Harmonic Analysis and Additive Combinatorics on Fractals

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Etta Z. Falconer Lecture
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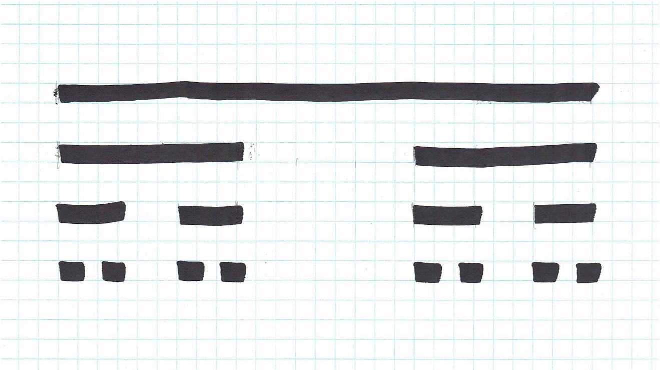
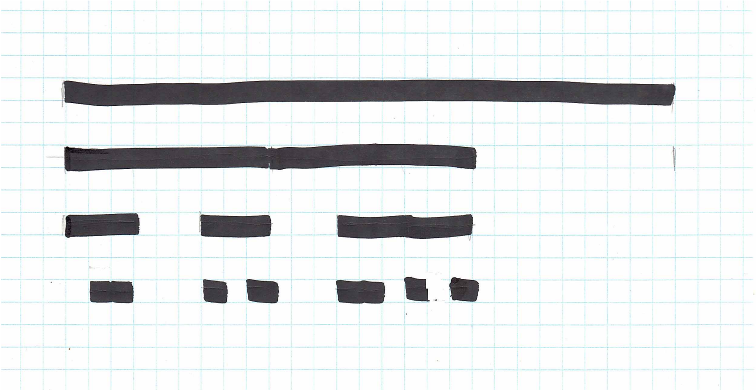
Fractals:
“random” vs. “structured”

Example: Ternary Cantor set



Dimension (similarity and Hausdorff) $\frac{\log 2}{\log 3}$. If we rescale the set by $1/3$, we get $1/2$ of it.

Example: Random ternary Cantor set



Choose the same number of intervals, but at random. Dimension (Hausdorff) still $\frac{\log 2}{\log 3}$, but the set is no longer self-similar.

Fractals in nature

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Random or not?

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It turns out to be easier to say what a random set should *not* look like.

Human activity: "structured" fractals

Structure vs. randomness

"Structured" fractals might have:

- Preferred directions or length units, on many scales.
- Many large segments that are exact (not just approximate) translates of each other.
- Exact correlations or resonances between different scales.

Absence of such features suggests randomness.

Pseudorandomness vs. additive structure

In additive combinatorics, *pseudorandomness* quantifies the absence of additive structure in discrete sets. For a set $A \subset \mathbb{Z}$, it might mean:

- A does not correlate well with long arithmetic progressions.
- A has small intersections with its own translates.
- Additive equations such as $a + b = c + d$ have the "expected" (based on probability) number of solutions with $a, b, c, d \in A$.
- $A + A$ is much larger than A.

We are looking for something similar in a fractal setting.

Fourier decay

Fourier analysis on fractals

The Fourier transform of a measure μ on $\mathbb{R}^d$ is

$$\widehat{\mu}(\xi) = \int e^{-2\pi i \xi \cdot x} d\mu(x), \quad \xi \in \mathbb{R}^d.$$

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Measure μ on a Cantor set: if $E = \bigcap_{j=1}^{\infty} E_j$ via Cantor iteration, let $\mu_j = \frac{1}{|E_j|} \mathbf{1}_{E_j} dx$ (normalized Lebesgue densities), then μ_j converge weakly to μ, a probability measure on E.

Fourier decay, first for smooth manifolds

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- Not possible if μ supported on a hyperplane of dimension less than d.
- Note that a hyperplane has more additive structure than a sphere. (Contains long arithmetic progressions, has more translational invariance.)

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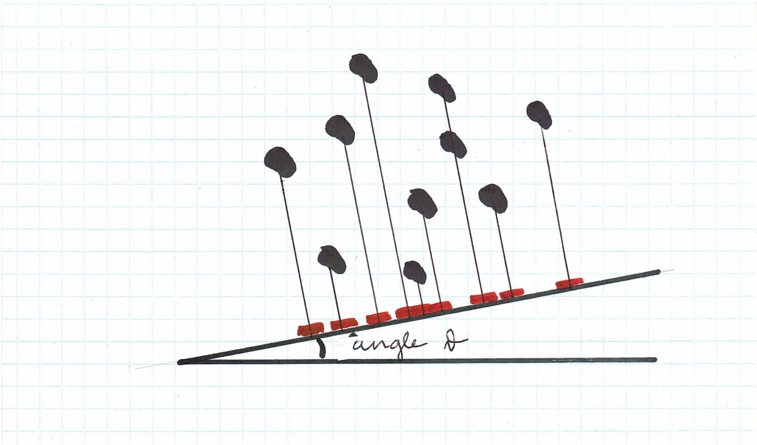
- Random Cantor sets typically have Fourier decay. (They behave like curved hypersurfaces such as spheres.)
- *Salem measures*: have optimal Fourier decay. Most constructions (Salem, Kahane, Bluhm, Łaba-Pramanik,...) are probabilistic, but deterministic examples are also known (Kaufman, Hambrook).

Is Fourier decay a *useful* measure of randomness?

Marstrand's projection theorem, special case: If $E \subset \mathbb{R}^2$ has Hausdorff dimension $\alpha < 1$, then the projected set $\pi_\theta(E)$ has dimension α for Lebesgue-almost all θ.



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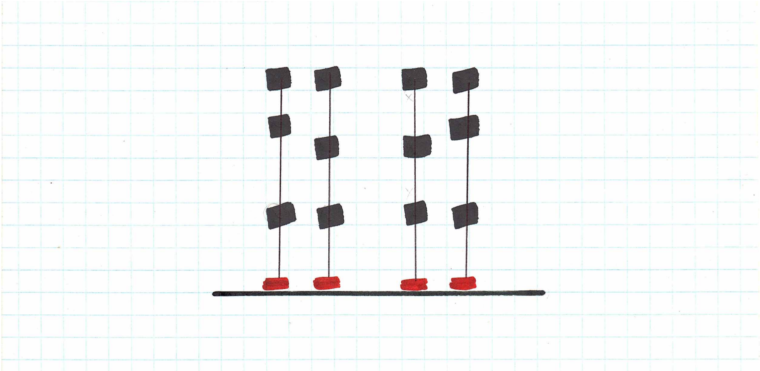
In general, there can be exceptional directions for which the projected set has lower dimension.



But this does not tend to happen for random sets.

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For Salem sets (optimal rate of Fourier decay), the projected set has dimension α for all directions. Also partial results if Fourier decay holds for *some* exponent (not necessarily optimal). In this context, Fourier decay does emulate randomness.

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Similar arguments apply to many other questions about projections, arithmetic sumsets, distance sets, etc.

Other criteria?

There are situations where even the best possible pointwise Fourier decay is not sufficient. There are also sets where we do not have Fourier decay but nonetheless expect random-like behaviour. Other criteria?

- Continue with Fourier analysis, but use it differently. (Example: restriction estimates, next slide.)
- Alternatively, look for pseudorandomness criteria that are based directly on additive structure, without using the Fourier transform. (Example: the "correlation conditions" used to prove a differentiation theorem for fractals, Łaba-Pramanik.)

There is ongoing work in both directions.

Restriction estimates

Restriction estimates

Fix a probability measure μ on $\mathbb{R}^d$. Define

$$\widehat{fd\mu}(\xi) = \int f(x) e^{-2\pi i \xi \cdot x} d\mu(x).$$

Looking for estimates of the form

$$\|\widehat{fd\mu}\|_{L^p(\mathbb{R}^d)} \leq C_{p,q} \|f\|_{L^q(\mu)} \quad \forall f \in L^q(\mu)$$

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- No need for pointwise decay estimates on $|\widehat{fd\mu}(\xi)|$, global L^p bounds are sufficient.
- But we want them for $\widehat{fd\mu}$, not just for $\widehat{\mu}$. For example, we could take $f = \mathbf{1}_F$, where F is a very small subset of $\operatorname{supp} \mu$.

Restriction for manifolds

Large body of work in classical harmonic analysis (Stein, Tomas, Fefferman, Bourgain, Tao, Wolff, Christ, Vargas, Carbery, Seeger, Bak, Oberlin, Guth, ...).

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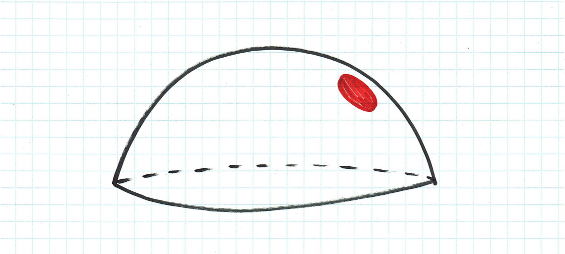
- Here, we focus on $q = 2$. (Improvements for $q > 2$ more difficult, require Kakeya-type geometric information.)

Range of restriction exponents for manifolds

If μ is the surface measure on the sphere, the Stein-Tomas range of exponents is optimal. Seen from *Knapp example*: f is the characteristic function of a small spherical cap.



The sphere is curved, but small spherical caps are almost flat. Same happens for other smooth manifolds.

Restriction exponents for fractal measures

The Stein-Tomas L^2 restriction theorem can be extended to more general measures with Fourier decay, including fractal measures (Mockenhaupt, Mistis, Bak-Seeger). The range of exponents for fractals matches the Stein-Tomas range for surfaces with the same dimension and rate of Fourier decay (where this makes sense).

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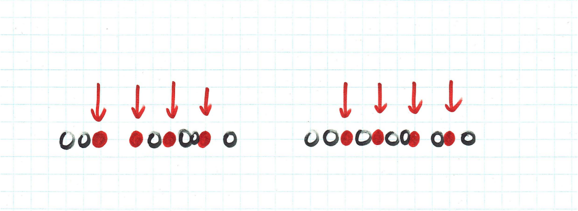
But for fractal measures, the Stein-Tomas exponent range is not always the best possible. Restriction estimates and Fourier decay capture different types of information.

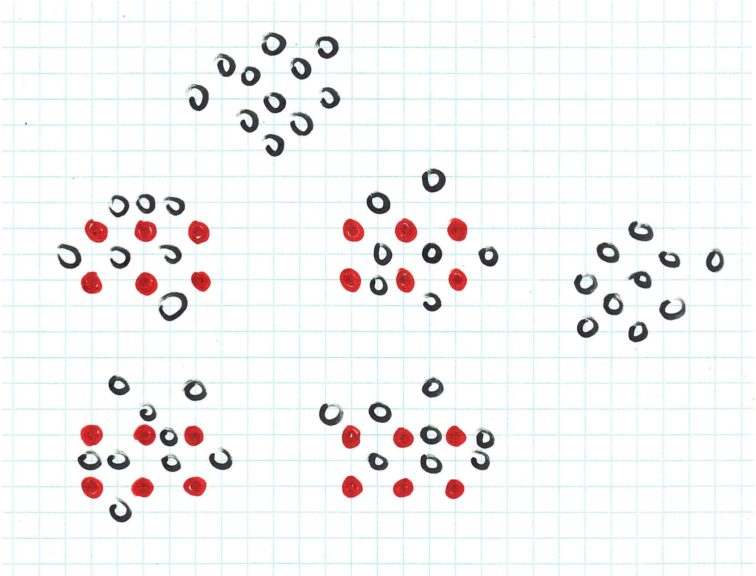
“Knapp examples” for fractals

Restriction with exponents better than Stein-Tomas cannot be proved using only Fourier decay. Random Cantor sets can contain much smaller subsets that are additivelly structured (multiscale arithmetic progressions). Fourier decay still holds, but restriction fails beyond Stein-Tomas range.



(Hambrook-Łaba, Chen)

Random sets with small structured subsets



Improved restriction for fractal measures

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- *Regularity of convolutions:* If $\alpha = d/k$ for $k \in \{2, 3, \dots\}$ and the k-fold convolution $\mu * \cdots * \mu$ is absolutely continuous, restriction holds for all $p \geq 2d/\alpha$. (Chen, Chen-Seeger, Shmerkin-Suomala)

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- *Multiscale $\Lambda(p)$ sets:* Use Bourgain's theorem on $\Lambda(p)$ sets as single-scale optimal restriction estimate, construct a multiscale version using the decoupling method of Bourgain-Demeter. Works for all α but without the endpoint $p = 2d/\alpha$. (Łaba-Wang)

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- Such measures might or might not have optimal Fourier decay.

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- Fractal analogues of results from additive combinatorics: restriction estimates are useful in proving Szemerédi-type theorems. (Chen, Henriot, Łaba, Pramanik)

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- Fractal analogues of results from additive combinatorics: restriction estimates are useful in proving Szemerédi-type theorems. (Chen, Henriot, Łaba, Pramanik)
- Applications to dynamical systems are forthcoming. (Example: Dyatlov-Zahl used additive combinatorics to prove estimates on spectral gaps for hyperbolic surfaces. Follow-up work requires more harmonic analysis on fractals.)

Thank you!