 \int_1^{\infty} e^{-t} \sqrt{14} \, dt \\ &= 2\sqrt{14} + 2\sqrt{14} [-e^{-t}]_1^{\infty} = 4\sqrt{14}, \end{aligned}$$

which again is finite.

Solution 3. Since $(0, 0, 0)$ lies on both planes, the line C passes through it. Now, notice that $f(x, y, z) = e^{-(x^2+y^2+z^2)}$ is radially symmetric! Therefore the integral of f on any line passing through the origin is the same. So, instead of integrating on the line indicated, we can make our lives easier and integrate on the x -axis instead, with $\mathbf{r} = t\mathbf{i}$ and $f(x, 0, 0) = e^{-x^2}$:

$$\int_C f \, ds = \int_{-\infty}^{\infty} e^{-t^2} dt = \sqrt{\pi}.$$

2.4 Path Independence

1. Let $\mathbf{F}(x, y, z) = e^{2y}\mathbf{i} + (axe^{2y} + be^z)\mathbf{j} - 3ye^z\mathbf{k}$. Find the values of a, b for which $\mathbf{F}$ is conservative, and the corresponding potential function. For these values of a and b , evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where C is a piecewise smooth curve from $(1, 1, 0)$ to $(0, 1, 3)$.

For $\mathbf{F}$ to be conservative, we must have

$$\frac{\partial}{\partial y}e^{2y} = \frac{\partial}{\partial x}(axe^{2y} + be^z), \quad 2e^{2y} = ae^{2y}, \quad a = 2,$$

$$\frac{\partial}{\partial z}(axe^{2y} + be^z) = \frac{\partial}{\partial y}(-3ye^z), \quad be^z = -3e^z, \quad b = -3.$$

Let $a = 2, b = -3$, and solve for ϕ such that $\mathbf{F} = \nabla\phi$:

$$\frac{\partial\phi}{\partial x} = e^{2y}, \quad \phi = xe^{2y} + f(y, z)$$

$$\frac{\partial}{\partial y}(xe^{2y} + f(y, z)) = 2xe^{2y} + \frac{\partial f}{\partial y} = 2xe^{2y} - 3e^z, \quad f(y, z) = -3ye^z + g(z),$$

$$\frac{\partial}{\partial z}(xe^{2y} - 3ye^z + g(z)) = -3ye^z + g'(z) = -3ye^z, \quad g(z) = C$$

In particular, we may take $C = 0$, so that

$$\phi = xe^{2y} - 3ye^z.$$

Finally,

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \phi(0, 1, 3) - \phi(1, 1, 0) = (0 - 3e^3) - (e^2 - 3) = 3 - 3e^3 - e^2.$$

2. Find all values of the parameters a, b, c such that the vector field

$$\mathbf{F}(x, y, z) = 4xy^3\mathbf{i} + (ax^2y^2 + b\cos z)\mathbf{j} + (y\sin z + c\cos z)\mathbf{k}$$

is conservative on $\mathbb{R}^3$. For all such a, b, c :

(a) find a potential function f such that $\mathbf{F} = \nabla f$,

(b) evaluate the integral $\int_C \mathbf{F} \cdot d\mathbf{r}$, if C is the line segment from $(0, 1, 0)$ to $(1, 1, \pi/2)$.

(a) Suppose that $\mathbf{F}$ is conservative, with $\mathbf{F} = \nabla f$. Then we must have

$$\frac{\partial}{\partial x}f(x, y, z) = F_1 = 4xy^3,$$

so that $f(x, y, z) = 2x^2y^3 + g(y, z)$. We must also have

$$\frac{\partial}{\partial y}f(x, y, z) = 6x^2y^2 + \frac{\partial}{\partial y}g(y, z) = F_2 = ax^2y^2 + b \cos z,$$

so that $a = 6$ and $g(y, z) = by \cos z + h(z)$. So far, we have $f(x, y, z) = 2x^2y^3 + by \cos z + h(z)$. Finally,

$$\frac{\partial}{\partial z}f(x, y, z) = -by \sin z + h'(z) = F_3 = y \sin z + c \cos z,$$

so that $b = -1$ and we can take $h(z) = c \sin z$, with no restrictions on c . The answer is $a = 6$, $b = -1$, $c \in \mathbb{R}$, and $f(x, y, z) = 2x^2y^3 - y \cos z + c \sin z$.

(b) With f as above, we have

$$\begin{aligned} \int_C \mathbf{F} \cdot d\mathbf{r} &= f(1, 1, \pi/2) - f(0, 1, 0) \\ &= (2 - 0 + c) - (0 - 1 + 0) = 3 + c. \end{aligned}$$

3. Let $\mathbf{F} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be a C^1 vector field. Assume that the following holds: for any $P, Q \in \mathbb{R}^2$, and for any two curves C_1, C_2 from P to Q **that are unions of at most 5 line segments each**, we have $\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_2} \mathbf{F} \cdot d\mathbf{r}$. Does it follow that $\mathbf{F}$ is conservative? Prove your answer.

The answer is yes, and we can prove it by modifying the argument from class just slightly. Assume that $\mathbf{F}$ has the path independence property on $\mathbb{R}^2$, and define

$$\phi(x, y) = \int_C \mathbf{F} \cdot d\mathbf{r},$$

where C is any path connecting the origin $O = (0, 0)$ to $P = (x, y)$ and consisting of at most 5 line segments.

We need to prove that

$$\frac{\partial \phi}{\partial x} = F_1, \quad \frac{\partial \phi}{\partial y} = F_2.$$

To prove the first identity, let C be the path consisting of 2 line segments: C_1 from $(0, 0)$ to $(0, y)$, and C_2 from $(0, y)$ to (x, y) . (This is a valid choice because the domain of $\mathbf{F}$ is all of $\mathbb{R}^2$.) Then

$$\phi(x, y) := \int_{C_1} \mathbf{F} \cdot d\mathbf{r} + \int_{C_2} \mathbf{F} \cdot d\mathbf{r}.$$

We now keep y constant and differentiate in x . The integral on C_1 is independent of x , so the derivative is 0. For C_2 , we have

$$\int_{C_2} \mathbf{F} \cdot d\mathbf{r} = \int_0^x F_1(t, y) dt$$

so that

$$\frac{\partial}{\partial x} \int_{C_2} \mathbf{F} \cdot d\mathbf{r} = F_1(x, y)$$

as claimed. The proof of the second part is identical but with x and y variables interchanged, so that we use a path consisting of the line segments from $(0, 0)$ to $(x, 0)$, and C_2 from $(x, 0)$ to (x, y) . Path independence guarantees that both of the above paths get us the same function ϕ .

Additional comments:

- We did not even need to use 5 line segments—paths consisting of at most 2 line segments would have been sufficient.
- It is also possible to solve this by approximating arbitrary piecewise smooth curves by paths consisting of increasing numbers of line segments. This is fine if done correctly.

4. Let f and g be two C^1 functions on D . Let $\mathbf{F} = f\nabla g$ and $\mathbf{G} = g\nabla f$. Prove that $\mathbf{F}$ has path-independent line integrals if and only if so does $\mathbf{G}$.

We have

$$\mathbf{F} + \mathbf{G} = f\nabla g + g\nabla f = \nabla(fg),$$

so that $\mathbf{F} + \mathbf{G}$ has path independent line integrals.

Let $P_1, P_2 \in D$, and let C_1, C_2 be two piecewise smooth curves C_1, C_2 from P_1 to P_2 . Then

$$\begin{aligned} & \int_{C_1} \mathbf{G} \cdot d\mathbf{r} - \int_{C_2} \mathbf{G} \cdot d\mathbf{r} \\ &= \left(\int_{C_1} (\mathbf{F} + \mathbf{G}) \cdot d\mathbf{r} - \int_{C_1} \mathbf{F} \cdot d\mathbf{r} \right) - \left(\int_{C_2} (\mathbf{F} + \mathbf{G}) \cdot d\mathbf{r} - \int_{C_2} \mathbf{F} \cdot d\mathbf{r} \right) \\ &= \left(\int_{C_1} (\mathbf{F} + \mathbf{G}) \cdot d\mathbf{r} - \int_{C_2} (\mathbf{F} + \mathbf{G}) \cdot d\mathbf{r} \right) - \left(\int_{C_1} \mathbf{F} \cdot d\mathbf{r} - \int_{C_2} \mathbf{F} \cdot d\mathbf{r} \right) \\ &= \int_{C_2} \mathbf{F} \cdot d\mathbf{r} - \int_{C_1} \mathbf{F} \cdot d\mathbf{r}. \end{aligned}$$

Therefore the left side is always zero if and only if the right side is always zero, and the conclusion follows.

Chapter 3

Parametric Surfaces and Surface Integration

3.1 Area Integrals

1. Evaluate the surface integral $\iint_{\mathcal{S}} \sqrt{x} dS$, if $\mathcal{S}$ is the parametrized surface $\mathbf{r}(u, v) = (u^2, v, u + v)$ with $0 \leq u \leq 1$, $0 \leq v \leq 1$.

We have $\mathbf{r}_u = (2u, 0, 1)$, $\mathbf{r}_v = (0, 1, 1)$, so

$$\mathbf{n} = \mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2u & 0 & 1 \\ 0 & 1 & 1 \end{vmatrix} = -\mathbf{i} - 2u\mathbf{j} + 2u\mathbf{k},$$

$$|\mathbf{n}| = \sqrt{1 + 4u^2 + 4u^2} = \sqrt{1 + 8u^2}.$$

Therefore, with the substitution $8u^2 + 1 = w$, $16udu = dw$,

$$\begin{aligned} \iint_{\mathcal{S}} \sqrt{x} dS &= \int_0^1 \int_0^1 u \sqrt{1 + 8u^2} dv du \\ &= \int_0^1 u \sqrt{1 + 8u^2} du \\ &= \int_1^9 \sqrt{w} \frac{dw}{16} \\ &= \frac{w^{3/2}}{(3/2) \cdot 16} \Big|_1^9 = \frac{27 - 1}{24} = \frac{26}{24} = \frac{13}{12}. \end{aligned}$$

2. A parametric surface is given by the equations $x = uv$, $y = u^2 - v^2$, $z = u^2 + v^2$, with $u, v \geq 0$. Find the equation of the plane tangent to the surface at the point P where $(u, v) = (2, 1)$.

We have $\mathbf{r}_u = \langle v, 2u, 2u \rangle$ and $\mathbf{r}_v = \langle u, -2v, 2v \rangle$, so that the normal vector is

$$\mathbf{n} = \mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ v & 2u & 2u \\ u & -2v & 2v \end{vmatrix} = 8uv\mathbf{i} - (2v^2 - 2u^2)\mathbf{j} + (-2v^2 - 2u^2)\mathbf{k}.$$

The point P has coordinates $(2, 3, 5)$, and the normal vector at P is $\mathbf{n} = 16\mathbf{i} + 6\mathbf{j} - 10\mathbf{k}$. Therefore the tangent plane is

$$16(x - 2) + 6(y - 3) - 10(z - 5) = 0,$$

which can be simplified to $8(x - 2) + 3(y - 3) - 5(z - 5) = 0$, $8x + 3y - 5z = 0$.

3. Let $\mathcal{S}$ be the surface $\mathbf{r}(u, v) = (u \cos v, u \sin v, v)$ with $0 \leq u \leq 1$, $0 \leq v \leq \pi$.

(a) Find the tangent plane to the surface at the point where $u = 1/2$, $v = \pi/6$.

We have $\mathbf{r}_u = \cos v \mathbf{i} + \sin v \mathbf{j}$ and $\mathbf{r}_v = -u \sin v \mathbf{i} + u \cos v \mathbf{j} + \mathbf{k}$, so that

$$\mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos v & \sin v & 0 \\ -u \sin v & u \cos v & 1 \end{vmatrix} = \sin v \mathbf{i} - \cos v \mathbf{j} + u \mathbf{k}$$

At $(u, v) = (1/2, \pi/6)$,

$$\mathbf{r} = \left\langle \frac{1}{2} \cos \frac{\pi}{6}, \frac{1}{2} \sin \frac{\pi}{6}, \frac{\pi}{6} \right\rangle = \left\langle \frac{\sqrt{3}}{4}, \frac{1}{4}, \frac{\pi}{6} \right\rangle,$$

$$\mathbf{n} = \left\langle \sin \frac{\pi}{6}, -\cos \frac{\pi}{6}, \frac{1}{2} \right\rangle = \frac{1}{2} \langle 1, -\sqrt{3}, 1 \rangle,$$

so the equation of the tangent plane is

$$\left(x - \frac{\sqrt{3}}{4}\right) - \sqrt{3} \left(y - \frac{1}{4}\right) + \left(z - \frac{\pi}{6}\right) = 0,$$

which simplifies to $x - \sqrt{3}y + z = \pi/6$.

(b) Evaluate the surface integral $\iint_{\mathcal{S}} (x^2 + y^2)^{1/2} dS$.

From (a), we have $|\mathbf{n}| = (\sin^2 v + \cos^2 v + u^2)^{1/2} = \sqrt{1 + u^2}$. Now

$$\iint_{\mathcal{S}} (x^2 + y^2)^{1/2} dS = \int_0^\pi \int_0^1 u \sqrt{1 + u^2} du dv$$

We substitute $t = 1 + u^2$, $2u du = dt$, $u^2 = t - 1$, so that the last integral is

$$\frac{1}{2} \int_0^\pi \int_1^2 \sqrt{t} dt dv = \frac{1}{2} \int_0^\pi \left. \frac{2}{3} t^{3/2} \right|_1^2 dv = \frac{1}{2} \int_0^\pi \frac{2}{3} (2^{3/2} - 1) dv = \frac{\pi}{3} (2^{3/2} - 1)$$

4. Let $\mathcal{S}$ be the surface parametrized by $\mathbf{r}(u, v) = (u + \sin u, v + \sin v, 1 - \cos(u + v))$, $0 < u < \pi$, $0 < v < \pi$ (the boundary of the rectangle is not included).

(a) Find all (u, v) such that the tangent plane to the surface at $\mathbf{r}(u, v)$ is parallel to the xy -plane.

We calculate the normal vector:

$$\mathbf{r}_u = \langle 1 + \cos u, 0, \sin(u + v) \rangle,$$

$$\mathbf{r}_v = \langle 0, 1 + \cos v, \sin(u + v) \rangle,$$

$$\begin{aligned} \mathbf{n} &= \mathbf{r}_u \times \mathbf{r}_v \\ &= -(1 + \cos v) \sin(u + v) \mathbf{i} - (1 + \cos u) \sin(u + v) \mathbf{j} + (1 + \cos u)(1 + \cos v) \mathbf{k}. \end{aligned}$$

For the tangent plane to be parallel to the xy -plane, $\mathbf{n}$ needs to be parallel to $\mathbf{k}$, so its first two components should be 0:

$$(1 + \cos v) \sin(u + v) = (1 + \cos u) \sin(u + v) = 0.$$

Since $1 + \cos u$ and $1 + \cos v$ are nonzero for u, v in $(0, \pi)$ (remember that we are excluding the boundary), we must have $\sin(u + v) = 0$, $u + v = \pi$. The corresponding points on the surface are

$$\mathbf{r}(u, \pi - u) = (u + \sin u, \pi - u + \sin u, 2), \quad 0 < u < \pi.$$

(b) Prove that the area of $\mathcal{S}$ is less than $5\pi^2$.

The area is $\int_0^\pi \int_0^\pi |\mathbf{n}(u, v)| du dv$. From part (a), we have

$$|\mathbf{n}|^2 = (1 + \cos v)^2 \sin(u + v)^2 + (1 + \cos u)^2 \sin(u + v)^2 + (1 + \cos u)^2 (1 + \cos v)^2.$$

Since sine and cosine are always between -1 and 1 , the first term on the right is at most 4, the second term is at most 4, and the third term is at most $4 \cdot 4 = 16$. Hence $|\mathbf{n}| \leq \sqrt{4 + 4 + 16} = \sqrt{24}$, so that

$$\text{area} \leq \int_0^\pi \int_0^\pi \sqrt{24} du dv = \sqrt{24} \pi^2 < 5\pi^2.$$

5. Evaluate the integral $\iint_{\mathcal{S}} (x^2 + 4y^2 + 1) dS$, if $\mathcal{S}$ is the part of the paraboloid $z = \frac{x^2}{2} + y^2$ that lies above the region $D = \{(x, y) : x^2 + 4y^2 \leq 1 \text{ and } y \geq 0\}$ in the xy -plane.

We parametrize the surface by $\mathbf{r}(x, y) = \langle x, y, \frac{x^2}{2} + y^2 \rangle$. Then $\mathbf{r}_x = \langle 1, 0, x \rangle$ and $\mathbf{r}_y = \langle 0, 1, 2y \rangle$, so that

$$\mathbf{n} = \mathbf{r}_x \times \mathbf{r}_y = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & x \\ 0 & 1 & 2y \end{vmatrix} = -x\mathbf{i} - 2y\mathbf{j} + \mathbf{k},$$

$$|\mathbf{n}| = \sqrt{x^2 + 4y^2 + 1},$$

$$\begin{aligned} \iint_{\mathcal{S}} (x^2 + 4y^2 + 1) dS &= \iint_D (x^2 + 4y^2 + 1) \sqrt{x^2 + 4y^2 + 1} dA \\ &= \iint_D (x^2 + 4y^2 + 1)^{3/2} dA \end{aligned}$$

To evaluate this integral, we use modified polar coordinates $x = r \cos \theta$, $y = \frac{1}{2}r \sin \theta$,

$$\frac{\partial(x, y)}{\partial(r, \theta)} = \begin{vmatrix} \cos \theta & \frac{1}{2} \sin \theta \\ -r \sin \theta & \frac{1}{2} r \cos \theta \end{vmatrix} = \frac{1}{2} r \cos^2 \theta + \frac{1}{2} r \sin^2 \theta = \frac{r}{2}$$

With that substitution, $x^2 + 4y^2 = r^2 \cos^2 \theta + 4 \cdot \frac{1}{4} r^2 \sin^2 \theta = r^2$, so that

$$\begin{aligned} \iint_{\mathcal{S}} (x^2 + 4y^2 + 1) dS &= \int_0^1 \int_0^\pi (r^2 + 1)^{3/2} \frac{r}{2} d\theta dr \\ &= \frac{\pi}{4} \int_0^1 (r^2 + 1)^{3/2} (2r) dr \\ &= \frac{\pi}{4} \frac{(r^2 + 1)^{5/2}}{5/2} \Big|_{r=0}^1 = \frac{\pi}{4} \frac{2^{5/2} - 1}{5/2} = \frac{\pi(2^{5/2} - 1)}{10} \end{aligned}$$

6. Let $\mathcal{C}$ be a smooth simple curve in the plane $z = 0$, parametrized by $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$ with $a \leq t \leq b$. Construct a surface $\mathcal{S}$ in three dimensions by taking the union of all line segments that connect points $\mathbf{r}(t)$ on the curve to the fixed point $P = (0, 0, 1)$. Prove that

$$\frac{1}{2}(\text{length of } \mathcal{C}) \leq (\text{area of } \mathcal{S}).$$

Give an example of $\mathcal{C}$ such that equality holds in the above formula.

(Hint: How would you parametrize the surface? You may want to start by parametrizing each line segment separately.)

First, the length of the curve is

$$L = \int_a^b |\mathbf{r}'(t)| dt = \int_a^b \sqrt{(x')^2 + (y')^2} dt \quad (3.1.1)$$

Next, we parametrize the surface by the vector-valued function $\mathbf{R}(t, s)$:

$$\mathbf{R}(t, s) = (1 - s)\mathbf{k} + s\mathbf{r}(t), \quad a \leq t \leq b, \quad 0 \leq s \leq 1.$$

Using this parametrization, we have

$$\mathbf{n} = \mathbf{r}_t \times \mathbf{r}_s = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ sx'(t) & sy'(t) & 0 \\ x(t) & y(t) & -1 \end{vmatrix} = -sy'\mathbf{i} + sx'\mathbf{j} + (sx'y - sxy')\mathbf{k},$$

so that

$$\begin{aligned} |\mathbf{n}| &= s\sqrt{(x')^2 + (y')^2 + (x'y - xy')^2} \\ &\geq s\sqrt{(x')^2 + (y')^2}. \end{aligned}$$

The area of the surface is

$$\begin{aligned} A &= \iint_{\mathcal{S}} dS = \int_a^b \int_0^1 s\sqrt{(x')^2 + (y')^2 + (x'y - xy')^2} ds dt \\ &= \int_a^b \left. \frac{s^2}{2} \right|_{s=0}^1 \sqrt{(x')^2 + (y')^2 + (x'y - xy')^2} dt \\ &= \frac{1}{2} \int_a^b \sqrt{(x')^2 + (y')^2 + (x'y - xy')^2} dt \\ &\geq \frac{1}{2} \int_a^b \sqrt{(x')^2 + (y')^2} dt = \frac{1}{2} L, \end{aligned}$$

as required.

For an example where the equality holds, let $\mathcal{C}$ be the line segment from $(0, 0, 0)$ to $(1, 0, 0)$. Then $\mathcal{C}$ has length 1. The surface $\mathcal{S}$ is the right triangle with vertices at $(0, 0, 0)$, $(1, 0, 0)$, and $(0, 0, 1)$, and has area $1/2$, which is half of the length of $\mathcal{C}$.

7. Let D be a planar domain such that

$$D \subset \{(x, y) : x^2 + y^2 \leq 64 \text{ and } \frac{x^2}{36} + \frac{y^2}{49} \geq 1\}.$$

(This is just an inclusion and D does not have to be equal to the set on the right.)

Let $\mathcal{S}$ be the part of the sphere $x^2 + y^2 + z^2 = 64$ that lies above D . Prove that

$$(\text{area of } D) \leq \frac{3}{4}(\text{area of } \mathcal{S}).$$

We have

$$(\text{area of } \mathcal{S}) = \iint_{\mathcal{S}} d\mathbf{S} = \iint_D \frac{|\nabla G|}{|G_z|} dx dy,$$

where $G(x, y, z) = x^2 + y^2 + z^2$. We have for $(x, y, z) \in \mathcal{S}$,

$$|\nabla G| = \sqrt{4x^2 + 4y^2 + 4z^2} = 2\sqrt{x^2 + y^2 + z^2} = 16,$$

$$|G_z| = 2z = 2\sqrt{64 - x^2 - y^2}.$$

On D , we have

$$\frac{x^2}{36} + \frac{y^2}{36} \geq \frac{x^2}{36} + \frac{y^2}{49} \geq 1,$$

so that $x^2 + y^2 \geq 36$ and $\sqrt{64 - x^2 - y^2} \leq \sqrt{64 - 36} = \sqrt{28}$. Therefore

$$\frac{|\nabla G|}{|G_z|} \geq \frac{16}{2\sqrt{28}} = \frac{4}{\sqrt{7}} > \frac{4}{3},$$

$$(\text{area of } \mathcal{S}) > \iint_D \frac{4}{3} dx dy = \frac{4}{3}(\text{area of } D).$$

8. (Let $\mathcal{S}$ be the part of the sphere $x^2 + y^2 + z^2 = 1$ that lies above the cone $z = \sqrt{x^2 + y^2}$. Consider the integral

$$\iint_{\mathcal{S}} \sqrt{x^2 + y^2} dS.$$

A student has evaluated this integral as follows.

The unit normal to $\mathcal{S}$ is the vector $\mathbf{N} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$. The angle γ between $\mathbf{N}$ and $\mathbf{k}$ is $\gamma = \phi$ in spherical coordinates, so that the area element is $\frac{1}{\cos \gamma} = \frac{1}{\cos \phi}$. Also in spherical coordinates, $\sqrt{x^2 + y^2} = \sin \phi$ on the sphere. Therefore

$$\begin{aligned} \iint_{\mathcal{S}} \sqrt{x^2 + y^2} dS &= \int_0^{2\pi} \int_0^{\pi/4} \frac{\sqrt{x^2 + y^2}}{\cos \gamma} d\phi d\theta \\ &= \int_0^{2\pi} \int_0^{\pi/4} \frac{\sin \phi}{\cos \phi} d\phi d\theta \\ &= 2\pi \left[-\ln(\cos \phi) \right]_0^{\pi/4} \\ &= 2\pi \left(-\ln(\cos(\pi/4)) + \ln(\cos 0) \right) \\ &= 2\pi(-\ln(\sqrt{2}/2) - \ln(1)) = -2\pi \ln(\sqrt{2}/2). \end{aligned}$$

Is this solution correct? If not, explain where the student has made a mistake (if there is more than one error, indicate all of them), and provide the correct solution.

The mistake is that the area element is $dS = \frac{1}{\cos \gamma} dx dy = \frac{1}{\cos \phi} dx dy$ in *rectangular* coordinates. In spherical coordinates, we have $dS = \sin \phi d\phi d\theta$, and

$$\begin{aligned} \iint_S \sqrt{x^2 + y^2} dS &= \int_0^{2\pi} \int_0^{\pi/4} \sqrt{x^2 + y^2} \sin \phi d\phi d\theta \\ &= \int_0^{2\pi} \int_0^{\pi/4} \sin^2 \phi d\phi d\theta \\ &= 2\pi \int_0^{\pi/4} \frac{1 - \cos(2\phi)}{2} d\phi \\ &= \pi \left[\phi - \frac{\sin(2\phi)}{2} \right]_0^{\pi/4} \\ &= \pi \left[\frac{\pi}{4} - \frac{1}{2} \right]. \end{aligned}$$

3.2 Flux Integrals

1. Let $\mathcal{S}$ be the surface $\mathbf{r}(u, v) = (u \cos v, u \sin v, 2v)$ with $0 \leq u \leq 2$, $0 \leq v \leq 2\pi$, oriented with $\mathbf{N}$ pointing up. Evaluate the integral $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F} = -y\mathbf{i} + (x + y^2)\mathbf{j} + z\mathbf{k}$

We have $\mathbf{r}_u = (\cos v, \sin v, 0)$, $\mathbf{r}_v = (-u \sin v, u \cos v, 2)$, so

$$\mathbf{n} = \mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos v & \sin v & 0 \\ -u \sin v & u \cos v & 2 \end{vmatrix} = 2 \sin v \mathbf{i} - 2 \cos v \mathbf{j} + u \mathbf{k}.$$

Using this $\mathbf{n}$, we get

$$\begin{aligned} \iint_S \mathbf{F} \cdot d\mathbf{S} &= \int_0^2 \int_0^{2\pi} [(-u \sin v)(2 \sin v) + (u \cos v + u^2 \sin^2 v)(-2 \cos v) + 2vu] dv du \\ &= \int_0^2 \int_0^{2\pi} (-2u - 2u^2 \sin^2 v \cos v + 2uv) dv du \\ &= \int_0^2 \left(-2uv - \frac{2}{3} u^2 \sin^3 v + uv^2 \right) \Big|_{v=0}^{2\pi} du \\ &= \int_0^2 (-4\pi u + 4\pi^2 u) du \\ &= 4\pi(\pi - 1) \int_0^2 u du = 2\pi(\pi - 1) u^2 \Big|_0^2 = 8\pi(\pi - 1) \end{aligned}$$

2. Evaluate the integral $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F} = z\mathbf{i} + y^2\mathbf{j}$ and S is the part of the cylinder $x^2 + y^2 = 4$ where $x \geq 0$, $y \geq 0$, $0 \leq z \leq 1$, with the normal pointing away from the z -axis.

We parametrize the surface using $\mathbf{r}(\theta, z) = \langle 2 \cos \theta, 2 \sin \theta, z \rangle$, with $0 \leq \theta \leq \pi/2$, $0 \leq z \leq 1$. Then $\mathbf{r}_\theta = \langle -2 \sin \theta, 2 \cos \theta, 0 \rangle$, $\mathbf{r}_z = \langle 0, 0, 1 \rangle$, so

$$\mathbf{n} = \mathbf{r}_\theta \times \mathbf{r}_z = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 \sin \theta & 2 \cos \theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = 2 \cos \theta \mathbf{i} + 2 \sin \theta \mathbf{j}.$$

This vector points away from the z -axis as required. We have

$$\mathbf{F} \cdot \mathbf{n} = 2z \cos \theta + 8 \sin^2 \theta \sin \theta.$$

Therefore,

$$\begin{aligned} \iint_S \mathbf{F} \cdot d\mathbf{S} &= \int_0^1 \int_0^{\pi/2} (2z \cos \theta + 8 \sin^2 \theta \sin \theta) d\theta dz \\ &= \int_0^1 \int_0^{\pi/2} (2z \cos \theta + 8(1 - \cos^2 \theta) \sin \theta) d\theta dz \\ &= \int_0^1 \left(2z \sin \theta - 8 \cos \theta + \frac{8}{3} \cos^3 \theta \right) \Big|_{\theta=0}^{\pi/2} dz \\ &= \int_0^1 \left(2z + 8 - \frac{8}{3} \right) dz = \int_0^1 \left(2z + \frac{16}{3} \right) dz = z^2 + \frac{16z}{3} \Big|_0^1 = \frac{19}{3} \end{aligned}$$

3. Evaluate the surface integral $\iint_S \mathbf{F} \cdot d\mathbf{S}$, if $\mathbf{F} = z\mathbf{k}$ and S is the rectangle with vertices $(0, 2, 0)$, $(0, 0, 4)$, $(5, 2, 0)$, $(5, 0, 4)$, oriented so that the normal vector points upward.

The plane in which the rectangle lies has the equation $2y + z = 4$ (by inspection, or you can use any method you know for finding the equation of a plane passing through any three of these points). We can rewrite this as $z = 4 - 2y$, so that

$$d\mathbf{S} = (2\mathbf{j} + \mathbf{k})dxdy.$$

We need to evaluate the area of it that lies above the rectangle $0 \leq x \leq 5$, $0 \leq y \leq 2$. This is

$$\begin{aligned} \int_0^5 \int_0^2 \mathbf{F} \cdot (2\mathbf{j} + \mathbf{k}) dy dx &= \int_0^5 \int_0^2 z dy dx = \int_0^5 \int_0^2 (4 - 2y) dy dx \\ &= \int_0^5 (4y - y^2) \Big|_0^2 dx = \int_0^5 4 dx = 20. \end{aligned}$$

4. Evaluate the surface integral $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F} = x^2\mathbf{i}$ and $\mathcal{S}$ is the part of the plane $x + \frac{y}{2} + \frac{z}{3} = 1$ that lies in the first octant ($x, y, z \geq 0$), with the normal vector pointing up.

With $\mathcal{S}$ parametrized by x, y , and given by the equation $G(x, y, z) = 0$ for $G(x, y, z) = 3x + \frac{3}{2}y + z - 3$, we can use the formula

$$d\mathbf{S} = \frac{\nabla G}{G_z} dx dy = \langle 3, \frac{3}{2}, 1 \rangle dx dy.$$

(Note that the $\mathbf{k}$ -component is $1 > 0$, so the normal vector points up.) The range of x, y is $0 \leq x \leq 1$, $0 \leq y \leq 2 - 2x$. Therefore

$$\begin{aligned} \iint_S \mathbf{F} \cdot d\mathbf{S} &= \int_0^1 \int_0^{2-2x} 3x^2 dy dx \\ &= \int_0^1 3x^2(2 - 2x) dx \\ &= \left(2x^3 - \frac{3x^4}{2} \right) \Big|_{x=0}^1 = 2 - \frac{3}{2} = \frac{1}{2}. \end{aligned}$$

5. Evaluate the integral $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F} = \mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$ and $\mathcal{S}$ is the surface parametrized by $\mathbf{r}(u, v) = \langle uv, u^2 - v^2, u^2 + v^2 \rangle$ with $0 \leq u \leq 1$, $0 \leq v \leq u$, with the normal vector pointing up.

We have $\mathbf{r}_u = (v, 2u, 2u)$, $\mathbf{r}_v = (u, -2v, 2v)$, so

$$\mathbf{n} = \mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ v & 2u & 2u \\ u & -2v & 2v \end{vmatrix} = 8uv\mathbf{i} - (2v^2 - 2u^2)\mathbf{j} + (-2v^2 - 2u^2)\mathbf{k}.$$

The last component of this is negative, so we need to use $-\mathbf{n}$ instead. We have

$$\mathbf{F} \cdot \mathbf{n} = 8uv + 2(-2v^2 + 2u^2) + 2(-2v^2 - 2u^2) = 8uv - 8v^2.$$

Therefore,

$$\begin{aligned} \iint_S \mathbf{F} \cdot d\mathbf{S} &= - \int_0^1 \int_0^u (8uv - 8v^2) dv du \\ &= - \int_0^1 \left(4uv^2 - \frac{8}{3}v^3 \right) \Big|_{v=0}^u du \\ &= - \int_0^1 \left(4u^3 - \frac{8}{3}u^3 \right) du = - \int_0^1 \frac{4}{3}u^3 du = \frac{u^4}{3} \Big|_0^1 = -\frac{1}{3}. \end{aligned}$$

6. Evaluate the integral $\iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F} = x^2\mathbf{i} + y^4\mathbf{j}$ and $\mathcal{S}$ is the part of the surface $xy^2z = 3$ that lies above the rectangle $1 \leq x \leq 2$, $1 \leq y \leq 3$, with the normal vector pointing up.

Our surface $\mathcal{S}$ has the form $G(x, y, z) = 3$ with $G = xy^2z$, so that

$$\begin{aligned} d\mathbf{S} &= \frac{\nabla G}{G_z} = \frac{y^2z\mathbf{i} + 2xyz\mathbf{j} + xy^2\mathbf{k}}{xy^2} = \frac{z}{x}\mathbf{i} + \frac{2z}{y}\mathbf{j} + \mathbf{k} \\ &= \frac{3/xy^2}{x}\mathbf{i} + \frac{6/xy^2}{y}\mathbf{j} + \mathbf{k} = \frac{3}{x^2y^2}\mathbf{i} + \frac{6}{xy^3}\mathbf{j} + \mathbf{k} \\ \mathbf{F} \cdot d\mathbf{S} &= \frac{3}{y^2} + \frac{6y}{x} \end{aligned}$$

$$\begin{aligned} \iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S} &= \int_1^2 \int_1^3 \left(\frac{3}{y^2} + \frac{6y}{x} \right) dy dx = \int_1^2 \left(-\frac{3}{y} + \frac{3y^2}{x} \right) \Big|_1^3 dx \\ &= \int_1^2 \left(-1 + 3 + \frac{27}{x} - \frac{3}{x} \right) dx = \int_1^2 \left(2 + \frac{24}{x} \right) dx \\ &= 2x + 24 \ln x \Big|_1^2 = 2 + 24 \ln 2. \end{aligned}$$

Chapter 4

Main Theorems of Vector Calculus

4.1 Divergence and Curl

1. Let $\mathbf{F} = (y + z)^2\mathbf{i} + y\mathbf{j} + z\mathbf{k}$. Find the divergence and curl of $\mathbf{F}$.

$$\operatorname{div} \mathbf{F} = 0 + 1 + 1 = 2,$$

$$\operatorname{curl} \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ (y + z)^2 & y & z \end{vmatrix} = 0\mathbf{i} + 2(y + z)\mathbf{j} - 2(y + z)\mathbf{k}$$

2. (a) Let $\mathbf{F} = x \cos y \mathbf{i} + e^z \mathbf{j} + \cos y \mathbf{k}$. Find the curl and divergence of $\mathbf{F}$.

$$\operatorname{div} \mathbf{F} = \cos y + 0 + 0 = \cos y$$

$$\operatorname{curl} \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ x \cos y & e^z & \cos y \end{vmatrix} = (-\sin y - e^z)\mathbf{i} + x \sin y \mathbf{k}$$

(b) Is it true that $\nabla \times \mathbf{F}$ is always perpendicular to $\mathbf{F}$ for all C^1 vector fields $\mathbf{F}$ in $\mathbb{R}^3$? Explain why or why not.

For the vector field $\mathbf{F}$ from (a), we have

$$\mathbf{F} \cdot (\nabla \times \mathbf{F}) = x \cos y (-\sin y - e^z) + x \sin y \cos y = -xe^z \cos y$$

which is clearly not always zero. Therefore the statement is false.

3. Prove that if f and g are two C^2 functions on an open set $X \subset \mathbb{R}^3$, then $\nabla \cdot (\nabla f \times \nabla g) = 0$.

We first compute

$$\nabla f \times \nabla g = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x f & \partial_y f & \partial_z f \\ \partial_x g & \partial_y g & \partial_z g \end{vmatrix} = (f_y g_z - f_z g_y)\mathbf{i} - (f_x g_z - f_z g_x)\mathbf{j} + (f_x g_y - f_y g_x)\mathbf{k},$$

so that

$$\begin{aligned} \nabla \cdot (\nabla f \times \nabla g) &= (f_{xy}g_z + f_y g_{xz} - f_{xz}g_y - f_z g_{xy}) \\ &\quad - (f_{xy}g_z + f_x g_{yz} - f_{yz}g_x - f_z g_{xy}) \\ &\quad + (f_{xz}g_y + f_x g_{yz} - f_{yz}g_x - f_y g_{xz}) \\ &= 0. \end{aligned}$$

4. Let $\mathbf{F}$ be a 2-dimensional C^2 vector field on an open set U in the xy -plane. Let $P \in U$, and let $\mathbf{r}(t)$ be the field line of $\mathbf{F}$ that passes through P . Assume that:

- $\mathbf{F}$ has constant magnitude 1 on a neighbourhood of P ,
- the field line $\mathbf{r}(t)$ has curvature a at P .

Find the curl of $\mathbf{F}$ at P . Prove your answer.

Let $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j}$, and let $\mathbf{r}(t)$ be the field line as indicated. We use the curvature formula

$$a = \frac{|\mathbf{r}' \times \mathbf{r}''|}{|\mathbf{r}'|^3}. \quad (4.1.1)$$

For a field line, we have

$$\begin{aligned} \mathbf{r}'(t) &= \mathbf{F}(\mathbf{r}) = \langle F_1(\mathbf{r}), F_2(\mathbf{r}) \rangle \\ \mathbf{r}''(t) &= \langle \nabla F_1(\mathbf{r}) \cdot \mathbf{r}', \nabla F_2(\mathbf{r}) \cdot \mathbf{r}' \rangle \\ &= \langle \nabla F_1(\mathbf{r}) \cdot \mathbf{F}(\mathbf{r}), \nabla F_2(\mathbf{r}) \cdot \mathbf{F}(\mathbf{r}) \rangle. \end{aligned}$$

We have $|\mathbf{r}'| = |\mathbf{F}| = 1$ in a neighbourhood of P . The cross product is

$$\begin{aligned} \mathbf{r}' \times \mathbf{r}'' &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ F_1(\mathbf{r}) & F_2(\mathbf{r}) & 0 \\ \nabla F_1(\mathbf{r}) \cdot \mathbf{F}(\mathbf{r}) & \nabla F_2(\mathbf{r}) \cdot \mathbf{F}(\mathbf{r}) & 0 \end{vmatrix} \\ &= \left(F_1(\nabla F_2(\mathbf{r}) \cdot \mathbf{F}(\mathbf{r})) - F_2(\nabla F_1(\mathbf{r}) \cdot \mathbf{F}(\mathbf{r})) \right) \mathbf{k}. \end{aligned}$$

Therefore

$$|\mathbf{r}' \times \mathbf{r}''| = |F_1(F_{2,x}F_1 + F_{2,y}F_2) - F_2(F_{1,x}F_1 + F_{1,y}F_2)|, \quad (4.1.2)$$

with everything evaluated at $\mathbf{r}$. Now we use that $|\mathbf{F}| = 1$ is constant in the neighbourhood of P . Therefore in that neighbourhood,

$$\begin{aligned} 0 &= \frac{\partial}{\partial x} |\mathbf{F}|^2 = \frac{\partial}{\partial x} (F_1^2 + F_2^2) = F_1 F_{1,x} + F_2 F_{2,x}, \\ 0 &= \frac{\partial}{\partial y} |\mathbf{F}|^2 = \frac{\partial}{\partial y} (F_1^2 + F_2^2) = F_1 F_{1,y} + F_2 F_{2,y}, \end{aligned}$$

so that $F_1 F_{1,x} = -F_2 F_{2,x}$ and $F_2 F_{2,y} = -F_1 F_{1,y}$. Going back with this to (4.1.1) and (4.1.2), we get

$$\begin{aligned} a &= |\mathbf{r}' \times \mathbf{r}''| = |F_1(F_{2,x}F_1 - F_1 F_{1,y}) - F_2(-F_2 F_{2,x} + F_1 F_{1,y})| \\ &= |(F_1^2 + F_2^2)F_{2,x} - (F_1^2 + F_2^2)F_{1,y}| \\ &= |\mathbf{F}|^2 |\text{curl} \mathbf{F}| \\ &= |\text{curl} \mathbf{F}| \end{aligned}$$

Therefore $\text{curl} \mathbf{F}(P) = \pm a \mathbf{k}$, with the sign depending on which way the field line is turning.

4.2 Green's Theorem in Two Dimensions

1. The curve $x = \sin t$, $y = \pi^2 - t^2$, $-\pi \leq t \leq \pi$, is simple and closed. Use Green's Theorem to find the area bounded by this curve.

The orientation of the curve is clockwise (this can be checked in various ways, e.g. y is increasing for $-\pi \leq t \leq 0$ when $x \leq 0$, and decreasing for $0 \leq t \leq \pi$ when $x \geq 0$). Using Green's theorem, and taking the orientation into account, we compute the area:

$$\begin{aligned} \text{Area} &= - \int_C x dy = - \int_{-\pi}^{\pi} \sin t \cdot (-2t) dt = \int_{-\pi}^{\pi} 2t \sin t dt \\ &= -2t \cos t \Big|_{-\pi}^{\pi} + \int_{-\pi}^{\pi} 2 \cos t dt = -2(-2\pi) + 2 \sin t \Big|_{-\pi}^{\pi} = 4\pi. \end{aligned}$$

2. Use Green's Theorem to evaluate the line integral $\int_C (xy^2 + y^2) dx + (4xy + x^2y + \cos(y^2)) dy$, where C is the boundary of the region enclosed by the parabola $y = x^2$ and the line $y = x$, oriented counterclockwise.

Let $\mathbf{F}$ be the vector field $\mathbf{F} = (xy^2 + y^2)\mathbf{i} + (4xy + x^2y + \cos(y^2))\mathbf{j}$. Then

$$(\nabla \times \mathbf{F}) \cdot \mathbf{k} = \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = (4y + 2xy) - (2xy + 2y) = 2y,$$

so that

$$\begin{aligned} \int_C (xy^2 + y^2) dx + (4xy + x^2y + \cos(y^2)) dy &= \int_0^1 \int_{x^2}^x 2y dy dx = \int_0^1 y^2 \Big|_{x^2}^x dx \\ &= \int_0^1 (x^2 - x^4) dx = \frac{x^3}{3} - \frac{x^5}{5} \Big|_0^1 = \frac{1}{3} - \frac{1}{5} = \frac{2}{15}. \end{aligned}$$

3. Use Green's Theorem to evaluate the integral $\int_C ydx + xdy$, where C is the arc of the circle $(x-1)^2 + y^2 = 1$ from $(0,0)$ to $(1,1)$ in the first quadrant of the xy -plane.

Since $\mathbf{curl} \mathbf{F} \cdot \mathbf{k} = \frac{\partial x}{\partial x} - \frac{\partial y}{\partial y} = 0$, we have $\int_C = \int_{C_1} + \int_{C_2}$, where C_1 is the line segment from $(0,0)$ to $(1,0)$ and C_2 is the line segment from $(1,0)$ to $(1,1)$. On C_1 , $y = dy = 0$, so $\int_{C_1} = 0$. On C_2 , $dx = 0$ and $x = 1$, so that

$$\int_{C_2} ydx + xdy = \int_0^1 1dy = y \Big|_{y=0}^1 = 1.$$

Therefore $\int_C ydx + xdy = \int_{C_2} ydx + xdy = 1$.

Optional: For comparison, here is an alternative solution where we compute the integral directly. Parametrize the arc by $x = 1 - \cos \theta$, $y = \sin \theta$, $0 \leq \theta \leq \pi/2$. Then

$$\begin{aligned} \int_C ydx + xdy &= \int_0^{\pi/2} (\sin \theta)(\sin \theta) + (1 - \cos \theta)(\cos \theta) d\theta \\ &= \int_0^{\pi/2} (\sin^2 \theta + \cos \theta - \cos^2 \theta) d\theta = \int_0^{\pi/2} (\cos \theta - \cos(2\theta)) d\theta \\ &= \sin \theta - \frac{\sin(2\theta)}{2} \Big|_0^{\pi/2} = 1 - 0 = 1. \end{aligned}$$

4. Use Green's Theorem to evaluate the integral $\int_C y^3 dx - x^3 dy$, where C is the circle $x^2 + y^2 = 4$ oriented counterclockwise.

Let D be the disc $x^2 + y^2 \leq 4$, then

$$\begin{aligned}\int_C y^3 dx - x^3 dy &= \iint_D \left(\frac{\partial(-x^3)}{\partial x} - \frac{\partial(y^3)}{\partial y} \right) dx dy \\ &= \iint_D (-3x^2 - 3y^2) dx dy \\ &= \int_0^{2\pi} \int_0^2 -3r^2 \cdot r dr d\theta \\ &= 2\pi \cdot \frac{-3r^4}{4} \Big|_0^2 = 2\pi \cdot (-12) = -24\pi.\end{aligned}$$

5. Let $\mathbf{F} = (e^{x^2} + 4x^2y)\mathbf{i} + (e^{-4y^2} - y^2x + 4x)\mathbf{j}$. Use Green's Theorem to evaluate the integral $\int_C \mathbf{F} \cdot d\mathbf{r}$, where C is the boundary of the rectangle $0 \leq x \leq 1$, $0 \leq y \leq 3$, oriented counterclockwise.

We have

$$\mathbf{curl} \mathbf{F} \cdot \mathbf{k} = \frac{\partial}{\partial x}(e^{-4y^2} - y^2x + 4x) - \frac{\partial}{\partial y}(e^{x^2} + 4x^2y) = -y^2 + 4 - 4x^2.$$

Thus by Green's Theorem, the integral in question is equal to

$$\begin{aligned}\int_0^1 \int_0^3 (4 - 4x^2 - y^2) dy dx &= \int_0^1 (4y - 4x^2y - \frac{y^3}{3}) \Big|_{y=0}^3 dx \\ &= \int_0^1 (12 - 12x^2 - 9) dx = \int_0^1 (3 - 12x^2) dx = (3x - 4x^3) \Big|_0^1 = 3 - 4 = -1.\end{aligned}$$

6. Let $\mathbf{F} = (y^3 + 4x)\mathbf{i} + (3x - 4x^3)\mathbf{j}$. Find the simple, closed, oriented counterclockwise curve C for which the integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ is maximized (in the sense that $\int_C \mathbf{F} \cdot d\mathbf{r} \geq \int_{C'} \mathbf{F} \cdot d\mathbf{r}$ for any other simple, closed, oriented counterclockwise curve C'). If no such curve exists, explain why.

By Green's theorem, we have

$$\begin{aligned}\int_C (y^3 + 4x) dx + (3x - 4x^3) dy &= \iint_D \left(\frac{\partial(3x - 4x^3)}{\partial x} - \frac{\partial(y^3 + 4x)}{\partial y} \right) dx dy \\ &= \iint_D (3 - 12x^2 - 3y^2) dx dy,\end{aligned}$$

where D is the region bounded by C . The function $3 - 12x^2 - 3y^2 = 3(1 - 4x^2 - y^2)$ is positive inside the ellipse $4x^2 + y^2 = 1$ and negative outside it. Hence to maximize the last integral we should take C to be the ellipse $4x^2 + y^2 = 1$.

7. Let

$$g(x) = \begin{cases} 0 & \text{if } x < 0 \text{ or } x > 1, \\ x^3 \sin(\frac{\pi}{x}) & \text{if } 0 < x \leq 1. \end{cases} \quad (4.2.1)$$

Prove that the graph of $y = g(x)$, $x \in \mathbb{R}$, is a smooth curve, with smooth parametrization $\mathbf{r}(x) = x\mathbf{i} + g(x)\mathbf{j}$. (This means that $\mathbf{r}$ is C^1 and $\mathbf{r}'$ is never zero. The main part is to prove that $\mathbf{r}'(x)$ exists, is continuous, and is nonzero at $x = 0$.) Prove further that the graph of $y = g(x)$ for $-1 \leq x \leq 1$ has finite length.

We parametrize the graph by $\mathbf{r}(x) = x\mathbf{i} + g(x)\mathbf{j}$ as suggested. We first note that this graph is a continuous curve since

$$|g(x)| \leq |x^3| \rightarrow 0 = g(0) \text{ as } x \rightarrow 0.$$

Next, $\mathbf{r}'(x) = \mathbf{i} + g'(x)\mathbf{j}$. This is always nonzero as long as $g'(x)$ exists, but we do need to prove existence and continuity. To prove that $g'(0)$ exists, we need to examine the limit

$$\lim_{x \rightarrow 0} \frac{g(x) - 0}{x}.$$

But since $|g(x)| \leq |x^3|$, we have $|g(x)/x| \leq x^2 \rightarrow 0$ as $x \rightarrow 0$. Thus $g'(0) = 0$.

We now prove that $g'(x)$ is continuous. For $x < 0$, we have $g'(x) = 0$, so that g' is continuous for $x < 0$ and continuous at 0 from the left. For $x > 0$,

$$g'(x) = 3x^2 \sin(\frac{\pi}{x}) + x^3 \sin(\frac{\pi}{x}) \cdot (-\frac{\pi}{x^2}) = (3x^2 - \pi x) \sin(\frac{\pi}{x}).$$

This is clearly continuous at all $x > 0$, and

$$|g'(x)| \leq 3x^2 + \pi x \leq 7x \text{ for } 0 < x < 1. \quad (4.2.2)$$

Since $7x \rightarrow 0$ as $x \rightarrow 0$, $g'(x)$ is also continuous from the right at $x = 0$. Finally, the last inequality also gets us an estimate on the length of the graph for $-1 \leq x \leq 1$:

$$\begin{aligned} \text{length} &= \int_{-1}^1 \sqrt{1 + |g'(x)|^2} dx = 1 + \int_0^1 \sqrt{1 + |g'(x)|^2} dx \\ &\leq 1 + \int_0^1 \sqrt{1 + 49} dx \leq 1 + \int_0^1 8 dx = 9. \end{aligned}$$

We could get a better estimate, but here we just need to know that the length is finite.

8. Let $g(x)$ be the function in (4.2.1). Let C be the closed curve, oriented counterclockwise, consisting of:

- the graph of $y = g(x)$ for $-1 \leq x \leq 1$, oriented in the direction of decreasing x ,

- the line segment from $(-1, 0)$ to $(-1, -2)$,
- the line segment from $(-1, -2)$ to $(1, -2)$,
- the line segment from $(1, -2)$ to $(1, 0)$.

Let also $\mathbf{F}$ be a C^1 vector field on $\mathbb{R}^2$, and let D be the planar region enclosed by C . (This region cannot be split into finitely many regions that are both x -simple and y -simple.) Prove that Green's Theorem remains valid in this setting, as follows.

(a) For $n = 1, 2, \dots$, let C_n be the closed curve, oriented counterclockwise, consisting of:

- the graph of $y = g(x)$ for $n^{-1} \leq x \leq 1$, oriented in the direction of decreasing x ,
- the line segment from $(n^{-1}, 0)$ to $(-1, 0)$,
- the line segment from $(-1, 0)$ to $(-1, -2)$,
- the line segment from $(-1, -2)$ to $(1, -2)$,
- the line segment from $(1, -2)$ to $(1, 0)$.

Prove that $\oint_{C_n} \mathbf{F} \cdot d\mathbf{r} \rightarrow \oint_C \mathbf{F} \cdot d\mathbf{r}$ as $n \rightarrow \infty$.

After cancelling out the parts where C and C_n overlap, and reversing the orientation, the difference $\oint_{C_n} \mathbf{F} \cdot d\mathbf{r} - \oint_C \mathbf{F} \cdot d\mathbf{r}$ is equal to

$$\int_{C_{1,n}} \mathbf{F} \cdot d\mathbf{r} - \oint_{C_{2,n}} \mathbf{F} \cdot d\mathbf{r}, \quad (4.2.3)$$

where:

- $C_{1,n}$ is the graph of $y = g(x)$ for $0 \leq x \leq n^{-1}$,
- $C_{2,n}$ is the line segment from $(0, 0)$ to $(n^{-1}, 0)$,

both oriented in the direction of increasing x . (If you prefer to keep the original orientation with decreasing x , you need to switch the signs above.) We need to prove that (4.2.3) goes to 0 as $n \rightarrow \infty$. We will actually prove that this is true for each integral in (4.2.3) separately.

Since $\mathbf{F}$ is C^1 everywhere, it is bounded on any bounded and closed rectangle R . Taking R to be any rectangle containing C and C_n , say $R = \{(x, y) : |x| \leq 10, |y| \leq 10\}$, we see that there is a constant $M > 0$ such that $|F_1| \leq M$ and $|F_2| \leq M$ on C and C_n . Therefore

$$\begin{aligned} \left| \int_{C_{1,n}} \mathbf{F} \cdot d\mathbf{r} \right| &= \left| \int_0^{1/n} (F_1 + F_2 g'(x)) dx \right| \leq \int_0^{1/n} (|F_1| + |F_2| |g'(x)|) dx \\ &\leq \int_0^{1/n} 8M dx = 8M/n, \end{aligned}$$

which goes to 0 as $n \rightarrow \infty$. We used (4.2.2) to estimate $|g'| \leq 7$. The second integral is easier:

$$\begin{aligned} \left| \int_{C_{2,n}} \mathbf{F} \cdot d\mathbf{r} \right| &= \left| \int_0^{1/n} F_1 dx \right| \leq \int_0^{1/n} |F_1| dx \\ &\leq \int_0^{1/n} M dx = M/n, \end{aligned}$$

which also goes to 0 as $n \rightarrow \infty$.

(b) For $n = 1, 2, \dots$, let D_n be the planar region enclosed by C_n . Prove that

$$\iint_{D_n} (\nabla \times \mathbf{F}) \cdot \mathbf{k} dA \rightarrow \iint_D (\nabla \times \mathbf{F}) \cdot \mathbf{k} dA \text{ as } n \rightarrow \infty.$$

We have

$$\begin{aligned} &\iint_{D_n} (\nabla \times \mathbf{F}) \cdot \mathbf{k} dA - \iint_D (\nabla \times \mathbf{F}) \cdot \mathbf{k} dA \\ &= \iint_{D_n \setminus D} (\nabla \times \mathbf{F}) \cdot \mathbf{k} dA - \iint_{D \setminus D_n} (\nabla \times \mathbf{F}) \cdot \mathbf{k} dA. \end{aligned}$$

We will prove that the last two integrals go to 0 as $n \rightarrow \infty$. Since both $D_n \setminus D$ and $D \setminus D_n$ are subsets of the set

$$D'_n := \left\{ (x, y) : 0 \leq x \leq \frac{1}{n}, |y| \leq 2 \right\},$$

we have

$$\left| \iint_{D_n \setminus D} (\nabla \times \mathbf{F}) \cdot \mathbf{k} dA \right| \leq \iint_{D_n \setminus D} |(\nabla \times \mathbf{F}) \cdot \mathbf{k}| dA \leq \iint_{D'_n} |(\nabla \times \mathbf{F}) \cdot \mathbf{k}| dA$$

and similarly for $D \setminus D_n$. So, it suffices to prove that the last integral above goes to 0 as $n \rightarrow \infty$. Since $\mathbf{F}$ is C^1 everywhere, its partial derivatives are bounded on any bounded and closed rectangle R . Taking R to be any rectangle containing C and C_n , say $R = \{(x, y) : |x| \leq 10, |y| \leq 10\}$, we see that there is a constant $M' > 0$ such that

$$\left| \frac{\partial F_1}{\partial y} \right| \leq M', \quad \left| \frac{\partial F_2}{\partial x} \right| \leq M' \quad \text{on } R.$$

Since $D'_n \subset R$ for all n , we have

$$\iint_{D'_n} |(\nabla \times \mathbf{F}) \cdot \mathbf{k}| dA \leq \iint_{D'_n} 2M' dA \leq 2M' \cdot 4n^{-1} = 8M'n^{-1},$$

which goes to 0 as claimed.

(c) Assuming that Green's Theorem holds for each C_n and D_n , conclude that it also holds for C and D .

Applying Green's Theorem to each C_n and D_n , we get

$$\oint_{C_n} \mathbf{F} \cdot d\mathbf{r} = \iint_{D_n} (\nabla \times \mathbf{F}) \cdot \mathbf{k} dA.$$

Taking the limit as $n \rightarrow \infty$ on both sides, and using (a) and (b), we get Green's Theorem for C and D , as claimed.

4.3 The Divergence Theorem

1. Compute the flux of the vector field $\mathbf{F} = x\mathbf{i} + y\mathbf{j} - x\mathbf{k}$ out of a closed surface consisting of the part of the cylinder $x^2 + y^2 = 4$ between the planes $z = 0$ and $z = 3$, together with the discs at the top and bottom, with the normal vector pointing outward.

Direct computation: We write the flux integral as $\iint_S \mathbf{F} \cdot \mathbf{N} dS = \iint_{S_1} \mathbf{F} \cdot \mathbf{N} dS + \iint_{S_2} \mathbf{F} \cdot \mathbf{N} dS + \iint_{S_3} \mathbf{F} \cdot \mathbf{N} dS$, where S_1 is the disc at the top, S_2 is the disc at the bottom, and S_3 is the surface of the cylinder.

- On S_1 , we have $\mathbf{N} = \mathbf{k}$, so that

$$\iint_{S_1} \mathbf{F} \cdot \mathbf{N} dS = \iint_{S_1} (-x) dS.$$

By symmetry, this integral is 0.

- On S_2 , we have $\mathbf{N} = -\mathbf{k}$, so that

$$\iint_{S_2} \mathbf{F} \cdot \mathbf{N} dS = \iint_{S_2} x dS.$$

This integral is also 0 by symmetry.

- On S_3 , the unit normal $\mathbf{N}$ points in the direction of $x\mathbf{i} + y\mathbf{j}$, so that

$$\mathbf{N} = \frac{x\mathbf{i} + y\mathbf{j}}{\sqrt{x^2 + y^2}} = \frac{1}{2}(x\mathbf{i} + y\mathbf{j}).$$

Thus $\mathbf{F} \cdot \mathbf{N} = \frac{1}{2}(x^2 + y^2) = 2$, and

$$\iint_{S_3} \mathbf{F} \cdot \mathbf{N} dS = 2(\text{area of } S_3) = 2 \cdot 4\pi \cdot 3 = 24\pi.$$

Adding up, we get that $\iint_{\mathcal{S}} \mathbf{F} \cdot \mathbf{N} dS = 24\pi$.

Alternative solution using Divergence Theorem: We have

$$\operatorname{div} \mathbf{F} = \frac{\partial}{\partial x}x + \frac{\partial}{\partial y}y + \frac{\partial}{\partial z}(-x) = 1 + 1 + 0 = 2.$$

Applying the Divergence Theorem to the closed surface $\mathcal{S}$, we get

$$\iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S} = \iiint_D 2dV = 2(\text{volume of } D) = 2(12\pi) = 24\pi$$

where D is the part of the solid cylinder between the given planes.

2. Evaluate $\iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S}$, if $\mathbf{F} = (e^{y^2} - x^3)\mathbf{i} + (y^2 + e^{2z^2})\mathbf{j} + e^{x^2}\mathbf{k}$ and $\mathcal{S}$ is the sphere $x^2 + y^2 + z^2 = 1$, oriented so that the normal vector points outward.

We use the Divergence Theorem, with $\mathcal{S}$ as above and $D = \{(x, y, z) : x^2 + y^2 + z^2 \leq 1\}$:

$$\begin{aligned} \iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S} &= \iiint_D \nabla \cdot \mathbf{F} dV = \iiint_D (-3x^2 + 2y) dV \\ &= \iiint_D (-3x^2) dV \quad (\text{since } \iiint_D y dV = 0) \\ &= - \iiint_D (x^2 + y^2 + z^2) dV \quad (\text{by symmetry in the } x, y, z \text{ variables}) \\ &= - \int_0^1 \int_0^{2\pi} \int_0^\pi R^2 R^2 \sin \phi d\phi d\theta dR \\ &= - \int_0^1 R^4 dR \cdot 2\pi \cdot \int_0^\pi \sin \phi d\phi \\ &= - \left. \frac{R^5}{5} \right|_0^1 \cdot 2\pi \cdot (-\cos \phi) \Big|_0^\pi \\ &= -\frac{1}{5} \cdot 2\pi \cdot 2 = -\frac{4\pi}{5}. \end{aligned}$$

3. Evaluate $\iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S}$, if $\mathbf{F} = (x^2 + y^2)\mathbf{i} + z^2\mathbf{j} + \mathbf{k}$ and $\mathcal{S}$ is the half-sphere $x^2 + y^2 + z^2 = 1$, $z \geq 0$, oriented so that the normal vector points away from the origin. (Hint: complete the sphere to a closed surface, then apply the Divergence Theorem.)

We apply the Divergence Theorem to the region $R = \{(x, y, z) : x^2 + y^2 + z^2 \leq 1, z \geq 0\}$. The divergence of $\mathbf{F}$ is $2x$, so

$$\iiint_R \nabla \cdot \mathbf{F} dV = \iiint_R 2x dV = 0,$$

by symmetry. The oriented boundary of R consists of $\mathcal{S}$ and $D := \{(x, y, 0) : x^2 + y^2 \leq 1\}$, with the normal on D equal to $-\mathbf{k}$. So

$$\begin{aligned} 0 &= \iiint_R \nabla \cdot \mathbf{F} \, dV = \iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S} + \iint_D \mathbf{F} \cdot d\mathbf{S}, \\ \iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S} &= - \iint_D \mathbf{F} \cdot (-\mathbf{k}) dS = \iint_D 1 dS = \pi. \end{aligned}$$

4. Evaluate the integral $\iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S}$, if $\mathbf{F} = y^2 \sin y \cos z \mathbf{i} + x^2 \cos x \sin z \mathbf{j} + (z^2 + 1) \mathbf{k}$ and $\mathcal{S}$ is the surface consisting of the cylinder $x^2 + y^2 = 1$, $0 \leq z \leq 2$, together with the disc at the top, oriented so that the normal vector points away from the origin.

We apply the Divergence Theorem to the region $R = \{(x, y, z) : x^2 + y^2 \leq 1, 0 \leq z \leq 2\}$. The divergence of $\mathbf{F}$ is $2z$, so

$$\begin{aligned} \iiint_R \nabla \cdot \mathbf{F} &= \iiint_R 2z \, dV \\ &= \iint_{x^2+y^2 \leq 1} \int_0^2 2z \, dz \, dy \, dx \\ &= \iint_{x^2+y^2 \leq 1} z^2 \Big|_0^2 \, dy \, dx \\ &= \iint_{x^2+y^2 \leq 1} 4 \, dy \, dx = 4\pi. \end{aligned}$$

The oriented boundary of R consists of $\mathcal{S}$ and the disc at the bottom $D := \{(x, y, 0) : x^2 + y^2 \leq 1\}$, with the normal on D equal to $-\mathbf{k}$. On D , we have $z = 0$, so that $\mathbf{F} \cdot \mathbf{N} = -1$ and

$$\iint_D \mathbf{F} \cdot d\mathbf{S} = \iint_{x^2+y^2 \leq 1} (-1) dA = -\pi.$$

Therefore, by the Divergence Theorem,

$$\begin{aligned} 4\pi &= \iiint_R \nabla \cdot \mathbf{F} = \iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S} - \pi, \\ \iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S} &= 4\pi - (-\pi) = 5\pi. \end{aligned}$$

5. Suppose that $g(x, y, z)$ is a smooth function on all of $\mathbb{R}^3$, that the level surface $\mathcal{S} = \{(x, y, z) : g(x, y, z) = 0\}$ is closed, and that $\nabla g \neq 0$ on $\mathcal{S}$.
- (a) Prove that there is a point where the vector field $\mathbf{F} = \nabla g$ has nonzero divergence.

Suppose for contradiction that $\nabla \cdot \mathbf{F} \equiv 0$. By the Divergence Theorem, it follows that

$$0 = \iiint_R \nabla \cdot \mathbf{F} = \iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_S \mathbf{F} \cdot \mathbf{N} dS,$$

where $\mathbf{N}$ is the unit normal vector pointing outward. Since $\mathcal{S}$ is a level surface of g with $\nabla g \neq 0$, we have $\mathbf{N} = \pm \frac{\nabla g}{|\nabla g|}$, with the choice of $\pm$ sign depending on whether ∇g points inward or outward. Since both $\mathbf{N}$ and $\frac{\nabla g}{|\nabla g|}$ are continuous on $\mathcal{S}$, the $\pm$ sign is in fact the same for all points on $\mathcal{S}$. So

$$0 = \iint_S \mathbf{F} \cdot \mathbf{N} dS = \pm \iint_S \nabla g \cdot \frac{\nabla g}{|\nabla g|} dS = \pm \iint_S |\nabla g| dS.$$

This is a contradiction, since $\nabla g \neq 0$ on $\mathcal{S}$.

(b) Does there have to exist a point where $\mathbf{F} = \nabla g$ has positive divergence? If yes, explain why. If no, give a counterexample.

No. For example, let $g(x, y, z) = 4 - x^2 - y^2 - z^2$ with $\mathcal{S} = \{(x, y, z) : x^2 + y^2 + z^2 = 4\}$. Then $\mathbf{F} = \nabla g = -2x\mathbf{i} - 2y\mathbf{j} - 2z\mathbf{k} \neq 0$ on $\mathcal{S}$, and $\nabla \cdot \mathbf{F} \equiv -6 < 0$.

6. Assume that a solid D in $\mathbb{R}^3$ and its boundary $\mathcal{S}$ satisfy the assumptions of the Divergence Theorem. Let $f, g : \mathbb{R}^3 \rightarrow \mathbb{R}$ be C^2 functions. Prove that

$$\iint_S (f\nabla g - g\nabla f) \cdot d\mathbf{S} = \iiint_D (f\nabla^2 g - g\nabla^2 f) dV.$$

Here $\nabla^2 f = \nabla \cdot (\nabla f) = f_{xx} + f_{yy} + f_{zz}$.

Let $\mathbf{F} = f\nabla g - g\nabla f$. We have

$$\nabla \cdot (f\nabla g) = \frac{\partial(fg_x)}{\partial x} + \frac{\partial(fg_y)}{\partial y} + \frac{\partial(fg_z)}{\partial z}$$

$$= (f_x g_x + f g_{xx}) + (f_y g_y + f g_{yy}) + (f_z g_z + f g_{zz}) = \nabla f \cdot \nabla g + f \nabla^2 g,$$

and similarly, $\nabla \cdot (g\nabla f) = \nabla g \cdot \nabla f + g \nabla^2 f$. Hence

$$\begin{aligned} \nabla \cdot \mathbf{F} &= \nabla \cdot (f\nabla g - g\nabla f) = \nabla f \cdot \nabla g + f \nabla^2 g - \nabla g \cdot \nabla f - g \nabla^2 f \\ &= f \nabla^2 g - g \nabla^2 f, \end{aligned}$$

which together with the Divergence Theorem implies the claimed equality.

7. Let $\mathbf{F}$ be an incompressible smooth vector field such that its field lines are the hyperbolas

$$x^2 + y^2 = c(z^2 + 1), \quad ax + by = 0,$$

for $c \geq 0$ and $a, b \in \mathbb{R}$, all oriented towards increasing z . (For $c = 0$, this is the z -axis.) Suppose that

$$\iint_{D_A} \mathbf{F} \cdot \mathbf{k} \, dS = \ln(A + 1)$$

for all $A \geq 0$, where $D_A = \{(x, y, 0) : x^2 + y^2 \leq A^2\}$.

What is $\iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S}$, if $\mathcal{S}$ is the disc $\{(x, y, 4) : x^2 + y^2 \leq 9\}$ with the normal vector pointing up? Justify your answer.

Let $\mathcal{S}' = \mathcal{S} \cup \mathcal{S}_1 \cup \mathcal{S}_2$, where:

- $\mathcal{S}_1$ is the part of the hyperboloid $x^2 + y^2 = \frac{9}{17}(z^2 + 1)$ that lies between the planes $z = 0$ and $z = 4$, oriented with the normal vector pointing away from the z -axis,
- $\mathcal{S}_2$ is the disc $x^2 + y^2 = \frac{9}{17}$ in the plane $z = 0$, with $\mathbf{N} = -\mathbf{k}$.

Then $\mathcal{S}'$ is an oriented closed surface bounding a region $R \subset \mathbb{R}^3$. By the Divergence Theorem,

$$\iint_{\mathcal{S}'} \mathbf{F} \cdot d\mathbf{S} = \iiint_D \nabla \cdot \mathbf{F} \, dV = 0,$$

where we used that $\mathbf{F}$ is incompressible. Therefore

$$0 = \iint_{\mathcal{S}'} \mathbf{F} \cdot d\mathbf{S} = \iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S} + \iint_{\mathcal{S}_1} \mathbf{F} \cdot d\mathbf{S} + \iint_{\mathcal{S}_2} \mathbf{F} \cdot d\mathbf{S}.$$

From the assumption above, we have

$$\iint_{\mathcal{S}_2} \mathbf{F} \cdot d\mathbf{S} = - \iint_{\sqrt{9/17}} \mathbf{F} \cdot \mathbf{k} \, dS = -\ln(\sqrt{9/17} + 1).$$

Also, since the field lines of $\mathbf{F}$ passing through points of the hyperboloid $x^2 + y^2 = \frac{9}{17}(z^2 + 1)$ lie on that hyperboloid, it follows that the vectors of $\mathbf{F}$ are tangent to the hyperboloid, therefore perpendicular to the vector normal to the hyperboloid. Therefore

$$\iint_{\mathcal{S}_1} \mathbf{F} \cdot d\mathbf{S} = 0.$$

Putting everything together, we get

$$\iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S} = - \iint_{\mathcal{S}_2} \mathbf{F} \cdot d\mathbf{S} = \ln(\sqrt{9/17} + 1).$$

4.4 Stokes's Theorem

1. Evaluate $\iint_{\mathcal{S}} \mathbf{curl} \mathbf{F} \cdot d\mathbf{S}$, if $\mathbf{F} = (x^4 - 2y)\mathbf{i} + (\cos y)\mathbf{j} + z \sin(xy)\mathbf{k}$ and $\mathcal{S}$ is the part of the paraboloid $z = 9 - x^2 - y^2$ that lies above the xy -plane.

We apply Stokes's Theorem twice, first to $\mathcal{S}$ and its boundary $C = \{(x, y, z) : x^2 + y^2 = 9, z = 0\}$ oriented counterclockwise, then to C and $D = \{(x, y, z) : x^2 + y^2 \leq 9, z = 0\}$:

$$\iint_{\mathcal{S}} \mathbf{curl} \mathbf{F} \cdot d\mathbf{S} = \int_C \mathbf{F} \cdot d\mathbf{r} = \iint_D \mathbf{curl} \mathbf{F} \cdot \mathbf{k} dA.$$

On D , we have $\mathbf{curl} \mathbf{F} \cdot \mathbf{k} = \frac{\partial(\cos y)}{\partial x} - \frac{\partial(x^4 - 2y)}{\partial y} = 2$, so that the last integral is

$$\iint_D 2dA = 2 \cdot (\text{area of } D) = 2 \cdot 9\pi = 18\pi.$$

2. Use Stokes's Theorem to evaluate $\int_C (x-z)dx + (x+y)dy + (y+z)dz$, if C is the ellipse where the plane $z = y$ intersects the cylinder $x^2 + y^2 = 1$, oriented counterclockwise as viewed from above.

Let $\mathbf{F} = (x - z)\mathbf{i} + (x + y)\mathbf{j} + (y + z)\mathbf{k}$. By Stokes's Theorem, we have

$$\int_C \mathbf{F} \cdot d\mathbf{s} = \iint_{\mathcal{S}} (\nabla \times \mathbf{F}) \cdot d\mathbf{S},$$

where $\mathcal{S}$ is the region in the plane $z = y$ bounded by $\mathcal{C}$. We parametrize $\mathcal{S}$ as $\{(x, y, y) : (x, y) \in D\}$, where D is the disc $x^2 + y^2 = 1$ in the xy -plane. Then $\mathbf{n} = -\mathbf{j} + \mathbf{k}$ and $\nabla \times \mathbf{F} = \mathbf{i} - \mathbf{j} + \mathbf{k}$ (you should verify this!), so that

$$\iint_{\mathcal{S}} (\nabla \times \mathbf{F}) \cdot d\mathbf{S} = \iint_D 2dxdy = 2(\text{area of } D) = 2\pi.$$

3. Evaluate the integral $\int_C \mathbf{F} \cdot d\mathbf{r}$, where C is the curve of intersection of the cylinder $x^2 + y^2 = 1$ and the paraboloid $z = 100 - (x-2)^2 - (y+4)^2$, oriented counterclockwise when viewed from above, and $\mathbf{F} = (3x^2y + y^3)\mathbf{i} + z^4\mathbf{k}$.

We apply Stokes's Theorem to C and to the surface $\mathcal{S}$ consisting of:

- the part $\mathcal{S}_1$ of the cylinder $x^2 + y^2 = 1$ between C and the xy -plane, with normal pointing inward,
- and the disc $D = \{(x, y, z) : x^2 + y^2 \leq 1, z = 0\}$, with normal $\mathbf{k}$.

By Stokes's Theorem, $\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \mathbf{curl} \mathbf{F} \cdot d\mathbf{S}$. The curl of $\mathbf{F}$ is

$$\mathbf{curl} \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ 3x^2y + y^3 & 0 & z^4 \end{vmatrix} = -(3x^2 + 3y^2)\mathbf{k}.$$

On the cylindrical surface, $\mathbf{curl} \mathbf{F} \cdot \mathbf{N} = 0$. Therefore

$$\begin{aligned} \int_C \mathbf{F} \cdot d\mathbf{r} &= - \int_D (3x^2 + 3y^2) dA \\ &= - \int_0^1 \int_0^{2\pi} 3r^2 \cdot r \, d\theta dr \\ &= -2\pi \cdot \frac{3r^4}{4} \Big|_0^1 = -\frac{3\pi}{2}. \end{aligned}$$

4. Let R be the cube whose one face is the square in the xy -plane with vertices $(\pm 1, \pm 1, 0)$, and the opposite face lies in the plane $z = 2$ and has vertices $(\pm 1, \pm 1, 2)$. Let

$$\mathbf{F} = xyz^2\mathbf{i} + xy^2\mathbf{j} + x^3yz\mathbf{k}.$$

Evaluate $\iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S}$, if $\mathcal{S}$ is the surface consisting of the 4 vertical faces and the top of R (without the bottom), oriented so that the normal vector points away from the origin.

We apply Stokes's Theorem twice:

$$\begin{aligned} \iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} &= \int_C \mathbf{F} \cdot d\mathbf{r} \\ &= \iint_D (\nabla \times \mathbf{F}) \cdot d\mathbf{S}, \end{aligned}$$

where D is the square with vertices $(\pm 1, \pm 1, 0)$ in the xy -plane, oriented with $\mathbf{N} = \mathbf{k}$, and C is the common oriented boundary of both $\mathcal{S}$ and D . The curl of $\mathbf{F}$ is

$$\mathbf{curl} \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ xyz^2 & xy^2 & x^3yz \end{vmatrix} = x^3z\mathbf{i} - (3x^2yz - 2xyz)\mathbf{j} + (y^2 - xz^2)\mathbf{k}.$$

In the xy -plane, we have $z = 0$, so that $\mathbf{curl} \mathbf{F} = y^2\mathbf{k}$. Therefore

$$\begin{aligned} \iint_D (\nabla \times \mathbf{F}) \cdot \mathbf{k} dA &= \int_D y^2 dA \\ &= \int_{-1}^1 \int_{-1}^1 y^2 \, dx dy = \int_{-1}^1 2y^2 dy = \frac{2y^3}{3} \Big|_{-1}^1 = \frac{4}{3}. \end{aligned}$$

Chapter 5

Differential Forms

5.1 Introduction to Differential Forms

1. Let $\mathbf{u} = (1, -1, 0, 5)$, $\mathbf{v} = (3, 2, 6, -1)$, $\mathbf{w} = (5, -2, 3, 1)$. Evaluate $dx_1 \wedge dx_3 \wedge dx_4(\mathbf{u}, \mathbf{v}, \mathbf{w})$.

$$dx_1 \wedge dx_3 \wedge dx_4(\mathbf{u}, \mathbf{v}, \mathbf{w}) = \begin{vmatrix} 1 & 3 & 5 \\ 0 & 6 & 3 \\ 5 & -1 & 1 \end{vmatrix} = 6 + 45 - 150 + 3 = -96$$

2. Let $\mathbf{u} = (2, 1, 4, -1)$, $\mathbf{v} = (1, 3, 5, 0)$, $\mathbf{w} = (2, 7, -2, 3)$. Evaluate:

(a)

$$\begin{aligned} dx_1 \wedge dx_2 \wedge dx_4(\mathbf{u}, \mathbf{v}, \mathbf{w}) &= \begin{vmatrix} 2 & 1 & 2 \\ 1 & 3 & 7 \\ -1 & 0 & 3 \end{vmatrix} \\ &= (-1) \begin{vmatrix} 1 & 2 \\ 3 & 7 \end{vmatrix} + 3 \begin{vmatrix} 2 & 1 \\ 1 & 3 \end{vmatrix} \\ &= (-1)(1) + 3(5) \\ &= 14, \end{aligned}$$

where we expanded the 3×3 determinant along the third row.

(b)

$$\begin{aligned} dx_2 \wedge dx_3(\mathbf{u}, \mathbf{v}) - 2dx_1 \wedge dx_4(\mathbf{u}, \mathbf{w}) &= \begin{vmatrix} 1 & 3 \\ 4 & 5 \end{vmatrix} - 2 \begin{vmatrix} 2 & 2 \\ -1 & 3 \end{vmatrix} \\ &= -7 - 2(8) \\ &= -23. \end{aligned}$$

3. Calculate and simplify the wedge product $\phi \wedge \psi$. Write the dx_i 's in the answers in increasing subscript order.

(a) $\phi = dx_1 \wedge dx_3 \wedge dx_4$, $\psi = a_1 dx_1 + a_2 dx_2 + a_3 dx_3$.

$$\begin{aligned}
 \phi \wedge \psi &= (dx_1 \wedge dx_3 \wedge dx_4) \wedge (a_1 dx_1 + a_2 dx_2 + a_3 dx_3) \\
 &= a_1(dx_1 \wedge dx_3 \wedge dx_4 \wedge dx_1) + a_2(dx_1 \wedge dx_3 \wedge dx_4 \wedge dx_2) \\
 &\quad + a_3(dx_1 \wedge dx_3 \wedge dx_4 \wedge dx_3) \\
 &= a_1(-1)^2(dx_1 \wedge dx_1 \wedge dx_3 \wedge dx_4) + a_2(-1)^2(dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4) \\
 &\quad + a_3(-1)(dx_1 \wedge dx_3 \wedge dx_3 \wedge dx_4) \\
 &= a_2(dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4)
 \end{aligned}$$

(b) $\phi = dx_2 \wedge dx_3 + dx_1 \wedge dx_4$, $\psi = 4dx_1 \wedge dx_2 - dx_1 \wedge dx_3 + 5dx_2 \wedge dx_3$.

$$\begin{aligned}
 \phi \wedge \psi &= (dx_2 \wedge dx_3 + dx_1 \wedge dx_4) \wedge (4dx_1 \wedge dx_2 - dx_1 \wedge dx_3 + 5dx_2 \wedge dx_3) \\
 &= 4(dx_2 \wedge dx_3 \wedge dx_1 \wedge dx_2) - (dx_2 \wedge dx_3 \wedge dx_1 \wedge dx_3) \\
 &\quad + 5(dx_2 \wedge dx_3 \wedge dx_2 \wedge dx_3) + 4(dx_1 \wedge dx_4 \wedge dx_1 \wedge dx_2) \\
 &\quad - (dx_1 \wedge dx_4 \wedge dx_1 \wedge dx_3) + 5(dx_1 \wedge dx_4 \wedge dx_2 \wedge dx_3) \\
 &= 5(-1)^2(dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4) \\
 &= 5 dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4
 \end{aligned}$$

4. Calculate and simplify the wedge product $\phi \wedge \psi$. Write the dx_i 's in the answers in increasing subscript order.

(a) $\phi = 3dx_2 \wedge dx_3 - 7dx_2 \wedge dx_4$, $\psi = 2dx_1 + 3dx_2 - dx_4$

$$\begin{aligned}
 \phi \wedge \psi &= (3dx_2 \wedge dx_3 - 7dx_2 \wedge dx_4) \wedge (2dx_1 + 3dx_2 - dx_4) \\
 &= 6dx_2 \wedge dx_3 \wedge dx_1 - 3dx_2 \wedge dx_3 \wedge dx_4 - 14dx_2 \wedge dx_4 \wedge dx_1 \\
 &= 6dx_1 \wedge dx_2 \wedge dx_3 - 14dx_1 \wedge dx_2 \wedge dx_4 - 3dx_2 \wedge dx_3 \wedge dx_4,
 \end{aligned}$$

where in the second line the three missing terms vanish because they each have two appearances of some dx_i .

(b) $\phi = dx_2 \wedge dx_3 \wedge dx_4$, $\psi = 3dx_1 - 5dx_2$

$$\begin{aligned}
\phi \wedge \psi &= (dx_2 \wedge dx_3 \wedge dx_4) \wedge (3dx_1 - 5dx_2) \\
&= 3dx_2 \wedge dx_3 \wedge dx_4 \wedge dx_1 \\
&= -3dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4.
\end{aligned}$$

$$(c) \quad \phi = 2dx_1 \wedge dx_3, \quad \psi = -2dx_1 \wedge dx_4 + 3dx_2 \wedge dx_4.$$

$$\begin{aligned}
\phi \wedge \psi &= (2dx_1 \wedge dx_3) \wedge (-2dx_1 \wedge dx_4 + 3dx_2 \wedge dx_4) \\
&= 6dx_1 \wedge dx_3 \wedge dx_2 \wedge dx_4 \\
&= -6dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4.
\end{aligned}$$

5. Compute the exterior product $\Phi_1 \wedge \Phi_2$, if $\Phi_1 = x_1 dx_1 \wedge dx_2 \wedge dx_4$ and $\Phi_2 = x_4 dx_1 + x_2 dx_3 + x_3 dx_4$ in $\mathbb{R}^4$. Give your answer in the form $f(x_1, x_2, x_3, x_4)dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4$.

$$\begin{aligned}
\Psi \wedge \Phi &= x_1 x_2 dx_1 \wedge dx_2 \wedge dx_4 \wedge dx_3 \\
&= -x_1 x_2 dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4.
\end{aligned}$$

5.2 Exterior Derivative

1. Let $\Psi = x_1^2 dx_1 \wedge dx_3 - x_3 dx_2 \wedge dx_4$ in $\mathbb{R}^4$.

(a) Compute $\Psi \wedge \Phi$, if $\Phi = x_3^5 dx_2 \wedge dx_4$. Give your answer in the form $f(x_1, x_2, x_3, x_4)dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4$.

(b) Find $d\Psi$. Simplify your answer.

(a)

$$\begin{aligned}
\Psi \wedge \Phi &= x_1^2 x_3^5 dx_1 \wedge dx_3 \wedge dx_2 \wedge dx_4 \\
&= -x_1^2 x_3^5 dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4.
\end{aligned}$$

(b)

$$\begin{aligned}
d\Psi &= 2x_1 dx_1 \wedge dx_1 \wedge dx_3 - dx_3 \wedge dx_2 \wedge dx_4 \\
&= dx_2 \wedge dx_3 \wedge dx_4
\end{aligned}$$

2. Calculate the exterior derivative of the differential forms:

(a) $\Phi = xydy + x^2z^2dz$

$$\begin{aligned} d\Phi &= d(xydy + x^2z^2dz) \\ &= d(xy) \wedge dy + d(x^2z^2) \wedge dz \\ &= (ydx + xdy) \wedge dy + (2xz^2dx + 2x^2zdz) \wedge dz \\ &= ydx \wedge dy + 2xz^2dx \wedge dz. \end{aligned}$$

(b) $\Phi = (x_1 + x_2^2)dx_1 \wedge dx_2 + x_2 \cos(x_3)dx_2 \wedge dx_4$

$$\begin{aligned} d\Phi &= d((x_1 + x_2^2)dx_1 \wedge dx_2 + x_2 \cos(x_3)dx_2 \wedge dx_4) \\ &= d(x_1 + x_2^2) \wedge dx_1 \wedge dx_2 + d(x_2 \cos(x_3)) \wedge dx_2 \wedge dx_4 \\ &= (dx_1 + 2x_2dx_2) \wedge dx_1 \wedge dx_2 + (\cos(x_3)dx_2 - x_2 \sin(x_3)dx_3) \wedge dx_2 \wedge dx_4 \\ &= -x_2 \sin(x_3)dx_3 \wedge dx_2 \wedge dx_4 \\ &= x_2 \sin(x_3)dx_2 \wedge dx_3 \wedge dx_4. \end{aligned}$$

3. Calculate the exterior derivative of the differential forms:

(a) $\Phi = 5x_2 dx_1 \wedge dx_3 + (3x_3 - x_1) dx_3 \wedge dx_4$

$$\begin{aligned} d\Phi &= 5 dx_2 \wedge dx_1 \wedge dx_3 - dx_1 \wedge dx_3 \wedge dx_4 \\ &= -5 dx_1 \wedge dx_2 \wedge dx_3 - dx_1 \wedge dx_3 \wedge dx_4 \end{aligned}$$

(b) $\Psi = x_1x_2^2 dx_1 \wedge dx_3 \wedge dx_4 + x_3^2x_4 dx_2 \wedge dx_3 \wedge dx_4.$

$$\begin{aligned} d\Psi &= 2x_1x_2 dx_2 \wedge dx_1 \wedge dx_3 \wedge dx_4 \\ &= -2x_1x_2 dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4 \end{aligned}$$

(c) $\Phi = e^{x_1-x_3}dx_1 \wedge dx_2 \wedge dx_4 + e^{4x_2-5x_3}dx_1 \wedge dx_3 \wedge dx_4.$

$$\begin{aligned} d\Phi &= -e^{x_1-x_3}dx_3 \wedge dx_1 \wedge dx_2 \wedge dx_4 + 4e^{4x_2-5x_3}dx_2 \wedge dx_1 \wedge dx_3 \wedge dx_4 \\ &= (-e^{x_1-x_3} - 4e^{4x_2-5x_3}) dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4. \end{aligned}$$

4. Find the exterior derivative of the 3-form $x_1x_2^2x_3^4dx_1 \wedge dx_3 \wedge dx_4$ in $\mathbb{R}^4$. Give your answer in the form $g(x_1, x_2, x_3, x_4)dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4$.

$$\begin{aligned} d\Psi &= 2x_1x_2x_3^4dx_2 \wedge dx_1 \wedge dx_3 \wedge dx_4 \\ &= -2x_1x_2x_3^4dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4 \end{aligned}$$

5. Let $\Psi = xydx + z^2dy$ and $\Phi = (2y + z)dx \wedge dy + (xy + z^2)dy \wedge dz$.

(a) Find $\Psi \wedge \Phi$. Simplify your answer.

(b) Find $d\Phi$ and $d\Psi$.

(a)

$$\begin{aligned} \Psi \wedge \Phi &= xy(2y + z)dx \wedge dx \wedge dy + xy(xy + z^2)dx \wedge dy \wedge dz \\ &\quad + z^2(2y + z)dy \wedge dx \wedge dy + z^2(xy + z^2)dy \wedge dy \wedge dz \\ &= xy(xy + z^2)dx \wedge dy \wedge dz. \end{aligned}$$

(b)

$$\begin{aligned} d\Phi &= dz \wedge dx \wedge dy + ydx \wedge dy \wedge dz \\ &= (1 + y)dx \wedge dy \wedge dz \end{aligned}$$

$$\begin{aligned} d\Psi &= xdy \wedge dx + 2zdz \wedge dy \\ &= -xdx \wedge dy - 2zdy \wedge dz \end{aligned}$$

5.3 Integrating Differential Forms

1. Compute $\int_M \Psi$, if M is the parametrized manifold

$$\mathbf{r}(u, v) = (u^3 - v, v^3, uv^2, u^2v), \quad 0 \leq u \leq 3, \quad 0 \leq v \leq 2,$$

and $\Psi = dx_1 \wedge dx_2 + 2dx_3 \wedge dx_4$.

We have the tangent vectors $\mathbf{r}_u = (3u^2, 0, v^2, 2uv)$ and $\mathbf{r}_v = (-1, 3v^2, 2uv, u^2)$. Now

$$\begin{aligned} \Psi(\mathbf{r}_u, \mathbf{r}_v) &= dx_1 \wedge dx_2(\mathbf{r}_u, \mathbf{r}_v) + 2dx_3 \wedge dx_4(\mathbf{r}_u, \mathbf{r}_v) \\ &= \begin{vmatrix} 3u^2 & -1 \\ 0 & 3v^2 \end{vmatrix} + 2 \begin{vmatrix} v^2 & 2uv \\ 2uv & u^2 \end{vmatrix} \\ &= (9u^2v^2) + 2(-3u^2v^2) \\ &= 3u^2v^2 \end{aligned}$$

Thus

$$\begin{aligned}
 \int_M \Psi &= \int_0^2 \int_0^3 \Psi(\mathbf{r}_u, \mathbf{r}_v) \, du \, dv \\
 &= \int_0^2 \int_0^3 3u^2 v^2 \, du \, dv \\
 &= \int_0^2 [u^3 v^2]_{u=0}^3 \, dv \\
 &= \int_0^2 27v^2 \, dv \\
 &= [9v^3]_0^2 \\
 &= 72.
 \end{aligned}$$

2. Integrate $\Phi = 4dx_1 \wedge dx_2 - x_2^2 dx_3 \wedge dx_4$ on the 2-manifold $\mathbf{x} = \langle t, s, t \cos s, t \sin s \rangle$, $0 \leq t \leq 1$, $0 \leq s \leq 2\pi$, in $\mathbb{R}^4$.

The tangent vectors are $\mathbf{x}_t = \langle 1, 0, \cos s, \sin s \rangle$ and $\mathbf{x}_s = \langle 0, 1, -t \sin s, t \cos s \rangle$. We first evaluate the 2-forms:

$$dx_1 \wedge dx_2(\mathbf{x}_t, \mathbf{x}_s) = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1,$$

$$x_2^2 dx_3 \wedge dx_4(\mathbf{x}_t, \mathbf{x}_s) = s^2 \begin{vmatrix} \cos s & \sin s \\ -t \sin s & t \cos s \end{vmatrix} = s^2 t,$$

so that the integral is

$$\int_0^1 \int_0^{2\pi} (4 - s^2 t) ds dt = 8\pi - \frac{4\pi^3}{3}.$$

5.4 Orientation on Manifolds

1. Compute $\int_M \Psi$, if $\Psi = x_1^2 dx_1 \wedge dx_3 - x_3 dx_2 \wedge dx_4$ and M is the parametrized manifold $\mathbf{r}(u, v) = (u, v, uv, u^2 - v^2)$, $0 \leq u \leq 1$, $0 \leq v \leq 2$, oriented consistently with this parametrization.

We have $\mathbf{r}_u = (1, 0, v, 2u)$, $\mathbf{r}_v = (0, 1, u, -2v)$, so

$$dx_1 \wedge dx_3(\mathbf{r}_u, \mathbf{r}_v) = \begin{vmatrix} 1 & v \\ 0 & u \end{vmatrix} = u,$$

$$dx_2 \wedge dx_4(\mathbf{r}_u, \mathbf{r}_v) = \begin{vmatrix} 0 & 2u \\ 1 & -2v \end{vmatrix} = -2u$$

Therefore,

$$\begin{aligned}
 \iint_M \Psi &= \int_0^1 \int_0^2 u^3 + 2u^2v \, dv \, du \\
 &= \int_0^1 (u^3v + u^2v^2) \Big|_{v=0}^2 \, du \\
 &= \int_0^1 2u^3 + 4u^2 \, du \\
 &= \frac{u^4}{2} + \frac{4u^3}{3} \Big|_0^1 = \frac{1}{2} + \frac{4}{3} = \frac{11}{6}.
 \end{aligned}$$

2. Compute $\int_M \Psi$, if M is the parametrized manifold $\mathbf{r}(u, v) = (u^2, v^2, u + v^3, u^3 - v)$, $0 \leq u \leq 1$, $0 \leq v \leq 1$, oriented consistently with this parametrization, and $\Psi = 3x_1dx_2 \wedge dx_4 + x_2dx_3 \wedge dx_4$.

We have the tangent vectors $\mathbf{r}_u = (2u, 0, 1, 3u^2)$ and $\mathbf{r}_v = (0, 2v, 3v^2, -1)$. Now

$$\begin{aligned}
 \Psi(\mathbf{r}_u, \mathbf{r}_v) &= 3u^2dx_2 \wedge dx_4(\mathbf{r}_u, \mathbf{r}_v) + v^2dx_3 \wedge dx_4(\mathbf{r}_u, \mathbf{r}_v) \\
 &= 3u^2 \begin{vmatrix} 0 & 2v \\ 3u^2 & -1 \end{vmatrix} + v^2 \begin{vmatrix} 1 & 3v^2 \\ 3u^2 & -1 \end{vmatrix} \\
 &= 3u^2(-6u^2v) + v^2(-1 - 9u^2v^2) \\
 &= -18u^4v - v^2 - 9u^2v^4.
 \end{aligned}$$

Thus

$$\begin{aligned}
 \int_M \Psi &= \int_0^1 \int_0^1 \Psi(\mathbf{r}_u, \mathbf{r}_v) \, du \, dv \\
 &= \int_0^1 \int_0^1 (-18u^4v - v^2 - 9u^2v^4) \, du \, dv \\
 &= \int_0^1 \left[-\frac{18}{5}u^5v - uv^2 - 3u^3v^4 \right]_0^1 \, dv \\
 &= \int_0^1 \left(-\frac{18}{5}v - v^2 - 3v^4 \right) \, dv \\
 &= \left[-\frac{9}{5}v^2 - \frac{1}{3}v^3 - \frac{3}{5}v^5 \right]_0^1 \\
 &= -\frac{9}{5} - \frac{1}{3} - \frac{3}{5} \\
 &= -\frac{41}{15}.
 \end{aligned}$$

3. Compute $\int_M \Psi$, if $\Psi = 2x_4 dx_1 \wedge dx_2 \wedge dx_4$ and M is the parametrized manifold $\mathbf{x}(u, v, t) = (u \cos t, u \sin t, u + v, v)$, $0 \leq u \leq 1$, $0 \leq v \leq 2$, $0 \leq t \leq 2\pi$, oriented consistently with this parametrization.

We have $\mathbf{r}_u = (\cos t, \sin t, 1, 0)$, $\mathbf{r}_v = (0, 0, 1, 1)$, $\mathbf{r}_t = (-u \sin t, u \cos t, 0, 0)$,

$$dx_1 \wedge dx_2 \wedge dx_4(\mathbf{r}_u, \mathbf{r}_v, \mathbf{r}_t) = \begin{vmatrix} \cos t & \sin t & 0 \\ 0 & 0 & 1 \\ -u \sin t & u \cos t & 0 \end{vmatrix} = - \begin{vmatrix} \cos t & \sin t \\ -u \sin t & u \cos t \end{vmatrix} = -u$$

$$\begin{aligned} \iint_M \Psi &= \int_0^1 \int_0^2 \int_0^{2\pi} -2uv \, dt \, dv \, du \\ &= -2\pi \int_0^1 uv^2 \Big|_{v=0}^2 \, du \\ &= -2\pi \int_0^1 4u \, du \\ &= -4\pi u^2 \Big|_0^1 = -4\pi. \end{aligned}$$

4. Let M be the manifold parametrized by

$$\mathbf{r}(u, v) = (u + v, u - v, u \cos v, u \sin v),$$

with $0 \leq u \leq 2$, $0 \leq v \leq \pi$.

(a) Prove that the differential form $\omega = dx_1 \wedge dx_2$ orients M .

(b) Compute $\int_M \Phi$, if $\Phi = x_1 x_2 dx_3 \wedge dx_4$ and M is oriented by the form ω from (a).

The tangent vectors are

$$\begin{aligned} \mathbf{r}_u &= (1, 1, \cos v, \sin v), \\ \mathbf{r}_v &= (1, -1, -u \sin v, u \cos v). \end{aligned}$$

(a) We have

$$\omega(\mathbf{r}_u, \mathbf{r}_v) = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -2$$

This is always nonzero, so that the form does orient the manifold.

(b) Since $\omega(\mathbf{r}_u, \mathbf{r}_v) < 0$, we need to reverse the order of the tangent vectors in the computation:

$$\Phi(\mathbf{r}_v, \mathbf{r}_u) = \begin{vmatrix} -u \sin v & u \cos v \\ \cos v & \sin v \end{vmatrix} = -u,$$

$$\begin{aligned}\iint_M \Psi &= \int_0^2 \int_0^\pi (u^2 - v^2)(-u) dv du = \int_0^2 \int_0^\pi (v^2 u - u^3) dv du \\&= \int_0^2 \left[\frac{v^3 u}{3} - u^3 v \right]_{v=0}^\pi du \\&= \int_0^2 \left[\frac{\pi^3 u}{3} - \pi u^3 \right] du \\&= \left[\frac{\pi^3 u^2}{6} - \frac{\pi u^4}{4} \right]_{u=0}^2 = \frac{2\pi^3}{3} - 4\pi.\end{aligned}$$