Harmonic analysis and the geometry of fractals

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Harmonic analysis has long studied singular and oscillatory integrals associated with surface measures on lower-dimensional manifolds in $\mathbb{R}^d$. The behaviour of such integrals depends on the geometric properties of the manifold: dimension, smoothness, curvature.

This is a well established, thriving and productive research area.

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Partial answer: "pseudorandomness," suggested by additive combinatorics.

In additive combinatorics, "pseudorandomness" refers to lack of additive structure. (The precise formulation depends on the problem at hand.) This is a key ingredient of major advances on Szemerédi-type problems (Gowers, Green-Tao), and we will draw on ideas from that work.

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For us, "random" fractals will behave like curved hypersurfaces such as spheres, whereas structured fractals (e.g. the middle-third Cantor set) behave like flat surfaces.

Outline of talk:

- Set-up: measures, dimensionality, curvature/randomness and Fourier decay.
- Restriction estimates.
- Maximal estimates and differentiation theorems.
- Szemerédi-type results.

Dimensionality of measures

Let μ be a finite, nonnegative Borel measure on $\mathbb{R}^d$.

- Let $0 \leq \alpha \leq d$. We say that μ obeys the α-dimensional ball condition if

$$\mu(B(x,r)) \leq Cr^\alpha \quad \forall x \in \mathbb{R}^d, \ r \in (0,\infty) \tag{1}$$

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- Connection to Hausdorff dimension via Frostman's Lemma: if $E \subset \mathbb{R}^d$ closed, then

  $\dim_H(E) = \sup\{\alpha \in [0,d] :$ E supports a probability measure $\mu = \mu_\alpha$ obeying (1)$\}$

Examples

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- The surface measure on a smooth k-dimensional submanifold of $\mathbb{R}^d$ obeys the ball condition with $\alpha = k$.
- Let E be the middle-third Cantor set on the line, then the natural self-similar measure on E obeys the ball condition with $\alpha = \frac{\log 2}{\log 3}$.

More general Cantor measures

Construct μ supported on $E = \bigcap_{j=1}^{\infty} E_j$ via Cantor iteration:

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- *The choices of subintervals might or might not be the same at all stages of the construction, or for all intervals of E_j.*
- Let $\mu_j = \frac{1}{|E_j|}\mathbf{1}_{E_j}$, then μ_j converge weakly to μ, a probability measure on E.

More general Cantor measures, cont.

For *any* choice of subintervals in the Cantor construction, E has Hausdorff dimension $\alpha = \frac{\log t}{\log N}$, and $\mu(B(x,r)) \leq Cr^{\alpha}$ for all $x \in \mathbb{R}$, $r > 0$.

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- If the choices of intervals are always the same (e.g. the middle-thirds Cantor set), E and μ have arithmetic structure; Fourier-analytic behaviour analogous to flat surfaces.
- "Random" Cantor sets (with the subintervals chosen through a randomized procedure) can behave like curved hypersurfaces such as spheres.

Pointwise Fourier decay: curvature vs. flatness, randomness vs. structure

Pointwise decay of $\widehat{\mu}$: curvature and randomness

Define the Fourier transform $\widehat{\mu}(\xi) = \int e^{-2\pi i \xi \cdot x} d\mu(x)$.
When do we have an estimate

$$|\widehat{\mu}(\xi)| \leq C(1 + |\xi|)^{-\beta/2} \tag{2}$$

for some $\beta > 0$?

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- But if $\mu = \sigma$ is the surface measure on the sphere S^{d-1}, then (2) holds with $\beta = d - 1$.
- For surface measures, estimates such as (2) depend on curvature.

Fourier decay for fractal sets

- Structured case: let μ be the middle-third Cantor measure, then $\widehat{\mu}(3) = \widehat{\mu}(3^2) = \cdots = \widehat{\mu}(3^j) = \dots$, hence no pointwise decay.

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- *Salem measures*: can take β arbitrarily close to $\dim_H(\operatorname{supp}\mu)$. (This is essentially the best possible decay.) Constructions due to Salem, Kahane, Kaufman, Bluhm, Łaba-Pramanik, Chen...

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- Most constructions of Salem measures (all except Kaufman) are probabilistic. Fourier decay is a measure of the "randomness" of μ.

Restriction estimates

Restriction estimates for measures on $\mathbb{R}^d$

For $f \in L^1(d\mu)$, define $\widehat{fd\mu}(\xi) = \int f(x)e^{-2\pi i\xi\cdot x}d\mu(x)$. When do we have an estimate

$$\|\widehat{fd\mu}\|_{L^p(\mathbb{R}^d)} \leq C_{p,q}\|f\|_{L^q(d\mu)}?$$

Large body of work in classical harmonic analysis (Stein, Tomas, Fefferman, Bourgain, Tao, Wolff, Christ, Vargas, Carbery, Seeger, Bak, Oberlin, Guth, ...). The range of exponents depends on the geometrical properties of μ.

L^2 restriction theorem

Let μ be a compactly supported probability measure on $\mathbb{R}^d$ such that for some $\alpha, \beta \in (0, d)$

- $\mu(B(x,r)) \leq C_1 r^\alpha$ for all $x \in \mathbb{R}^d$, $r > 0$,
- $|\widehat{\mu}(\xi)| \leq C_2(1+|\xi|)^{-\beta/2}$

Then for all $p \geq p_{d,\alpha,\beta} := \frac{2(2d-2\alpha+\beta)}{\beta}$,

$$\|\widehat{fd\mu}\|_{L^p(\mathbb{R}^d)} \leq C_p \|f\|_{L^2(d\mu)}$$

for all $f \in L^2(d\mu)$.

Stein-Tomas (1970s) for the sphere, Mockenhaupt, Mitsis (2000), Bak-Seeger (2011) general case.

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- Similar examples can be constructed for other manifolds. But for fractal measures, the situation is more complicated...

Range of restriction exponents: fractal measures

- "Knapp example for fractals" (Hambrook-Łaba 2012, further work by Chen): Random Cantor sets can contain much smaller subsets that are arithmetically structured. It follows that the range of exponents in Mockenhaupt's theorem is sharp.

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- The construction draws on ideas from additive combinatorics, especially restriction estimates for discrete sets (integers, finite fields).
- On the other hand, there are fractal measures for which restriction estimates hold for a better range of exponents. (Chen 2012, based on a construction of Körner.) Thus, a wider range of behaviours than for smooth manifolds.

Restriction estimates beyond L^2?

For manifolds, there are restriction estimates beyond $q = 2$. These carry additional geometric information beyond curvature and Fourier decay (e.g. Kakeya-type results in the case of the sphere).

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Open question: Is there an analogue of this for Cantor sets on the line?

Maximal operators and differentiation theorems

Classic result: Hardy-Littlewood maximal theorem

Given $f \in L^1(\mathbb{R}^d)$, define

$$Mf(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)| dy$$

Then $\|Mf\|_p \leq C_{p,d} \|f\|_p$ for all $1 < p \leq \infty$.

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Then $\|Mf\|_p \leq C_{p,d}\|f\|_p$ for all $1 < p \leq \infty$.

Corollary (Lebesgue differentiation theorem): for a.e. x,

$$\lim_{r\to 0} \frac{1}{|B(x,r)|} \int_{B(x,r)} f(y)dy = f(x).$$

Maximal theorems for singular measures

Let μ be a probability measure on $\mathbb{R}^d$, singular w.r.t. Lebesgue.

- Define
$$M_\mu f(x) = \sup_{r>0} \int |f(x+ry)| d\mu(y)$$
For what range of p is M_μ bounded on $L^p(\mathbb{R}^d)$?

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For what range of p is M_μ bounded on $L^p(\mathbb{R}^d)$?
- Is there a differentiation theorem: for $f \in L^p(\mathbb{R}^d)$, some range of p,
$$\lim_{r\to 0} \int f(x+ry)dy = f(x), \ a.e.\ x?$$

Maximal theorems for lower-dimensional manifolds

- The spherical maximal operator

$$M_\sigma f(x) = \sup_{r>0} \int_{\mathbb{S}^{d-1}} |f(x+ry)| d\sigma(y)$$

is bounded on $L^p(\mathbb{R}^d)$ for $d \geq 2$ and $p > \frac{d}{d-1}$. This range of p is optimal. (Stein 1978 for $d \geq 3$, Bourgain 1986 for $d = 2$)

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- Many other results on maximal and averaging operators associated with smooth lower-dimensional manifolds (Stein, Wainger, Nagel, Sogge, Phong, Iosevich, Seeger, Rubio de Francia, ...) Most are based on curvature via the decay of $\widehat{\mu}$. (Notable exception: Bourgain.)

Maximal theorems for fractals on the line

- Łaba-Pramanik 2011: For any $0 < \epsilon < \frac{1}{3}$, there is a probability measure $\mu = \mu_\epsilon$ supported on a set $E \subset [1, 2]$ of Hausdorff dimension $1 - \epsilon$, such that M_μ is bounded on $L^p(\mathbb{R})$ for $p > \frac{1+\epsilon}{1-\epsilon}$.

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- The given ranges of ϵ and p are not likely to be optimal. We require $\operatorname{supp} \mu \subset [1,2]$ so that $\mu = \delta_0$ is not allowed.
- Probabilistic construction. (Open question: deterministic examples?) Key property of μ_ϵ: a "correlation condition," reminiscent of higher-order uniformity conditions in additive combinatorics.

Szemerédi-type problems

Additive combinatorics: Szemerédi theorem

- **Szemerédi's Theorem**: Let $k \geq 3$, $A \subset \{1, 2, \ldots, N\}$, $|A| \geq \delta N$ for some $\delta > 0$. If N is sufficiently large (depending on δ, k), then A must contain a k-term arithmetic progression $\{x, x + r, \ldots, x + (k - 1)r\}$ with $r \neq 0$.

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- Brief history: Roth (1953, $k = 3$), Szemerédi (1969-74, all k), Furstenberg (1977), Gowers (1998), Gowers and Nagle-Rödl-Schacht-Skokan (2003-04); more...

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- Brief history: Roth (1953, $k = 3$), Szemerédi (1969-74, all k), Furstenberg (1977), Gowers (1998), Gowers and Nagle-Rödl-Schacht-Skokan (2003-04); more...
- Many extensions and generalizations, including a multidimensional version (Furstenberg-Katznelson 1978) and the polynomial Szemerédi theorem (Bergelson-Leibman 1996)

A continuous analogue?

- First attempt: Let $A \subset \mathbf{R}$ be a finite set, e.g. $A = \{0, 1, 2, \ldots, k-1\}$. Must any set $E \subset [0, 1]$ of positive Lebesgue measure contain an affine (i.e. rescaled and translated) copy of A?

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- **Erdős:** Same question if A is an infinite sequence. There are counterexamples for slowly decaying sequences (Falconer, Bourgain) but open e.g. for $A = \{2^{-n} : \ n = 1, 2, \ldots\}$.

A continuous analogue: finite patterns in sets of measure zero

Let $A \subset \mathbf{R}$ be a finite set, e.g. $A = \{0, 1, 2\}$. If a set $E \subset [0, 1]$ has Hausdorff dimension α sufficiently close to 1, must it contain an affine copy of A?

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- Keleti 1998: There is a closed set $E \subset [0, 1]$ of Hausdorff dimension 1 (but Lebesgue measure 0) which contains no affine copy of $\{0, 1, 2\}$.
- But additive combinatorics suggests that there should be positive results under additional pseudorandomness conditions on E.

Arithmetic progressions in fractal sets

Łaba-Pramanik 2009: Let $E \subset [0,1]$ compact. Assume that E supports a probability measure μ such that:

- $\mu(B(x,r)) \leq C_1 r^\alpha$ for all $x, r \in [0,1]$ (in particular, $\dim(E) \geq \alpha$),
- $|\widehat{\mu}(k)| \leq C_2(1+|k|)^{-\beta/2}$ for all $k \in \mathbb{Z}$ and some $\beta > 2/3$.

If α is close enough to 1 (depending on C_1, C_2), then E contains a non-trivial 3-term arithmetic progression.

Multidimensional version

Chan-Łaba-Pramanik 2013: multidimensional version. Full statement too technical but here is an example. Let $a, b, c \in \mathbb{R}^2$ distinct. Let $E \subset \mathbb{R}^2$ compact, supports a probability measure μ such that:

- $\mu(B(x,r)) \leq C_1 r^\alpha$ for all $x \in \mathbb{R}^2$, $r > 0$,
- $|\widehat{\mu}(\xi)| \leq C_2(1+|\xi|)^{-\beta/2}$ for all $\xi \in \mathbb{R}^2$.

If α, β are close enough to 2, then E must contain three distinct points x, y, z such that $\triangle xyz$ is a similar copy of $\triangle abc$.

(Maga 2012: not true without the Fourier decay assumption.)

Ideas from additive combinatorics

- Multilinear forms "counting" arithmetic progressions, defined via Fourier analysis.
- Decomposition of μ into "structured" and "random" parts. The structured part is absolutely continuous and contributes the main term to the multilinear form. The random part contributes small errors. (Idea from Green 2003, Green-Tao 2004.)

Thank you!