ON THE MAXIMAL DIRECTIONAL HILBERT TRANSFORM

IZABELLA ŁABA, ALESSANDRO MARINELLI, MALABIKA PRAMANIK

ABSTRACT. For any dimension $n \geq 2$, we consider the maximal directional Hilbert transform $\mathscr{H}_U$ on $\mathbb{R}^n$ associated with a direction set $U \subseteq \mathbb{S}^{n-1}$:

$$\mathscr{H}_U f(x) := \frac{1}{\pi} \sup_{v \in U} \left| \text{p.v.} \int f(x - tv) \, \frac{dt}{t} \right|.$$

The main result in this article asserts that for any exponent $p \in (1, \infty)$, there exists a positive constant $C_{p,n}$ such that for any finite direction set $U \subseteq \mathbb{S}^{n-1}$,

$$\|\mathscr{H}_U\|_{p \to p} \geq C_{p,n} \sqrt{\log \#U},$$

where $\#U$ denotes the cardinality of U. As a consequence, the maximal directional Hilbert transform associated with an infinite set of directions cannot be bounded on $L^p(\mathbb{R}^n)$ for any $n \geq 2$ and any $p \in (1, \infty)$. This completes a result of Karagulyan [11], who proved a similar statement for $n = 2$ and $p = 2$.

CONTENTS

1. Introduction

The fundamental and ubiquitous nature of the classical one-dimensional Hilbert transform has inspired the study of a large variety of operators that share some of its distinctive features. Among the numerous higher-dimensional variants of this transform that are available in the literature, the maximal directional Hilbert transform is of notable interest, in view of its connections with several central problems in harmonic analysis, such as Carleson's theorem on the convergence of Fourier series, estimates on maximal functions of Kakeya type and Stein's conjecture on the Hilbert transform along Lipschitz vector fields. The treatises [15, 16] of Lacey and Li contain an extensive survey of these connections.

Given a unit vector $v \in \mathbb{S}^{n-1}$, the *directional Hilbert transform* $\mathscr{H}_v$ is defined initially on Schwartz functions on $\mathbb{R}^n$ as follows,

$$(1.1) \qquad \mathscr{H}_v f(x) := \frac{1}{\pi} \text{p.v.} \int f(x - tv) \frac{dt}{t} = \frac{1}{\pi} \lim_{\epsilon \to 0^+} \int_{|t| > \epsilon} f(x - tv) \frac{dt}{t}, \quad x \in \mathbb{R}^n.$$

After a rotation that sends v to the first canonical basis vector $e_1 = (1, 0, \ldots, 0)$, this is essentially a tensor product of the classical Hilbert transform in x_1 with the identity operator in the remaining variables. As a result, Lebesgue mapping properties of $\mathscr{H}_v$ are easy consequences of its one-dimensional counterpart [9, 10, 17]; namely, $\mathscr{H}_v$ is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ if and only if $1 < p = q < \infty$.

The maximal version of the operator $\mathscr{H}_v$, termed the *maximal directional Hilbert transform*, is the primary object of study in this article. Given a set of unit vectors $U \subseteq \mathbb{S}^{n-1}$ and initially for a Schwartz function f, it is defined to be

$$(1.2) \qquad \mathscr{H}_U f(x) := \max_{v \in U} |\mathscr{H}_v f(x)|, \qquad x \in \mathbb{R}^n.$$

For finite sets U, the triangle inequality gives $\mathscr{H}_U f \leq \sum_{v \in U} \mathscr{H}_v f$. Thus $\mathscr{H}_U$ continues to be bounded on the same Lebesgue spaces as the classical Hilbert transform, with the trivial bound

$$(1.3) \qquad ||\mathscr{H}_U||_{p \to p} \leq \#U ||\mathscr{H}_{e_1}||_{p \to p}, \qquad p \in (1, \infty).$$

Here and throughout the paper, $||T||_{p \to p}$ will denote the operator norm of T from $L^p(\mathbb{R}^n)$ to itself. This gives rise to the following natural questions:

1. *To what extent can one improve upon the trivial estimate* (1.3)?

2. *Do there exist infinite sets U for which $||\mathscr{H}_U||_{p \to p}$ is finite for some $p \in (1, \infty)$?*

For $n = 2$, various aspects of question 1 above have been addressed in a large body of work [15, 5, 6, 7], encompassing results of two distinct types. With $U = \mathbb{S}^1$ and for $\mathscr{H}_U$ localized to a single frequency scale, Lacey and Li [15] have shown that the operator $f \mapsto \mathscr{H}_{\mathbb{S}^1}(\zeta * f)$ maps L^2 into weak L^2, and L^p to itself for $p > 2$. Here ζ is a Schwartz function with frequency support $\{1 \leq |\xi| \leq 2\}$. For finite U and in the unrestricted setting (i.e., without any Fourier localization), $\mathscr{H}_U$ has been studied in the more general context of maximal directional

singular integral operators, co-authored in part by Demeter, Di Plinio and Parissis. For instance, the main results in [5, 6] give that for a general direction set $U \subseteq \mathbb{S}^1$,

$$||\mathscr{H}_U||_{p \to p} \leq C_p \log \#U, \qquad 2 \leq p < \infty,$$

where C_p is a constant independent of U. For $p = 2$, this upper bound is in fact sharp for the uniformly distributed set of directions

$$U = \{e^{\frac{2\pi i k}{N}} : k = 1, \ldots, N\},$$

see [5, Section 3]. On the other hand, for lacunary sets $U \subseteq \mathbb{S}^1$ of finite order defined as in [6, 7], it has been shown that

(1.4)
$$||\mathscr{H}_U||_{p \to p} \leq C_p \sqrt{\log \#U}, \qquad 1 < p < \infty,$$

where the constant C_p also depends on the lacunarity order of U. For $n \geq 3$, partial results with $p = 2$ are due to Kim [13, Theorem 2]. Specifically, the estimate

$$||\mathscr{H}_U||_{2 \to 2} \leq CN^{\frac{n-2}{2}}$$

is shown to hold for a direction set $U \subseteq \mathbb{S}^{n-1}$ of cardinality N^{n-1} in general position contained inside the positive orthant. The bound is shown to be sharp for a member of this class.

In contrast, question 2 is much less studied in complete generality. Even though phrased in terms of infinite direction sets, after a finitary and quantitative reformulation it is really a question about lower bounds on $||\mathscr{H}_U||_{p \to p}$ for general U. A result of Karagulyan [11] addresses this question in the planar setting and for $p = 2$, obtaining a lower bound of order $\sqrt{\log \#U}$ for $||\mathscr{H}_U||_{2 \to 2}$ in this case. The goal of this paper is to establish this bound in far greater generality, extending it to all exponents $p \in (1, \infty)$ and to all dimensions $n \geq 2$. For convenience, all logarithms below will be taken to the base 2.

Theorem 1.1. *Let U be a finite set of unit vectors in $\mathbb{R}^n$ with $n \geq 2$. Then for $1 < p < \infty$, there exists a positive constant $C_{p,n}$ such that*

(1.5)
$$||\mathscr{H}_U||_{p \to p} \geq C_{p,n} \sqrt{\log \#U},$$

where $\#U$ is the cardinality of the set U.

Remarks:

1. Since the single-vector Hilbert transform $\mathscr{H}_v$ is not bounded as an operator on $L^1(\mathbb{R}^n)$ or on $L^\infty(\mathbb{R}^n)$ or from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ for $p \neq q$, the theorem is trivially true for these exponents.

2. The lower bound in (1.5) is in fact attained by certain direction sets U, as (1.4) shows. This gives rise to an interesting question: which geometric properties of a direction set U dictate the growth rate of $||\mathscr{H}_U||_{p \to p}$?

3. Our result extends easily to the periodic setting, with a similar proof. More explicitly, if $\mathscr{H}_U$ is viewed as an operator from $L^p(\mathbb{T}^n)$ to $L^q(\mathbb{T}^n)$, where $\mathbb{T}^n$ denotes the n-dimensional unit torus, then our arguments show that

$$||\mathscr{H}_U||_{L^p(\mathbb{T}^n) \to \mathbb{L}^q(\mathbb{T}^n)} \geq C_{p,q,n} \sqrt{\log \#U},$$

for all $p, q \in (1, \infty)$ with $q \leq p$. The operator is unbounded for all other choices of p, q. The construction of test functions on the torus proceeds similarly, except the convolution in (4.1) is taken on $\mathbb{T}^n$ instead of $\mathbb{R}^n$.

4. As a consequence of (1.5), we are able to conclude the unboundedness of $\mathcal{H}_U$ for *all* infinite direction sets U in all dimensions and on all nontrivial Lebesgue spaces. We record this below.

Theorem 1.2. *For any infinite set of unit vectors U in $\mathbb{R}^n$ with $n \geq 2$, the operator $\mathcal{H}_U$ cannot be extended to a bounded operator on $L^p(\mathbb{R}^n)$ for any $1 < p < \infty$.*

This is in sharp contrast with the behaviour of the closely related maximal directional operator

$$\mathcal{M}_U f(x) := \sup_{v \in U} \sup_{r>0} \frac{1}{2r} \int_{-r}^{r} |f(x - tv)| \, dt,$$

whose Lebesgue boundedness is *not* connected with the finitude of U. For instance, the operator $\mathcal{M}_U$ is known to be L^p-bounded for all $p \in (1, \infty]$ if U is an infinite direction set of lacunary type in $\mathbb{R}^n$, see for example [1, 4, 18, 20, 21, 22]. For other types of direction sets that lack the feature of finite-type lacunarity, the operator $\mathcal{M}_U$ is known to be unbounded on L^p for all $p \in [1, \infty)$. This has been studied in [2, 3, 12, 14].

1.1. **Notation and a preliminary reduction.**

We recall the equivalent Fourier-analytic formulation of the problem. For functions $f, g \in L^2(\mathbb{R}^n)$, we will use $\widehat{f}$ and $g^\vee$ to denote the Fourier transform and inverse Fourier transform respectively,

$$\widehat{f}(\xi) := \int f(x) e(-\xi \cdot x) dx, \qquad g^\vee(x) := \int g(\xi) e(\xi \cdot x) d\xi,$$

where $e(t) := e^{2\pi i t}$ for $t \in \mathbb{R}$. If $E \subset \mathbb{R}^n$ is a measurable set, we will use χ_E to denote its characteristic function, and $|E|$ to denote its Lebesgue measure. Given a unit vector $v \in \mathbb{R}^n$, we will use Γ_v to denote the half-space

$$(1.6) \qquad\qquad \Gamma_v := \{x \in \mathbb{R}^n : x \cdot v > 0\}.$$

It is well known [9, 10, 17] that the classical one-dimensional Hilbert transform H can be expressed as a Fourier multiplier operator,

$$Hf(x) = (-i \operatorname{sgn}(\cdot)\widehat{f})^\vee(x).$$

For the directional Hilbert transform, this means that $\mathcal{H}_v f = -i \left[(2\chi_{\Gamma_v}\widehat{f})^\vee - f \right]$. Accordingly, we define

$$T_v f := (\chi_{\Gamma_v}\widehat{f})^\vee, \qquad \text{and} \qquad T_U f(x) := \max_{v \in U} |T_v f(x)|.$$

Thus the boundedness of (1.2) is equivalent to that of T_U. In particular, Theorem 1.1 is equivalent to the bound

$$(1.7) \qquad\qquad \|T_U\|_{p \to p} \geq C_{p,n} \sqrt{\log \#U}, \quad 1 < p < \infty.$$

1.2. **Overview of the proof.**

The proof of (1.7) relies on three main components. One is geometric. More precisely, a suitable pruning and ordering of the direction set $U = \{u_1, \cdots, u_{2^m}\}$ generates a finite number of mutually disjoint conic sectors $S_N \subseteq \mathbb{R}^n$, with the property that S_N is contained in Γ_{u_k} if $N \leq k$ and is disjoint from Γ_{u_k} otherwise. This part of the argument is greatly simplified in the planar setting, but needs a little more care in general dimensions. This geometric ingredient is contained in Lemma 3.3. Its proof is presented in Section 6.

The second ingredient is analytical. Following the general guidelines of [11] and given a fixed Lebesgue exponent p, the sectors S_N are used to construct a test function f of the form $f = \sum f_N$, based on which (1.7) will be verified. On one hand, the function f_N is frequency-supported in a large cube contained in the sector S_N. Not only are these cubes disjoint from one another, they are strongly separated in a way that ensures a high degree of orthogonality among the various summands f_N. On the other hand, the essential spatial support of f_N is in a set $E_N \subseteq [0,1]^n$, with the property that any two sets in the collection $\{E_N\}$ are either disjoint or nested. The critical features of this iterative construction of f have been laid out in Proposition 3.2 of Section 3, and the proof of (1.7) appears here, modulo the two main estimates

$$(1.8) \qquad ||f||_p \lesssim_p \sqrt{m} \quad \text{and} \quad |\{x \in \mathbb{R}^n : |T_U f(x) \gtrsim m\}| \gtrsim 1,$$

the details of which are given in later sections.

The proof of the estimates in (1.8) constitutes the combinatorial component of the argument. Section 5 contains the steps that lead to the first inequality in (1.8). The nested structure of the sets E_N is best encoded as a binary tree. Combined with the stringent frequency localizations imposed on f_N, this results in an upper bound on $||f||_p$ that is essentially comparable to $||f||_2$. Choosing p a large even integer without loss of generality allows us to express $||f||_p^p$ as the sum of a large number of terms of the form

$$\int h_{N_1} \cdots h_{N_p} \quad \text{where } h_N \text{ is either } f_N \text{ or } \overline{f}_N.$$

Many of these terms can be ignored, based on disjoint spatial and frequency support considerations. The language of trees aids greatly in the book-keeping, identifying strings of indices $(N_1, \cdots, N_p)$ that genuinely contribute to the norm. This segment of the proof has no corresponding counterpart in [11], where p was always 2.

In addition, the choice of modulation parameters in f_N endows the functions $\text{Re}(f_N) = \varphi_N$ with Haar function-like properties, termed "signed tree systems". Basic materials concerning trees and signed tree systems have been collected in Section 2. An important fact concerning a signed tree system $\{\varphi_N\}$ is proved in Section 2.3: namely, for a given m and despite obvious oscillations, there exists a universal permutation σ of $\{1, \cdots, 2^m - 1\}$ for which the largest partial sum

$$\max_{1 \leq l \leq 2^m} \left| \sum_{N=1}^{l} \varphi_{\sigma(N)} \right| \text{ is comparable to } \sum_{N=1}^{2^m - 1} |\varphi_N|.$$

This choice of σ dictates the ordering of the sectors S_N and is critical to the second estimate in (1.8).

2. TREES AND TREE SYSTEMS

2.1. **Trees.** Given a large positive integer m, we will use the following system of double-indexing to keep track of a large collection of sets and functions that arise in the sequel. Any positive integer $1 \leq N \leq 2^m$ will be identified with the pair (k, j), where

$$(2.1) \qquad N = 2^k + j - 1, \quad 1 \leq j \leq 2^k, \quad k = 0, 1, \ldots m - 1.$$

As indicated in the introduction, the language of binary trees is a convenient tool in depicting this double-indexing system. Consider a full binary tree $\mathcal{T}_m$ of height m, and label each tree

vertex as (k, j), where k is the *height* of the vertex (so that k ranges from 0 to $m-1$), and all vertices of height k are labelled lexicographically as $(k, 1), (k, 2), \ldots, (k, 2^k)$. Given a vertex (k, j),

- its *parent* can be identified as $\left(k - 1, \lfloor \frac{j+1}{2} \rfloor\right)$ if $k \geq 1$, and
- its *left and right children* can be identified respectively as $(k+1, 2j-1)$ and $(k+1, 2j)$ if $k \leq m - 2$.

A *ray* $\mathscr{R}$ of length $l+1$ rooted at (k, j_0) is a sequence of vertices $\{(k, j_0), (k + 1, j_1), \ldots (k + l, j_l)\}$ where, for each $i = 1, \ldots l$, the vertex $(k + i, j_i)$ is a child of $(k+i-1, j_{i-1})$. We will also say that the vertex (k', j') is a *descendant* of (k, j) if $k' > k$ and (k, j) lies on the ray connecting (k', j') to the root $(0, 1)$ of the tree.

In parallel to the double-numbering system in (2.1), we will use a similar convention for tree vertices, so that the vertex (k, j) will be alternatively labelled by the number $N(k, j) = 2^k + j - 1$. We will use $h(N) = k$ to denote the height of the vertex N.

2.2. **Tree systems.** Let $Q = [0, 1]^n$. We will consider finite sequences of functions $\{f_N : 1 \leq N \leq 2^m - 1\}$, where each f_N is a complex-valued function supported in Q, and use our double-numbering system from Section 2.1 to order the sequence. Thus

$$f_1 = f_1^{(0)},$$
$$f_2 = f_1^{(1)}, \quad f_3 = f_2^{(1)},$$
$$f_4 = f_1^{(2)}, \quad f_5 = f_2^{(2)}, \quad f_6 = f_3^{(2)}, \quad f_7 = f_4^{(2)}, \text{etc.}$$

In many of our applications, the sequences of functions f_N will satisfy at least one of the following properties:

- For any pair f_N, f_M with $N \neq M$, the supports of f_N and f_M are either nested or disjoint, or
- More specifically, if $M > N$, then f_N has constant sign on the support of f_M (up to sets of Lebesgue measure 0).

A prototype of such a system is provided by Haar functions (cf. [19]). The abstract formulation of the property we need was given by Karagulyan in [11]. We follow the rough outline of Karagulyan's presentation in the definitions below, but also modify the terminology and use the language of graph theory more extensively in order to accommodate the later parts of the proof that are not present in [11]. In particular, we use the term "signed tree systems" to refer to the "tree systems" of [11].

Definition 2.1. Let $f_N = f_j^{(k)}$, $1 \leq N \leq 2^m - 1$, be a finite sequence of functions, indexed as above with N, j, k related by (2.1).

(a) We say that $\{f_N\}$ is a *tree system* if the following holds Lebesgue almost everywhere on Q:

$$(2.2) \qquad \operatorname{supp} f_{2j-1}^{(k+1)} \cap \operatorname{supp} f_{2j}^{(k+1)} = \emptyset, \qquad \operatorname{supp} f_{2j-1}^{(k+1)} \cup \operatorname{supp} f_{2j}^{(k+1)} \subset \operatorname{supp} f_j^{(k)}.$$

(b) We say that the sequence $\{f_N\}$ is a *signed tree system* if the following holds Lebesgue almost everywhere on Q:

$$(2.3) \qquad\qquad\qquad \operatorname{supp} f_{2j-1}^{(k+1)} \subset \left\{ x \in Q : f_j^{(k)}(x) > 0 \right\},$$

(2.4) $$\operatorname{supp} f_{2j}^{(k+1)} \subset \left\{ x \in Q : f_j^{(k)}(x) < 0 \right\}.$$

Note that if $N = 2^k + j - 1$, then $2^{k-1} + \lfloor \frac{j+1}{2} \rfloor - 1 = \lfloor \frac{N}{2} \rfloor =: N^*$. In particular, (2.3) and (2.4) are equivalent to

(2.5) $$\operatorname{supp} f_N \subset \left\{ x \in Q : (-1)^{j-1} f_{N^*}(x) > 0 \right\}.$$

Figure 1 shows the relations among the supports. Clearly, every signed tree system is a tree system.
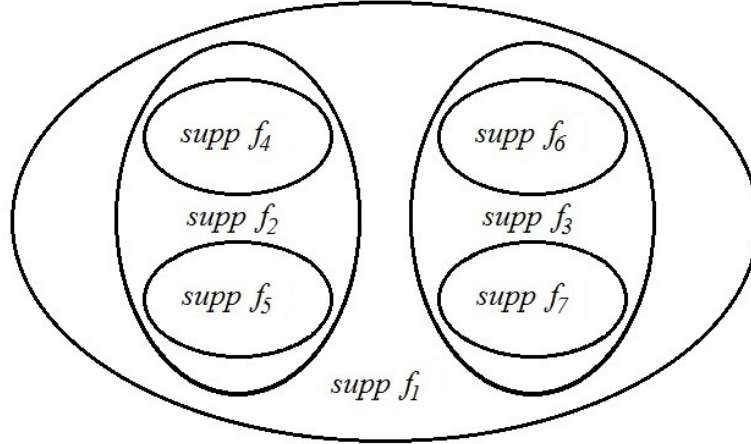

FIGURE 1. The nested supports of tree system functions.

The terminology of trees adapts easily to tree systems of functions. Thus for $k = 0, 1, \ldots, m-2$, each function $f_j^{(k)}$ in a tree system is identified with a vertex in a complete binary tree of height m, and has two children $f_{2j-1}^{(k+1)}$ and $f_{2j}^{(k+1)}$ with mutually disjoint (up to sets of measure 0) supports, both supported on supp $f_j^{(k)}$. In a signed tree system, we have the additional property that the left child $f_{2j-1}^{(k+1)}$ of $f_j^{(k)}$ is supported on the set where $f_j^{(k)}(x) > 0$, and the right child $f_{2j}^{(k+1)}$ is supported on the set where $f_j^{(k)}(x) < 0$. Iterating this, we get the following.

Lemma 2.2. *Let* $\{f_N : N = 1, \ldots, 2^m - 1\}$ *be a tree system. Then for* $(k, j) \neq (k', j')$, *the supports of* $f_j^{(k)}$ *and* $f_{j'}^{(k')}$ *are either disjoint or nested. Moreover:*

(a) If (k, j) *and* (k', j') *do not lie on the same tree ray (i.e. neither vertex is a descendant of the other), then the supports are disjoint.*

(b) If (k', j') *is a descendant of* (k, j), *then* $\operatorname{supp} f_{j'}^{(k')} \subset \operatorname{supp} f_j^{(k)}$.

(c) For each $x \in Q$ *such that* $f_1^{(0)}(x) \neq 0$, *there is a unique maximal ray*

$$\mathscr{R}(x) = \{(0, 1), (1, j_1), \ldots, (k, j_k)\} \text{ such that } f_{j_i}^{(i)}(x) \neq 0 \text{ for } i = 1, \ldots, k.$$

The ray terminates at $f_{j_k}^{(k)}$ *when either* $k = m - 1$ *or both children of* $f_{j_k}^{(k)}$ *take value 0 at* x.

(d) If $\{f_N : 1 \le N \le 2^m - 1\}$ is a signed tree system, then we have the additional property that $\mathscr{R}(x)$ encodes the sign of $f_1^{(0)}, f_{j_1}^{(1)}, \ldots, f_{j_{k-1}}^{(k-1)}$ at x:

$$\begin{cases} \text{it turns left at } (i, j_i) \text{ (i.e. goes to the left child } (i+1, 2j_i - 1)) \text{ if } f_{j_i}^{(i)}(x) > 0, \text{ and} \\[2em] \text{it turns right at } (i, j_i) \text{ (i.e. goes to the right child } (i+1, 2j_i)) \text{ if } f_{j_i}^{(i)}(x) < 0. \end{cases}$$
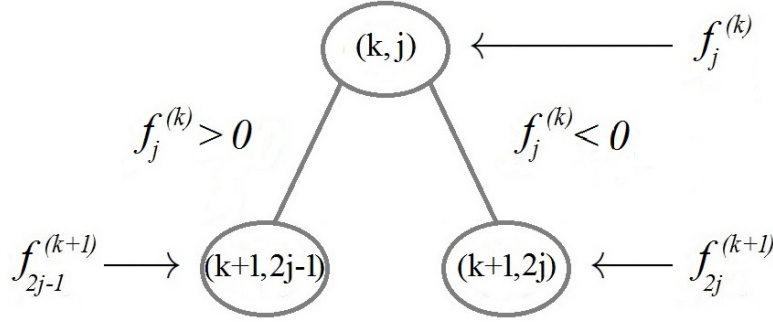

FIGURE 2. For a signed tree system, the signs of $f_j^{(k)}$ are encoded in the binary tree.

2.3. **Choice of the permutation σ.** Here we define a special permutation σ of $\{1, \ldots, 2^m - 1\}$ that plays a central role in the subsequent analysis. For $N = 2^k + j - 1$ as in (2.1), let

$$(2.6) \qquad t_N = t_j^{(k)} := \frac{2j - 1}{2^{k+1}} \in [0, 1].$$

Thus $t_1 = \frac{1}{2}$, $t_2 = \frac{1}{4}$, $t_3 = \frac{3}{4}$, $t_4 = \frac{1}{8}$, $t_5 = \frac{3}{8}$, $t_6 = \frac{5}{8}$, etc. Observe that a complete binary tree with m levels can be represented as a planar graph so that the vertex (k, j) has the x-coordinate $t_j^{(k)}$. Figure 3 illustrates this for $m = 3$.
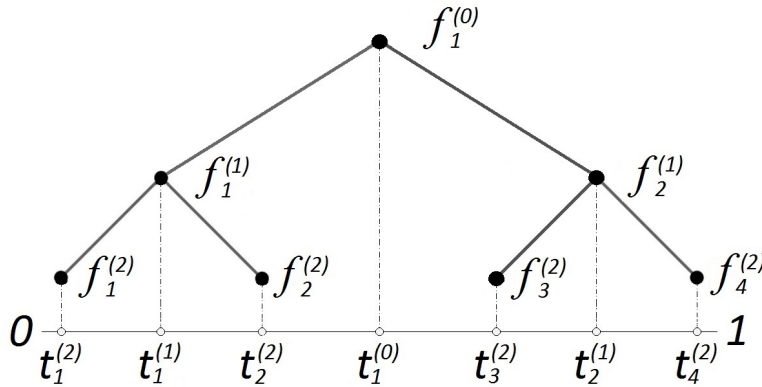

FIGURE 3. The full binary tree with $m = 3$ and with the numbers t_N.

We now rearrange the sequence $\{t_N\} \subset [0, 1]$ in increasing order. Specifically, there exists a unique permutation σ of the numbers $\{1, 2, \ldots 2^m - 1\}$, depending only on m, such that

$$(2.7) \qquad t_{\sigma(1)} < t_{\sigma(2)} < \ldots < t_{\sigma(2^m - 1)}.$$

(In the example in Figure 3, we have $\sigma(1) = 4, \sigma(2) = 2, \sigma(3) = 5$, etc.)

If $f_1, \ldots, f_{2^m-1}$ are Haar functions on the line, then for each $N = 1, \ldots, 2^m - 1$ the number t_N is the coordinate of the point where f_N changes sign from positive to negative, and the permutation σ arranges the sequence $\{t_N\}$ in increasing order. This observation leads directly to a special case, due to Nikishin and Ulyanov [19], of Lemma 2.3 below. The generalization of the lemma to general tree systems is due to Karagulyan [11]; while Karagulyan states it only for $n = 2$, the same proof works in all dimensions. We follow the argument of [11], with minor corrections[1] and expository changes.

Lemma 2.3. *([11, Lemma 1]) If σ is the permutation defined in (2.7), then for every signed tree system $f_1, \ldots, f_{2^m-1}$ in $\mathbb{R}^n$ we have*

$$(2.8) \qquad \max_{1 \le l < 2^m} \left| \sum_{N=1}^{l} f_{\sigma(N)}(x) \right| \ge \frac{1}{3} \sum_{N=1}^{2^m-1} |f_N(x)|.$$

for all $x \in \mathbb{R}^n$.

Proof. In view of (2.6), we find that

$$(2.9) \qquad t_{2j-1}^{(k+1)} = t_j^{(k)} - \frac{1}{2^{k+2}}, \quad t_{2j}^{(k+1)} = t_j^{(k)} + \frac{1}{2^{k+2}}.$$

Iterating over tree levels from $k+1$ to m, and using that $\sum_{i=k+2}^{k'} 2^{-i} < \sum_{i=k+2}^{\infty} 2^{-i} = 2^{-k-1}$ for any finite $k' \ge k+2$, we see that whenever $N' = N'(k', j')$ is a descendant of $N = N(k, j)$ in the binary tree, the corresponding numbers $t_{j'}^{(k')}$ obey

$$(2.10) \qquad t_{j'}^{(k')} \in \left(t_j^{(k)} - 2^{-k-1}, t_j^{(k)} + 2^{-k-1} \right).$$

Let $x \in Q$, and assume that $f_1(x) \ne 0$ since otherwise there is nothing to prove. Define

$$(2.11) \qquad l_x := \max \left\{ h : 1 \le h < 2^m, \ f_{\sigma(h)}(x) < 0 \right\},$$

with the convention that $l_x = 0$ if the set above is empty. It follow immediately from (2.11) that $f_{\sigma(h)}(x) \ge 0$ for all $h > l_x$. We claim that, furthermore,

$$(2.12) \qquad f_{\sigma(h)}(x) \le 0 \ \ \forall \, h \le l_x.$$

To prove this, suppose for contradiction that there exists an $h \le l_x$ such that $f_{\sigma(h)}(x) > 0$. By (2.11), we cannot have $h = l_x$. Let

$$\sigma(l_x) = N(k, j), \qquad \sigma(h) = N(s, i).$$

Consider the ray $\mathscr{R}(x)$ defined in Lemma 2.2 (c). By definition, $\mathscr{R}(x)$ contains the vertex $\sigma(l_x)$. If $\sigma(h) \notin \mathscr{R}(x)$, then $f_{\sigma(h)}(x) = 0$ and the claim is true. Thus we are reduced to the case where $\sigma(h)$ and $\sigma(l_x)$ both lie on $\mathscr{R}(x)$.

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[1] Karagulyan uses $f_{\sigma(h)}(x) \le 0$ instead of $f_{\sigma(h)}(x) < 0$ in his definition of l_x. With that definition, the property (2.12) does not necessarily hold. We have rewritten that part of the proof. Alternatively, Karagulyan's proof does work if the definition of l_x is changed to $l_x = \max \left\{ h : 1 \le h < 2^m, \ f_{\sigma(k)}(x) \le 0 \ \forall \, 1 \le k \le h \right\}$ instead. We thank an anonymous referee, as well as G. Karagulyan (private communication), for bringing that to our attention.

Suppose that $\sigma(h)$ is a descendant of $\sigma(l_x)$. Since $f_{\sigma(l_x)}(x) < 0$, Lemma 2.2 (d) dictates that the ray $\mathscr{R}(x)$ turns right at $\sigma(l_x)$, so that $\sigma(h)$ must be either $N(k+1, 2j)$ or one of its descendants. By (2.10) and then (2.9), we have

$$ t_{\sigma(h)} > t_{2j}^{(k+1)} - \frac{1}{2^{k+2}} = t_j^{(k)} = t_{\sigma(l_x)}, $$

which contradicts the assumption that $h < l_x$ and therefore $t_{\sigma(h)} < t_{\sigma(l_x)}$.

Finally, consider the case when $\sigma(l_x)$ is a descendant of $\sigma(h)$. If $f_{\sigma(h)}(x) > 0$, then $\mathscr{R}(x)$ turns left at $\sigma(h)$, so that $\sigma(l_x)$ must be either $N(s+1, 2i-1)$ or one of its descendants. Then, again by (2.10) and then (2.9), we have

$$ t_{\sigma(l_x)} < t_{2i-1}^{(s+1)} + \frac{1}{2^{s+2}} = t_i^{(s)} = t_{\sigma(h)}, $$
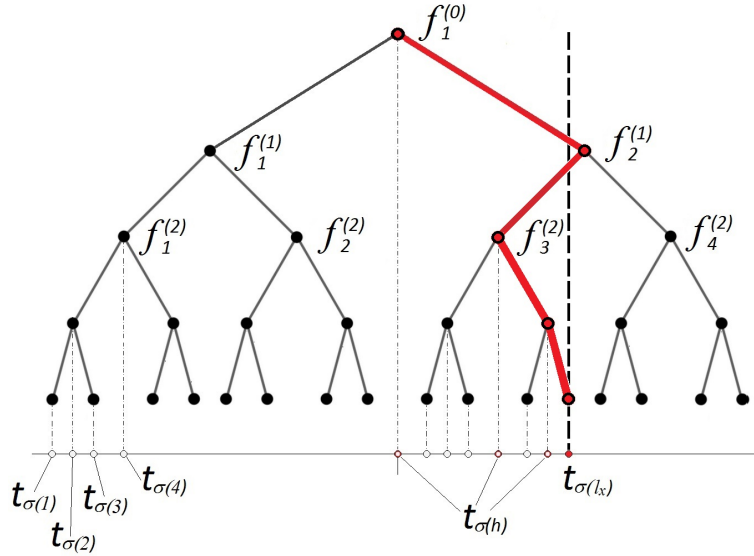
again contradicting our assumptions.

FIGURE 4. Choice of l_x for the example in Section 2.3.1 in the case $f_{12}^{(4)}(x) < 0$. The vertices in $\mathscr{R}^*(x)$ lie on the dashed line and to the left of it.

To recap, we have established the existence of an integer $l_x \geq 0$ such that

(2.13)
$$ \begin{cases} f_{\sigma(h)}(x) \leq 0 & \forall \, h \leq l_x \\ f_{\sigma(h)}(x) \geq 0 & \forall \, h > l_x. \end{cases} $$

Let

$$ S_1 = \sum_{N=1}^{l_x} -f_{\sigma(N)}(x), \quad S_2 = \sum_{N=l_x+1}^{2^m-1} f_{\sigma(N)}(x), $$

with the convention that $S_1 = 0$ if $l_x = 0$. Then $S_1, S_2 \geq 0$,

$$ \sum_{N=1}^{2^m-1} |f_N(x)| = S_1 + S_2, \text{ and} $$

$$\max_{1 \le l < 2^m} \left| \sum_{N=1}^{l} f_{\sigma(N)}(x) \right| = \max(S_1, S_2 - S_1).$$

If $S_1 \ge \frac{1}{3}(S_1 + S_2)$, then (2.8) follows immediately. Suppose now that $S_1 < \frac{1}{3}(S_1 + S_2)$. Then $S_2 > \frac{2}{3}(S_1 + S_2)$, and furthermore, $3S_1 < S_1 + S_2$ so that $2S_1 < S_2$. Hence

$$S_2 - S_1 > S_2 - \frac{S_2}{2} = \frac{S_2}{2} > \frac{1}{3}(S_1 + S_2),$$

and (2.8) again follows.   □

2.3.1. *An example.* The permutation σ arranges $t_{\sigma(N)}$ in increasing order. The integer l_x used in Lemma 2.3 then has a geometric interpretation in terms of the binary tree $\mathcal{T}_m$. Given $x \in Q$ and the ray $\mathscr{R}(x)$ as in Lemma 2.2(c), let $\mathscr{R}^*(x)$ be the subcollection of vertices on $\mathscr{R}(x)$ where the ray turns right. The maximal element (k, j_k) is included in $\mathscr{R}^*(x)$ if and only if $f_{j_k}^{(k)}(x) < 0$. Since the right child (and all its descendants) of any vertex N generate larger t-values than N itself, the relation (2.11) defining l_x is equivalent to the condition that $\sigma(l_x) = \max\{N : N \in \mathscr{R}^*(x)\}$.

We explain the choice of l_x in the context of an example given by Figure 4, with $m = 5$. Let $x \in Q$ be a point such that

$$\mathscr{R}(x) = \{(0,1), (1,2), (2,3), (3,6), (4,12)\}.$$

Then

$$\mathscr{R}^*(x) = \begin{cases} \{(0,1), (2,3), (3,6), (4,12)\} & \text{if } f_{27}(x) = f_{12}^{(4)}(x) < 0, \\ \{(0,1), (2,3), (3,6)\} & \text{if } f_{27}(x) = f_{12}^{(4)}(x) > 0, \end{cases}$$

and hence

$$\sigma(l_x) = \begin{cases} N(4,12) = 27 & \text{if } f_{27}(x) < 0, \\ N(3,6) = 13 & \text{if } f_{27}(x) > 0. \end{cases}$$

3. Proof of Theorem 1.1

A sector in $\mathbb{R}^n$ is an open conic region in Euclidean space bounded by a finite number of hyperplanes passing through the origin. More precisely,

Definition 3.1. Let $v_1, v_2, \dots v_r$ be distinct unit vectors in $\mathbb{R}^n$, and fix an integer $s \le r$. A *sector* in $\mathbb{R}^n$ is a nonempty set of the form

$$(3.1) \qquad X = \left\{ x \in \mathbb{R}^n : \begin{array}{l} x \cdot v_j > 0 \quad \forall \, 1 \le j \le s \\ x \cdot v_j < 0 \quad \forall \, s < j \le r \end{array} \right\}$$

Note that if $x \in X$, then $tx \in X$ for any $t > 0$. Thus a sector is infinite with nonempty interior, by definition.

We record in Section 3.1 two results (Proposition 3.2 and Lemma 3.3), one analytic and the other geometric, concerning sectors. These results are critical components of the proof of Theorem 1.1. We present the proof of this main theorem later in Section 3.2, modulo the two ingredients. The proofs of the two building blocks appear later in the paper (in Section 4 for Proposition 3.2 and Section 6 for Lemma 3.3).

3.1. **The ingredients of the proof.**

Proposition 3.2. *For any choice of integer $p_0 \geq 1$, there exist constants $C_1, C_2, C_3 > 0$ that depend only on p_0 and the ambient dimension n and satisfy the properties listed below.*

Let $\{X_N : N = 1, 2, \ldots, 2^m - 1\}$ be any finite collection of pairwise disjoint sectors in $\mathbb{R}^n$. Then there exists a corresponding sequence $\{f_N : N = 1, \ldots, 2^m - 1\}$ of smooth, integrable functions with compactly supported Fourier transforms such that:

(a) $\operatorname{supp} \widehat{f_N} \subset X_N$ *for each N.*

(b) For each $p \in [1, 2p_0]$,

$$(3.2) \qquad \Big\| \sum_{N=1}^{2^m-1} f_N \Big\|_p \leq C_1 \sqrt{m}.$$

(c) For the permutation σ defined in (2.7) and used in Lemma 2.3,

$$(3.3) \qquad \Big| \Big\{ x \in Q : \max_{1 \leq l < 2^m} \Big| \sum_{N=1}^{l} f_{\sigma(N)}(x) \Big| \geq C_2 m \Big\} \Big| > C_3.$$

Remark: The functions f_N given by Proposition 3.2 do not form a tree system as defined in Section 2.2. However, there are sequences of functions closely related to $\operatorname{Re}(f_N)$ that are in fact tree systems or signed tree systems. We elaborate on these connections in Section 4 where we prove the proposition; see specifically Lemma 4.2 (a) and (c).

Lemma 3.3. *Let U be a set of unit vectors in $\mathbb{R}^n$, all pointing in distinct directions. Assume that $\#U = M$ for some $M \geq 2$, and that all vectors $v \in U$ obey $v \cdot e_n > 0$, where $e_n = (0, \ldots 0, 1)$. Then there is an ordering $\{u_1, \ldots, u_M\}$ of vectors in U, and a collection of pairwise disjoint sectors $S_1, \ldots S_{M-1} \subset \mathbb{R}^n$ (see Definition 3.1), such that, up to sets of Lebesgue measure 0, we have for $l = 2, \ldots, M$*

$$(3.4) \qquad \Gamma_{u_l} \cap S_i = \begin{cases} S_i & \text{if } i < l, \\ \emptyset & \text{if } i \geq l. \end{cases}$$

3.2. **Completion of the proof.**

Proof of Theorem 1.1, assuming Proposition 3.2 and Lemma 3.3. As noted previously, it suffices to prove (1.7). Let $1 < p < \infty$, and let p_0 be an integer such that $p < 2p_0$. Assume without loss of generality that $\#U$ is sufficiently large relative to p_0, since the bound (1.7) is trivial otherwise. By rotational symmetry, we may assume (after passing to a subset of cardinality at least $(\#U)/2$ if necessary) that all vectors $v \in U$ obey $v \cdot e_n > 0$, where $e_n = (0, \ldots 0, 1)$. Passing to a further subset $\widetilde{U} \subseteq U$, we may also assume that $\#\widetilde{U} = 2^m$ with $m \in \mathbb{N}$ and $m \geq 0.1 \log(\#U)$. Since T_U dominates $T_{\widetilde{U}}$, we will henceforth work with $\widetilde{U}$, renaming it U.

Lemma 3.3 now yields an ordering $\{u_1, \ldots, u_{2^m}\}$ of vectors in U, and a collection of non-empty and pairwise disjoint sectors $S_1, \ldots S_{2^m-1} \subset \mathbb{R}^n$, such that for $l = 2, \ldots, 2^m$ we have

$$(3.5) \qquad \Gamma_{u_l} \cap S_i = \begin{cases} S_i & \text{if } i < l, \\ \emptyset & \text{if } i \geq l. \end{cases}$$

We now apply Proposition 3.2 to the sectors $X_N := S_{\sigma^{-1}(N)}$ for $N = 1, \ldots 2^m - 1$. Let $f = \sum_{N=1}^{2^m-1} f_N$, where f_N are the functions provided by Proposition 3.2. By (b), we have

$$(3.6) \qquad \qquad \|f\|_p \leq C_1 \sqrt{m}.$$

On the other hand, by Proposition 3.2(a), we have $\operatorname{supp} \widehat{f}_{\sigma(N)} \subset X_{\sigma(N)} = S_N$. Using this and (3.5), we get that for $l = 2, \ldots, 2^m$,

$$T_{u_l} f = (\chi_{\Gamma_{u_l}} \widehat{f})^\vee = \Big(\sum_{N=1}^{2^m-1} \chi_{\Gamma_{u_l}} \widehat{f_{\sigma(N)}} \Big)^\vee = \Big(\sum_{N=1}^{l-1} \widehat{f_{\sigma(N)}} \Big)^\vee = \sum_{N=1}^{l-1} f_{\sigma(N)}.$$

Hence

$$T_U f(x) \geq \max_{2 \leq l \leq 2^m} T_{u_l} f = \max_{1 \leq l \leq 2^m - 1} \Big| \sum_{N=1}^{l} f_{\sigma(N)}(x) \Big|.$$

By Proposition 3.2(c), it follows that

$$|\{x \in Q : T_U f(x) \geq C_2 m\}| > C_3,$$

so that for any $1 < p < \infty$, we have

$$\|T_U f\|_p \geq C_2 C_3^{1/p} m.$$

The estimate (1.7) follows from this and (3.6). $\qquad \square$

4. Proof of Proposition 3.2

4.1. The inductive construction of functions.
Proposition 3.2 asserts the existence of certain functions f_N; these will be of the following form,

$$(4.1) \qquad f_N(x) := e(\bar{p}_N \cdot x) g_N(x) \quad \text{with} \quad g_N := \phi_{\ell_N} * \chi_{E_N}.$$

We pause for a moment to clarify the notation in the preceding line. Here $\phi_\ell(x) := \ell^n \phi(\ell x)$, and ϕ is a Schwartz function on $\mathbb{R}^n$ such that

$$\phi \geq 0, \ \int_{\mathbb{R}^n} \phi(x)\,dx = 1 \ \text{ and } \ \operatorname{supp}(\widehat{\phi}) \subset [-1, 1]^n.$$

The sets E_N, the parameters ℓ_N and the vectors $\bar{p}_N$ appearing in (4.1) will be specified shortly in Proposition 4.1 below using an inductive mechanism and in the sequential order

$$E_1 \to \ell_1 \to \bar{p}_1 \to E_2 \to \ell_2 \to \bar{p}_2 \to \cdots,$$

subject to the defining condition $E_1 := Q = [0, 1]^n$, and

$$(4.2) \qquad E_N = E_j^{(k)} := \big\{ x \in E_{N^*} : (-1)^{j-1} \cos(2\pi x \cdot \bar{p}_{N^*}) > 0 \big\}, \quad N \geq 2, \quad N^* = \lfloor \tfrac{N}{2} \rfloor.$$

As we will see, the parameter p_N specifies the "location" and ℓ_N the "size" of the frequency support of f_N. These frequency supports will obey a number of constraints, one of which is pairwise disjointness. On the other hand, the spatial support of f_N, while not perfectly localized, is essentially contained in E_N. The set E_N will be shown to be nonempty and of positive measure, for every N. Here for sets as well as functions, we will continue to use the double-indexing notation from Section 2, identifying N with the pair (k, j) as given by the relation (2.1). We will also use $Q(\ell, x)$ to denote the axis-parallel cube with centre x and side length ℓ. For a multi-index $J = (j_1, \ldots, j_h) \in \mathbb{N}^h$, we will write $\|J\|_\infty = \max_i j_i$; additionally, if $J = (J_1, J_2)$ is a pair of such multi-indices, we will use $\|J\|_\infty$ to denote $\max(\|J_1\|_\infty, \|J_2\|_\infty)$.

Proposition 4.1. *Let p_0 and $\{X_N : N = 1, 2, \ldots 2^m - 1\}$ be as in Proposition 3.2. For any sufficiently large $C \gg p_0$, there exists a choice of large constants ℓ_N and vectors $\bar{p}_N \in \mathbb{Z}^n$ of large magnitude such that for all $N = 1, \ldots, 2^m - 1$ the following properties hold.*

(a) $Q_N := Q(\ell_N, \bar{p}_N) \subset X_N$.

(b) Given any $1 \leq h \leq p_0$, and any $2h$-tuple of indices $J = (J_1, J_2) \in \{1, \ldots, N\}^{2h}$, with $J_1 = (j_1, \ldots, j_h)$, $J_2 = (j'_1, \ldots, j'_h)$, we have

$$(4.3) \qquad \sum_{j \in J_1} Q_j \cap \sum_{j \in J_2} Q_j = \emptyset \quad \text{whenever} \quad \#\{r : j_r = ||J||_\infty\} \neq \#\{r : j'_r = ||J||_\infty\}.$$

   Here the sum of sets denotes the Minkowski sum, where $A + B = \{a + b : a \in A, b \in B\}$.

(c) For E_N defined as in (4.2), the vector $\bar{p}_N$ additionally satisfies

$$(4.4) \qquad \int_{E_N} |\cos(2\pi x \cdot \bar{p}_N)| \, dx \; > \; \frac{|E_N|}{3}.$$

(d) The functions $\chi_N := \chi_{E_N}$ and g_N in (4.1) obey

$$0 \leq g_N \leq 1, \quad supp(\widehat{g}_N) \subset [-\ell_N, \ell_N]^n, \quad \text{and}$$

$$(4.5) \qquad \left\|g_N - \chi_N\right\|_1 + \left\|g_N - \chi_N\right\|_{2p_0} \leq 2^{-Cm}.$$

(e) The function $x \mapsto \cos(2\pi x \cdot \bar{p}_N)$ changes sign in E_N. More precisely, the sets $\{x \in E_N : \cos(2\pi x \cdot \bar{p}_N) > 0\}$ and $\{x \in E_N : \cos(2\pi x \cdot \bar{p}_N) < 0\}$ both have positive Lebesgue measure.

Remark: Before embarking on the proof, let us rephrase the geometric condition (4.3) in an analytical form that is more convenient to check. Since

$$\sum_{j \in J_i} Q_j = Q(L_i, \bar{P}_i), \quad \text{with} \quad L_i = \sum_{j \in J_i} \ell_j, \; \bar{P}_i = \sum_{j \in J_i} \bar{p}_j,$$

the condition (4.3) is equivalent to $\bar{P}_1 - \bar{P}_2 \notin Q(L_1 + L_2, 0)$. If we set $j_0 = ||J||_\infty$, $\mu = \#\{r : j_r = j_0\}$ and $\nu = \#\{r : j'_r = j_0\}$, this in turn can be written as

$$(4.6) \qquad (\mu - \nu)\bar{p}_{j_0} + \sum_{j_r < j_0} \bar{p}_{j_r} - \sum_{j'_r < j_0} \bar{p}_{j'_r} \notin Q\Big(\sum_{j \in J} \ell_j, 0\Big).$$

If $\mu \neq \nu$, this condition specifies a set of possible $\bar{p}_{j_0}$ that ensures the disjointness condition (4.3). We will use this to define $\bar{p}_N$ in the sequel.

Proof of Proposition 4.1. The proof proceeds by induction on N. The sequence $\{\phi_\ell : \ell \geq 1\}$ is an approximation to the identity; hence setting $E_1 = E_1^{(0)} = Q$, we can choose $\ell_1 > 0$ large enough so that (4.5) holds with $N = 1$. Clearly $0 \leq g_1 \leq 1$. Further $\widehat{g}_1 = \widehat{\phi}_{\ell_1} \widehat{\chi}_{E_1}$, so we also have $\mathrm{supp}\,(\widehat{g}_1) \subset \mathrm{supp}\,\widehat{\phi}_{\ell_1} \subset [-\ell_1, \ell_1]^n$. This verifies the requirements of part (d). The condition (4.3) (or equivalently (4.6)), as required by part (b), is vacuous in this case, since the only cube available so far is Q_1, and hence $\mu = \nu$ for any choice of multi-index J. For (c), we observe that for any choice of nonzero $\bar{p}_1 \in \mathbb{Z}^n$ we have

$$\int_{E_1} |\cos(2\pi x \cdot \bar{p}_1)| \, dx = \int_Q |\cos(2\pi x \cdot \bar{p}_1)| \, dx \geq \int_Q \cos^2(2\pi x \cdot \bar{p}_1) \, dx$$

$$= \int_Q \frac{1 + \cos\left(4\pi x \cdot \bar{p}_1\right)}{2} dx = \frac{|Q|}{2} = \frac{|E_1|}{2} > \frac{|E_1|}{3}.$$

Thus any nontrivial choice of $\bar{p}_1$ would ensure (4.4). Since $E_1 = Q$, condition (e) is also trivially satisfied. With ℓ_1 already chosen as above and keeping in mind that X_1 is a sector with unbounded interior, we can now select $\bar{p}_1$ so that (a) holds. This completes the verification of the base case $N = 1$.

For the inductive step, assume that we have constructed $\bar{p}_i, l_i, E_i, g_i$ obeying all conclusions of the lemma for $i = 1, 2, \ldots, N - 1$. Define E_N via (4.2). Note that this is possible since E_{N^*} and $\bar{p}_{N^*}$ have already been set. Further, E_N thus defined is nonempty, measurable and of positive measure since condition (e) holds for N^*. Hence we can choose $\ell_N > 0$ large enough so that (4.5) holds. The properties $0 \le g_N \le 1$ and $\operatorname{supp}(\widehat{g}_N) \subset [-\ell_N, \ell_N]^n$ follow as in the case $N = 1$, establishing part (d). For (c), we argue as follows: given any $\bar{p} \in \mathbb{Z}^n$ we have

$$\int_{E_N} |\cos\left(2\pi x \cdot \bar{p}\right)| \, dx \ge \int_{E_N} |\cos\left(2\pi x \cdot \bar{p}\right)|^2 \, dx \ge \int_{E_N} \frac{1 + \cos\left(4\pi x \cdot \bar{p}\right)}{2} dx$$

$$= \frac{|E_N|}{2} + \frac{1}{2} \int_{E_N} \cos\left(4\pi x \cdot \bar{p}\right) dx = \frac{|E_N|}{2} + \frac{1}{2} \operatorname{Re} \widehat{\chi}_N(2\bar{p}).$$

By the Riemann-Lebesgue lemma, $\widehat{\chi}_N(\xi) \to 0$ as $|\xi| \to \infty$. Thus for any choice of $\xi = \bar{p}_N$ with $|\bar{p}_N|$ large enough, we can ensure $|\widehat{\chi}_N(\bar{p}_N)| < |E_N|/3$, resulting in (4.4).

For part (b), we must choose $\bar{p}_N$ so that (4.6) holds for all $2h$-dimensional multi-indices $J = (j_1, \cdots, j_h; j_1', \cdots, j_h')$ with $||J||_\infty \le N$ and $\mu \ne \nu$. If $||J||_\infty \le N - 1$, this is a consequence of the induction hypothesis. We may therefore assume $||J||_\infty = N$. This means that $j_0 = N$ in the notation of (4.6). In order to ensure (4.6), we must have for $s = \pm 1, \ldots, \pm p_0$,

$$s\bar{p}_N \notin -\sum_{j_r < N} \bar{p}_{j_r} + \sum_{j_r' < N} \bar{p}_{j_r'} + Q\Big(\sum_{j \in J} \ell_j, 0\Big).$$

Since $\bar{p}_1, \cdots, \bar{p}_{N-1}$ and $\ell_1, \cdots, \ell_N$ have been determined by the previous steps of the construction, the right hand side of the relation above gives us a finite number of known cubes that $s\bar{p}_N$ must avoid for $s = \pm 1, \ldots, \pm p_0$. This can be guaranteed if we assume that $|\bar{p}_N|$ is large enough.

To establish (e), we observe that the periodic function $x \mapsto \cos(2\pi x \cdot \bar{p}_N)$ alternately assumes positive and negative values on parallel strips separated by distance $\sim |p_N|^{-1}$ and of comparable thickness. Thus given any open ball in Q, one can always choose $\bar{p}_N$ large enough so that $\cos(2\pi x \cdot \bar{p}_N)$ changes sign on the ball. Since E_N is by definition open relative to Q, condition (e) follows.

Note that the possible choices of $\bar{p}_N$ so far only require the vector to be large in magnitude, with no restriction in direction. Now we choose a specific direction, and place $\bar{p}_N$ so that we additionally have $Q_N \subset X_N$, establishing (a). This completes the inductive step and hence the proof of the proposition. $\square$

4.2. Finer properties of f_N.

The algorithm described in Section 4.1 endows the resulting sets $E_N = E_j^{(k)}$ and functions $f_N = f_j^{(k)}$ with properties beyond those given in Proposition

4.1. A few of these finer properties are essential to the proof of Proposition 3.2. We record them here.

Lemma 4.2. *Let E_N be the sets defined in Proposition 4.1. Then the following conclusions hold.*

(a) The functions $\chi_N = \chi_j^{(k)}$ form a tree system, as defined in Definition 2.1(a)). In particular, they obey the conclusions of Lemma 2.2(a)-(c). Moreover, the following functional identities hold Lebesgue almost surely on Q.

$$\chi_{2j-1}^{(k+1)} + \chi_{2j}^{(k+1)} = \chi_j^{(k)}, \text{ and} \tag{4.7}$$

$$\sum_{N=1}^{2^m-1} \chi_N \equiv m \tag{4.8}$$

(b) Let $N_0 \in \{1, \ldots 2^m - 1\}$. Using the terminology of trees introduced in Section 2, let us denote by $\mathcal{T}^{(N_0)}$ the subtree of $\mathcal{T}_m$ having N_0 as root. Then for every fixed integer r with $h(N_0) \leq r \leq m$, we have:

$$\sum_{\substack{N \in \mathcal{T}^{(N_0)} \\ h(N)=r}} \chi_N = \chi_{N_0}. \tag{4.9}$$

(c) The family of functions $\{\tilde{f}_N := \cos(2\pi\bar{p}_N\cdot)\chi_N : N = 1, \cdots, 2^m - 1\}$ is a signed tree system in the sense of Definition 2.1(b).

Proof. Rewriting (4.2) in terms of j, k, we get that $E_1 = Q$ and

$$E_{2j-1}^{(k+1)} := \{x \in E_j^{(k)} : \cos(x \cdot \bar{p}_j^{(k)}) > 0\},$$
$$E_{2j}^{(k+1)} := \{x \in E_j^{(k)} : \cos(x \cdot \bar{p}_j^{(k)}) < 0\}.$$

Therefore, the sets $E_{2j-1}^{(k+1)}$ and $E_{2j}^{(k+1)}$ are disjoint and contained in $E_j^{(k)}$, so that the functions $\chi_j^{(k)}$ form a tree system. Since the set $\{x \in Q : \cos(x \cdot \bar{p}_j^{(k)}) = 0\}$ has Legesgue measure 0, we also have (4.7). Iterating (4.7), we get that the following holds almost surely,

$$\begin{aligned}
\chi_Q = \chi_1^{(0)} &= \chi_1^{(1)} + \chi_2^{(1)} \\
&= (\chi_1^{(2)} + \chi_2^{(2)}) + (\chi_3^{(2)} + \chi_4^{(2)}) \\
&= \cdots = \sum_{1 \leq j \leq 2^k} \chi_j^{(k)}
\end{aligned} \tag{4.10}$$

for every $k = 0, 1, \ldots, m - 1$. Summing over k yields (4.8).

We now turn to (4.9). If $h(N_0) = r$, the summation is over the single vertex N_0 and there is nothing to prove. If $h(N_0) < r \leq m$, then (4.9) follows from the same calculations as in (4.10), except we start from χ_{N_0} instead of χ_Q.

Regarding (c), let us observe that $\tilde{f}_N(x) = \cos(2\pi\bar{p}_N \cdot x)\chi_N$ and $\cos(2\pi\bar{p}_N \cdot x)$ have the same sign in the set E_N. In view of (4.2), this shows that we have, up to sets of Lebesgue measure zero

$$\text{supp}\, \tilde{f}_N = E_N = \left\{x \in E_{N^*} : (-1)^{j-1}\cos(2\pi x \cdot \bar{p}_{N^*}) > 0\right\}$$

$$= \left\{x \in E_{N^*} : (-1)^{j-1}\tilde{f}_{N^*}(x) > 0\right\} = \left\{x \in Q : (-1)^{j-1}\tilde{f}_{N^*}(x) > 0\right\}.$$

This is exactly the signed tree system condition (2.5). □

The confluence of spatial and frequency localization built into the definition of f_N results in a high degree of orthogonality amongst them. This interaction is manifested in the L^p-norms of their sums, for large exponents p. The following proposition, which offers an estimate of this norm, is a critical component in the proof of Proposition 3.2(b). The proof of the proposition is nontrivial and is relegated to Section 5.

Proposition 4.3. *For p_0 and $\{X_N\}$ as in Proposition 3.2, let $\{f_N : N = 1, \ldots, 2^m - 1\}$ be the family of functions given by (4.1) in Section 4.1. Then there exists a constant $C_0 > 0$ depending only on p_0 and n such that*

$$(4.11) \qquad \qquad || \sum_{N=1}^{2^m-1} f_N ||_{2p_0} \leq C_0\sqrt{m}.$$

Assuming this, the proof of Proposition 3.2 is completed in the next subsection.

4.3. **Proof of Proposition 3.2.**

Proof. Since $\mathrm{supp}(\widehat{f}_N) \subseteq Q_N$, part (a) of the proposition follows from Proposition 4.1(a).

Let us turn to (b). Given that $p \in [1, 2p_0]$, the desired conclusion follows from the log-convexity of Lebesgue norms, provided we have the correct estimates at the endpoints $p = 2p_0$ and $p = 1$. Proposition 4.3 asserts the necessary bound for $p = 2p_0$. Our claim is that the bound for $p = 1$ follows from the same proposition. The following chain of inequalities establishes this claim:

$$|| \sum_{N=1}^{2^m-1} f_N ||_1 \leq || \sum_{N=1}^{2^m-1} e(\bar{p}_N \cdot)\chi_N ||_1 + || \sum_{N=1}^{2^m-1} e(\bar{p}_N \cdot)(g_N - \chi_N) ||_1$$

$$\leq || \sum_{N=1}^{2^m-1} e(\bar{p}_N \cdot)\chi_N ||_1 + \sum_{N=1}^{2^m-1} ||g_N - \chi_N||_1$$

$$\leq || \sum_{N=1}^{2^m-1} e(\bar{p}_N \cdot)\chi_N ||_1 + 2^{m-Cm}$$

$$\leq || \sum_{N=1}^{2^m-1} e(\bar{p}_N \cdot)\chi_N ||_{2p_0} + 2^{m-Cm}$$

$$\leq || \sum_{N=1}^{2^m-1} f_N ||_{2p_0} + || \sum_{N=1}^{2^m-1} e(\bar{p}_N \cdot)(\chi_N - g_N) ||_{2p_0} + 2^{m-Cm}$$

$$\leq || \sum_{N=1}^{2^m-1} f_N ||_{2p_0} + 2 \cdot 2^{m-Cm}$$

$$\leq C_0\sqrt{m} + 1, \text{ which is } \leq C_1\sqrt{m} \text{ if } C_1 > 2C_0.$$

The third and the sixth inequality in the sequence above uses the error bound (4.5) proved in Proposition 4.1(d). The fourth inequality follows from the fact that $G = \sum_N e(p_N \cdot)\chi_N$ is supported on Q, and hence $\|G\|_1 \le \|G\|_{2p_0}$ by Hölder's inequality. The last inequality follows from the main estimate (4.11) in Proposition 4.3. The triangle inequality is used throughout. This completes the proof of (b).

It remains to prove (c). Recall from Lemma 4.2(c) that $\tilde{f}_N := \cos(2\pi\bar{p}_N \cdot)\chi_N$ is a signed tree system. Hence by Lemma 2.3 for signed tree systems, we have

$$
\max_{1 \le l < 2^m} \Big| \sum_{N=1}^{l} f_{\sigma(N)}(x) \Big| \ge \max_{1 \le l < 2^m} \Big| \sum_{N=1}^{l} \mathrm{Re}\, f_{\sigma(N)}(x) \Big|
$$

$$
\ge \max_{1 \le l < 2^m} \Big| \sum_{N=1}^{l} \tilde{f}_{\sigma(N)}(x) \Big| - \mathcal{E}(x)
$$

$$
\ge \frac{1}{3} \sum_{N=1}^{2^m-1} \big| \tilde{f}_N(x) \big| - \mathcal{E}(x),
$$

for all $x \in \mathbb{R}^n$, where $\mathcal{E}(x) = \sum_{N=1}^{2^m-1} \big| \tilde{f}_N(x) - \mathrm{Re}\, f_N(x) \big|$. The last inequality above follows from (2.8), the rest from the triangle inequality. We will show that

$$
(4.12) \qquad \Big| \Big\{ x : \sum_{N=1}^{2^m-1} |\tilde{f}_N(x)| \ge \frac{m}{10} \Big\} \Big| \ge \frac{1}{10}, \quad \text{and that}
$$

$$
(4.13) \qquad \big| \{ x : |\mathcal{E}(x)| > 1 \} \big| \le \frac{1}{100} \text{ if } C \text{ in Proposition 4.1 is sufficiently large.}
$$

For large m, this would ensure (3.3) and complete the proof, with constants $C_2 = 1/20$ and $C_3 = 9/100$, for instance.

To prove (4.12), let us set $\tilde{f}(x) := \sum_{N=1}^{2^m-1} \big| \tilde{f}_N(x) \big|$. On one hand, by Proposition 4.1(c),

$$
\int_Q |\tilde{f}_N(x)| dx = \int_{E_N} |\cos(2\pi x \cdot \bar{p}_N)| \, dx > \frac{|E_N|}{3}.
$$

Summing over all N and using (4.8) in Lemma 4.2, we obtain

$$
(4.14) \qquad \int_Q \tilde{f}(x)\, dx = \int_Q \sum_{N=1}^{2^m-1} |\tilde{f}_N(x)| dx \ge \frac{1}{3} \sum_{N=1}^{2^m-1} |E_N| = \frac{1}{3} \int_Q \sum_{N=1}^{2^m-1} \chi_N(x)\, dx = \frac{m}{3}.
$$

On the other hand,

$$
(4.15) \qquad \tilde{f}(x) \le \sum_{N=1}^{2^m-1} \chi_N(x) = m \text{ for a.e. } x \in Q.
$$

Set $E := \{ x \in Q : \tilde{f}(x) \ge \frac{m}{10} \}$. Combining (4.14) and (4.15), we see that

$$
m|E| + \frac{m}{10} \ge \int_E \tilde{f}(x)dx + \int_{Q \setminus E} \tilde{f}(x)dx = \int_Q \tilde{f}(x)dx \ge \frac{m}{3},
$$

This shows that $|E| > \frac{1}{3} - \frac{1}{10} > \frac{1}{10}$, establishing (4.12).

Regarding (4.13), we make use of (4.5) to deduce that

$$\|\mathcal{E}\|_1 = \Big\| \sum_{N=1}^{2^m-1} |\tilde{f}_N - \mathrm{Re}\, f_N| \Big\|_1 \leq \Big\| \sum_{N=1}^{2^m-1} |\cos(2\pi \bar{p}_N \cdot)(\chi_N - g_N)| \Big\|_1$$

$$\leq \sum_{N=1}^{2^m-1} ||\chi_N - g_N||_1 \leq 2^{m-Cm}.$$

Therefore, by Chebyshev's inequality

(4.16) $$\big|\{x \in Q : \mathcal{E}(x) > 1\}\big| \ \leq \ \big\|\mathcal{E}\big\|_1 \leq 2^{m-Cm},$$

which proves our claim (4.13) for $C > 0$ sufficiently large. $\qquad\square$

5. Norm estimate: Proof of Proposition 4.3

This section is given over to the estimation of the L^{2p_0} norm of the function $f := \sum_N f_N$, with the summands f_N defined as in Proposition 4.1. Parts of the argument are highly combinatorial, involving summations over index sets whose members are long sequences of integers. Two previously introduced tools will continue to be useful for book-keeping purposes; namely, the double-indexing notation relating N with the pair (k, j) as in (2.1), and the language of trees as described in Section 2. We begin by setting up some supplementary notation that will be convenient for handling sums over large index sets later on.

5.1. Notation.

5.1.1. *Small errors.* For any two quantities X and Y depending on m, we will write $X \cong Y$ if $|X - Y| \leq A2^{-Bm}$, where the multiplicative constant A and the exponent B may depend on p_0 and n, and may change from line to line but remain independent of m. Both A and B will always be sufficiently large. In our applications, B will depend on the large constant C from Proposition 4.1. Assuming that $C \gg p_0$ was chosen large enough, we will always be able to ensure that $B > \frac{C}{10}$.

The notation $X = O(Y)$ will be used to mean $|X| \leq A|Y|$, with the same conditions on the constant A as above.

5.1.2. *Grouping of vectors of vertices.* Our main estimate will be proved by expanding the L^{2p_0} norm of f as a sum of integrals of the form

(5.1) $$\int f_{m_1}^{\mu_1} \cdots f_{m_r}^{\mu_r} \overline{f_{n_1}^{\nu_1} \cdots f_{n_s}^{\nu_s}}\, dx$$

with $\mu_1 + \cdots + \mu_r = \nu_1 + \cdots + \nu_s = p_0$, then grouping these integrals appropriately to obtain cancellations and simplifications. The notation introduced in this subsection will facilitate that process.

Given an integer exponent $1 \leq p \leq p_0$ and an integer dimension $1 \leq r \leq p$, we define a *multiplicity vector for the exponent p of length r* to be of the form

$$\bar{\mu} = (\mu_1, \cdots, \mu_r) \in \mathbb{N}^r, \quad \text{where} \quad \mu_1 + \mu_2 + \cdots + \mu_r = p.$$

The use of a multiplicity vector allows us to rewrite a p-long integer vector with some possibly coincident entries in "collapsed form". For instance, all such sequences with r

distinct entries, where the i-th smallest element occurs with frequency μ_i, can be gathered into a single collection, as explained below.

Given integers $1 \leq h \leq m$, $1 \leq r \leq p \leq p_0$ we set

$$(5.2) \qquad \mathbb{A}_{p,h}[r] := \left\{ (\bar{m}, \bar{\mu}) \,\middle|\, \begin{array}{l} \bar{m} = (m_1, \cdots, m_r) \in \mathbb{N}^r, \ 1 \leq m_1 < \cdots < m_r < 2^h \\ \bar{\mu} \text{ is a multiplicity vector for } p \text{ of length } r \end{array} \right\}.$$

Observe that for every $(\bar{m}, \bar{\mu}) \in \mathbb{A}_{p,h}[r]$, there exist p-dimensional vectors $\mathbf{N} \in \mathbb{N}^p$ such that m_i occurs in the string $\mathbf{N}$ exactly μ_i times for each $1 \leq i \leq r$. For example, we can take $\mathbf{N}$ to be $\mathbf{N}[\bar{m}, \bar{\mu}]$, which is by definition a p-long vector whose first μ_1 entries are m_1, the next μ_2 entries are m_2, and so on. The relevance of $\mathbb{A}_{p,h}[r]$ lies in the following partition of the index set:

$$(5.3) \quad \{1, \cdots, 2^h - 1\}^p = \bigsqcup_{r=1}^{p} \{\mathbf{N} : \exists (\bar{m}, \bar{\mu}) \in \mathbb{A}_{p,h}[r] \text{ so that } \mathbf{N} \text{ is a permutation of } \mathbf{N}[\bar{m}, \bar{\mu}]\}.$$

Right now, an element of $\mathbb{A}_{p,h}[r]$ is a 2-tuple $(\bar{m}, \bar{\mu})$, whose first component $\bar{m}$ is a multi-index and whose second component $\bar{\mu}$ is an r-long multiplicity vector for p. The number of choices of $\bar{\mu}$ for a fixed p and r is bounded by a constant depending only on p and independent of $\#U$ (and hence h), whereas $\bar{m}$ ranges over an index set of cardinality $O(2^{hr})$, which is typically much larger. For the quantitative bounds that we seek, it is therefore no loss of generality to work with a fixed multiplicity vector $\bar{\mu}$ at a time. In order to keep track of the collection of all multi-indices $\bar{m}$ that generate elements of $\mathbb{A}_{p,h}[r]$ for a fixed multiplicity, we define

$$(5.4) \qquad\qquad\qquad \mathbb{A}_h[r, \bar{\mu}] := \{\bar{m} : (\bar{m}, \bar{\mu}) \in \mathbb{A}_{p,h}[r]\},$$

which is in effect the $\bar{\mu}$-fibre of $\mathbb{A}_{p,h}[r]$.

We will also need to stratify pairs of vectors according to the position of their combined maximal element in the binary tree. With that in mind and given multiplicity vectors $\bar{\mu}, \bar{\nu}$ of length r, s for the exponents p, q respectively, we set

$$(5.5) \qquad \mathbb{M}_h[r, \bar{\mu}; s, \bar{\nu}] := \left\{ \bar{\alpha} = (\bar{m}, \bar{n}) \,\middle|\, \begin{array}{l} \bar{m} \in \mathbb{A}_h[r, \bar{\mu}], \ \bar{n} \in \mathbb{A}_h[s, \bar{\nu}], \\ h(||\bar{\alpha}||_\infty) = h(\max(m_r, n_s)) = h \end{array} \right\}.$$

Recall that $h(N) = k$ denotes the height of the vertex $N = 2^k + j - 1$ in the binary tree $\mathcal{T}_m$. The parameters $r, s, \bar{\mu}, \bar{\nu}$ occurring in the argument of (5.5) will be suppressed if they are clear from the context.

Two special subclasses of $\mathbb{M}_h$ will be important for our analysis. They are:

$$(5.6) \qquad \mathbb{M}'_h := \{\bar{\alpha} \in \mathbb{M}_h : \exists \text{ a ray } \mathscr{R} \text{ in } \mathcal{T}_h \text{ such that all entries of } \bar{\alpha} \text{ lie on } \mathscr{R}\}$$

$$(5.7) \qquad \mathbb{M}^*_h := \begin{cases} \{\bar{\alpha} = (\bar{m}, \bar{n}) \in \mathbb{M}'_h : m_r = n_s\} & \text{if } \mu_r = \nu_s, \\ \emptyset & \text{otherwise.} \end{cases}$$

Figures 5 and 6 depict examples of multi-indices $\bar{\alpha}$ that lie in these special subclasses.
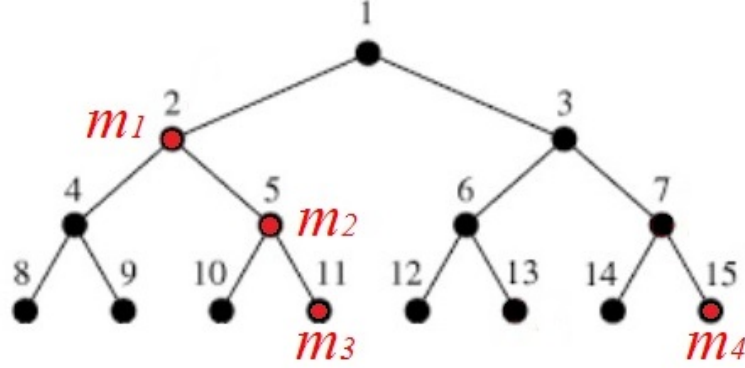
FIGURE 5. Fix $\bar{\mu}, \bar{\nu}$ with $r = s = 3$. Let $\bar{m} = (m_1, m_2, m_3)$, $\bar{m}' = (m_1, m_2, m_4)$. Then $\bar{\alpha} = (\bar{m}, \bar{m}) \in \mathbb{M}'_h$ but $\bar{\beta} = (\bar{m}, \bar{m}') \notin \mathbb{M}'_h$.
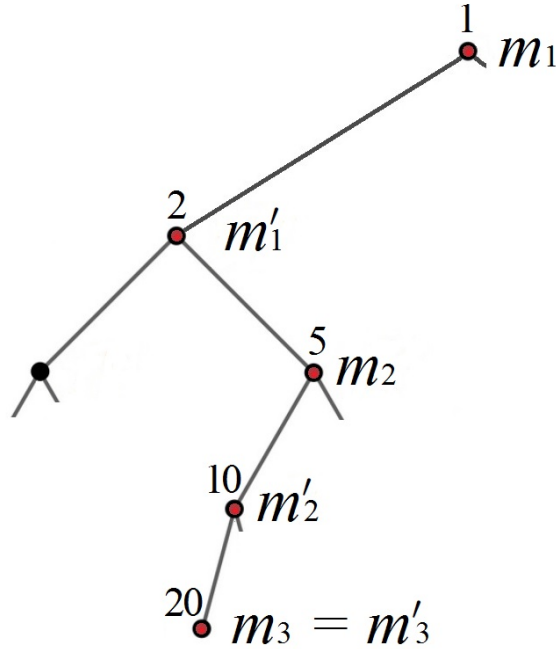

FIGURE 6. Fix multiplicity vectors $\bar{\mu}, \bar{\nu}$ with $r = s = 3$ and $\mu_3 = \nu_3$. Let $\bar{m} = (m_1, m_2, m_3)$, $\bar{n} = (m'_1, m'_2, m'_3)$ so that $m_3 = m'_3$ and all m_i, m'_i lie on the same ray of the tree. Then $\bar{\alpha} = (\bar{m}, \bar{n}) \in \mathbb{M}^*_h$ for these multiplicity vectors.

5.2. **Main steps.** The relevance of the aforementioned notation in the context of the norm estimation problem is clarified in the following sequence of lemmas, which provides the key ingredients.

Lemma 5.1. *Let* $f := \sum_N f_N$, *with* f_N *given by* (4.1) *in Section 4. Then*

$$(5.8) \qquad \|f\|_{2p_0}^{2p_0} = \sideset{}{'}\sum C(\bar{\mu}, \bar{\nu}) \sum_{h=1}^{m} \mathrm{F}_h[r, \bar{\mu}; s, \bar{\nu}], \qquad \text{where} \qquad \mathrm{F}_h := \sum_{\bar{\alpha} \in \mathbb{M}_h} \int F_{\bar{\alpha}}(x) \, dx.$$

To explain the notation in the above line,

- *The outer sum $\sum'$ ranges over all choices of positive integers $1 \le r, s \le p_0$ and all choices of multiplicity vectors $\bar{\mu}, \bar{\nu}$ for the exponent p_0 of lengths r and s respectively.*

- *The constants $C(\bar{\mu}, \bar{\nu})$ depend only on $\bar{\mu}, \bar{\nu}, r, s, p_0$ and are independent of m; specifically*

$$C(\bar{\mu}, \bar{\nu}) = \binom{p_0}{\bar{\mu}} \binom{p_0}{\bar{\nu}} = \frac{(p_0!)^2}{\mu_1! \cdots \mu_r! \nu_1! \cdots \nu_s!}.$$

- *Given $\bar{\alpha} = (\bar{m}, \bar{n}) \in \mathbb{M}_h[r, \bar{\mu}; s, \bar{\nu}]$,*

(5.9)
$$F_{\bar{\alpha}} := f_{m_1}^{\mu_1} \cdots f_{m_r}^{\mu_r} \overline{f_{n_1}^{\nu_1} \cdots f_{n_s}^{\nu_s}}.$$

We will continue to use the notation (5.9) even if $\bar{\mu}, \bar{\nu}$ are multiplicity vectors for different exponents.

Lemma 5.2. *Fix $1 \le h \le m$, and two multiplicity vectors $\bar{\mu}, \bar{\nu}$ for integer exponents $1 \le p, q \le p_0$ of lengths r, s respectively. Then the following conclusions hold:*

(a) For any $\bar{\alpha} = (\bar{m}, \bar{n}) \in \mathbb{M}_h[r, \bar{\mu}; s, \bar{\nu}]$,

(5.10)
$$\int F_{\bar{\alpha}} dx \cong \int G_{\bar{\alpha}} dx, \quad where$$

(5.11)
$$G_{\bar{\alpha}}(x) := e(v_{\bar{\alpha}} \cdot x) \chi_{E_{\bar{\alpha}}}(x), \quad with \quad v_{\bar{\alpha}} := \sum_{i=1}^{r} \mu_i \bar{p}_{m_i} - \sum_{i=1}^{s} \nu_i \bar{p}_{n_i}, \quad and$$

(5.12)
$$E_{\bar{\alpha}} := E_{m_1} \cap \ldots E_{m_r} \cap E_{n_1} \cap \ldots \cap E_{n_s}.$$

Here the notation $\cong$ denotes equality up to small errors, as explained in Section 5.1.1. The sets E_N on the right hand side of (5.12) are as in (4.2).

(b) If $E_{\bar{\alpha}} \ne \emptyset$, then $\bar{\alpha} \in \mathbb{M}_h'$. The converse is also true. For such $\bar{\alpha}$, we have

(5.13)
$$E_{\bar{\alpha}} = E_{||\bar{\alpha}||_\infty}.$$

(c) As a consequence, $\mathsf{F}_h \cong \mathsf{G}_h = \mathsf{G}_h'$, where

$$\mathsf{G}_h[r, \bar{\mu}; s, \bar{\nu}] := \sum_{\bar{\alpha} \in \mathbb{M}_h} \int G_{\bar{\alpha}}(x)\, dx \quad and \quad \mathsf{G}_h'[r, \bar{\mu}; s, \bar{\nu}] := \sum_{\bar{\alpha} \in \mathbb{M}_h'} \int G_{\bar{\alpha}}(x)\, dx.$$

Lemma 5.3. *In the notation of Lemma 5.2, we have*

(5.14)
$$\mathsf{G}_h[r, \bar{\mu}; s, \bar{\nu}] \cong \mathsf{G}_h^*[r, \bar{\mu}; s, \bar{\nu}] \cong \begin{cases} 0 & if \ \bar{\mu} \ne \bar{\nu}, \\ O\left(h^{r-1}\right) & if \ \bar{\mu} = \bar{\nu}. \end{cases}$$

Per our notational convention, we have used

$$\mathsf{G}_h^*[r, \bar{\mu}; s, \bar{\nu}] := \sum_{\bar{\alpha} \in \mathbb{M}_h^*} \int G_{\bar{\alpha}}(x)\, dx.$$

5.3. **Proof of Proposition 4.3.**

Proof. We complete the proof of the proposition assuming the three lemmas above. In view of Lemma 5.1, $||f||_{2p_0}^{2p_0}$ is given by (5.8). The number of summands in the outer sum on the right hand side of the equation (5.8) depends only on p_0; hence it suffices to show that

$$\sum_{h=1}^{m} \mathsf{F}_h[r, \bar{\mu}; s, \bar{\nu}] = O(m^{p_0})$$

for every fixed choice of integers $r, s \leq p_0$ and for each choice of multiplicity vectors $\bar{\mu}, \bar{\nu}$ for the exponent p_0 of lengths respectively r and s. Combining Lemma 5.2(c) with Lemma 5.3, we find that

$$\sum_{h=1}^{m} \mathsf{F}_h[r, \bar{\mu}; s, \bar{\nu}] \cong \sum_{h=1}^{m} \mathsf{G}_h[r, \bar{\mu}; s, \bar{\nu}] \cong \sum_{h=1}^{m} O(h^{r-1}) = O(m^r) = O(m^{p_0}) \text{ if } \bar{\mu} = \bar{\nu},$$

and $\cong 0$ otherwise, completing the proof. □

5.4. **Proof of Lemma 5.1.**

Proof. We start by expanding the L^{2p_0}-norm of f as follows,

$$(5.15) \quad ||f||_{2p_0}^{2p_0} = \int \left[\sum_N f_N\right]^{p_0} \overline{\left[\sum_N f_N\right]^{p_0}} dx = \int_{\mathbb{R}^n} \sum_{\mathbf{N},\mathbf{N}'} f_{N_1} \cdots f_{N_{p_0}} \overline{f_{N'_1} \cdots f_{N'_{p_0}}} dx,$$

where the summation ranges over all p_0-dimensional multi-indices

$$\mathbf{N} = (N_1, \cdots, N_{p_0}), \ \mathbf{N}' = (N'_1, \cdots, N'_{p_0}) \in \{1, \ldots, 2^m - 1\}^{p_0}.$$

The entries in $\mathbf{N}$ (and hence also $\mathbf{N}'$) need not be distinct. However, in view of (5.3) and the discussion leading up to it, for every $\mathbf{N}$ there exist

- a unique integer $1 \leq r = r(\mathbf{N}) \leq p_0$,
- a multiplicity vector $\bar{\mu} = \bar{\mu}(\mathbf{N})$ of length r for p_0, and
- a choice of $\bar{m} = \bar{m}(\mathbf{N}) = (m_1, \cdots, m_r) \in \mathbb{A}_m[r, \bar{\mu}]$ defined as in (5.4),

such that $\mathbf{N}[\bar{m}, \bar{\mu}]$ is a permutation of $\mathbf{N}$, i.e., $m_1 < m_2 < \cdots < m_r$, and m_i occurs exactly μ_i times in $\mathbf{N}$. Thus

$$f_{N_1} \cdots f_{N_{p_0}} = f_{m_1}^{\mu_1} \cdots f_{m_r}^{\mu_r}.$$

Further, given a fixed choice of $\bar{\mu}$ and $\bar{m}$, there are exactly $\binom{p_0}{\bar{\mu}} = p_0!/(\mu_1! \cdots \mu_r!)$-many possibilities of $\mathbf{N}$ that correspond to the same choice of $\mathbf{N}[\bar{m}, \bar{\mu}]$. Grouping the sum in (5.15) using these multiplicities, we obtain

$$(5.16) \qquad\qquad ||f||_{2p_0}^{2p_0} = \int {\sum_{r,s,\bar{\mu},\bar{\nu}}}' \binom{p_0}{\bar{\mu}} \binom{p_0}{\bar{\nu}} \sum_{\bar{\alpha}} F_{\bar{\alpha}}(x) \, dx,$$

where $F_{\bar{\alpha}}$ has been defined in (5.9), the outer sum is as in the statement of the lemma and the inner sum ranges over all multi-indices $\bar{\alpha} = (\bar{m}, \bar{n}) \in \mathbb{A}_m[r, \bar{\mu}] \times \mathbb{A}_m[s, \bar{\nu}]$. Finally, we note that

$$\mathbb{A}_m[r, \bar{\mu}] \times \mathbb{A}_m[s, \bar{\nu}] = \bigsqcup_{h=1}^{m} \mathbb{M}_h[r, \bar{\mu}; s, \bar{\nu}].$$

Decomposing the inner sum in (5.16) based on h therefore leads to (5.8). □

5.5. **Proof of Lemma 5.2.**

Proof. Part (a) of the lemma is based on an iterative application of the following estimate: for any measurable function H with $||H||_\infty \leq 1$, (4.5) gives

$$(5.17) \qquad \left| \int H(x)(g_N - \chi_N)(x)\, dx \right| \leq ||g_N - \chi_N||_1 \leq 2^{-Cm}.$$

We use this estimate to successively peel away each factor f_N occurring in $F_{\bar\alpha}$, replacing it by $e(p_N\cdot)\chi_N$ instead. Specifically, starting with any $\bar\alpha = (\bar m, \bar n) \in \mathbb{M}_h[r, \bar\mu; s, \bar\nu]$, we can write

$$F^{[1]} := F_{\bar\alpha} = f_{m_1} F^{[2]} = e(p_{m_1}\cdot)g_{m_1} F^{[2]}, \quad \text{where} \quad F^{[2]} = f_{m_1}^{\mu_1 - 1} \cdots f_{m_r}^{\mu_r} \overline{f_{n_1}^{\nu_1} \cdots f_{n_s}^{\nu_s}}$$

is a product of $p + q - 1$ factors. As a result,

$$
\begin{aligned}
(5.18) \qquad \int_{\mathbb{R}^n} F_{\bar\alpha}(x)dx = \int_{\mathbb{R}^n} F^{[1]}(x)\, dx &= \int_{\mathbb{R}^n} f_{m_1}(x)F^{[2]}(x)\, dx \\
&= \int_{E_{m_1}} e(\bar p_{m_1} \cdot x)F^{[2]}(x)\, dx + \int e(\bar p_{m_1} \cdot x)(g_{m_1} - \chi_{m_1})(x)F^{[2]}(x)\, dx \\
&\cong \int_{E_{m_1}} e(\bar p_{m_1} \cdot x)F^{[2]}(x)\, dx.
\end{aligned}
$$

The last step above uses (5.17) with $H = e(p_{m_1}\cdot)F^{[2]}$, which is bounded by 1 according to Proposition 4.1(d). Iterating the argument in (5.18) exactly $\mu_1 + \ldots + \mu_k + \nu_1 + \ldots \nu_l = p + q$ times (and using (5.17) with a different choice of H at each stage), we are able to remove all factors f_N and are left with the integrand $G_{\bar\alpha}$. This is the desired claim (5.10).

Regarding (b), we recall from Lemma 4.2(a) that the family of functions $\{\chi_N := \chi_{E_N}\}$ is a tree system. In particular, for any two indices $N < N'$, the sets E_N and $E_{N'}$ (which are non-empty by Proposition 4.1) are either disjoint or nested. Their intersection is nonempty precisely when $E_N \supseteq E_{N'}$, which in turn happens if and only if N is an ancestor of N', when represented as vertices on the binary tree $\mathcal{T}_h$. Thus $E_{\bar\alpha}$ is nonempty if and only if there is a strict lineage among the indices in $\bar\alpha$, i.e., for any two entries of $\bar\alpha$, one is either an ancestor or a descendant of the other. In other words, the vertices of $\bar\alpha$ lie on a ray of $\mathcal{T}_h$, i.e. $\bar\alpha \in \mathbb{M}'_h$. It follows from the definition (4.2) and more precisely from Lemma 4.2(a) that the sets E_N shrink as N proceeds down a ray of the tree. Thus $E_{\bar\alpha}$ must equal E_{N_0}, where $N_0 = ||\bar\alpha||_\infty$ is the terminating vertex of the ray that contains the indices of $\bar\alpha$. This leads to (5.13).

Part (c) is obtained by adding the estimates deduced in the first two parts of the lemma over all $\bar\alpha \in \mathbb{M}_h$. The verification is left to the interested reader. Note the importance of the large constant C in this step, as a result of which the error implicit in $\cong$ remains small even after summing over $\#(\mathbb{M}_h) = O(2^{2hp}) = O(2^{2mp_0})$ terms. $\qquad \square$

5.6. **Proof of Lemma 5.3.**

Proof. In view of Lemma 5.2(c), we know that $\mathsf{G}_h \cong \mathsf{G}'_h$. To establish the first relation in (5.14), it therefore suffices to show that $\mathsf{G}'_h \cong \mathsf{G}^*_h$. This in turn will follow from the estimate

below. For any choice of $h, p, q, r, s, \bar{\mu}, \bar{\nu}$,

(5.19)
$$\int G_{\bar{\alpha}}(x)\, dx \cong 0 \text{ for } \bar{\alpha} = (\bar{m}, \bar{n}) \in \mathbb{M}'_h[r, \bar{\mu}; s, \bar{\nu}] \setminus \mathbb{M}^*_h[r, \bar{\mu}; s, \bar{\nu}], \text{ i.e.,}$$
$$\int G_{\bar{\alpha}}(x)\, dx \cong 0 \text{ unless } \mu_r = \nu_s \text{ and } m_r = n_s.$$

By Lemma 5.2(a) combined with Plancherel's theorem, we obtain

$$\int G_{\bar{\alpha}}(x)\, dx \cong \int_{\mathbb{R}^n} F_{\bar{\alpha}}(x)\, dx = \int_{\mathbb{R}^n} \widehat{f}^{(\mu_1)}_{m_1} * \cdots * \widehat{f}^{(\mu_r)}_{m_r}(\xi) \times \overline{\widehat{f}^{(\nu_1)}_{n_1} * \cdots * \widehat{f}^{(\nu_s)}_{n_s}}(\xi)\, d\xi,$$

where we use the notation $h^{(l)}$ to denote the l-fold convolution of h with itself. The integrand on the right hand side above is supported in the set

$$\big(\mu_1 Q_{m_1} + \ldots + \mu_r Q_{m_r}\big) \cap \big(\nu_1 Q_{n_1} + \ldots + \nu_s Q_{n_s}\big).$$

By (4.3) in Lemma 4.1(b), this intersection is empty unless $m_r = n_s$ and $\mu_r = \nu_s$, establishing (5.19).

For the second relation in (5.14), we will rely on the following recursion formula, to be proven shortly:

(5.20)
$$\mathsf{G}'_h[r, \bar{\mu}; s, \bar{\nu}] \cong \sum_{h_1=1}^{h-1} \mathsf{G}'_{h_1}[r-1, \bar{\mu}^{[1]}; s-1, \bar{\nu}^{[1]}], \text{ where}$$
$$\bar{\mu}^{[1]} = (\mu_1, \cdots, \mu_{r-1}),\ \bar{\nu}^{[1]} = (\nu_1, \cdots, \nu_{s-1})$$

are multiplicity vectors of lengths $r - 1$ and $s - 1$ for the exponents $p - \mu_r$ and $q - \nu_s$ respectively. Assuming this for now, the proof is completed as follows.

First suppose that $\bar{\mu} \neq \bar{\nu}$, and that $0 \leq t \leq \min(r, s)$ is the smallest index such that $\mu_{r-t} \neq \nu_{s-t}$. If $t = 0$, then $\mathsf{G}_h \cong 0$ directly from (5.19). If $t > 0$, then a t-fold iteration of (5.20) yields

$$\mathsf{G}_h[r, \bar{\mu}; s, \bar{\nu}] \cong \mathsf{G}'_h[r, \bar{\mu}; s, \bar{\nu}] \cong \sum_{h_1=1}^{h-1} \mathsf{G}'_{h_1}[r-1, \bar{\mu}^{[1]}; s-1, \bar{\nu}^{[1]}] \cong \cdots$$

(5.21)
$$\cong \sum_{h_1=1}^{h-1} \cdots \sum_{h_t=1}^{h_{t-1}-1} \mathsf{G}'_{h_t}[r-t, \bar{\mu}^{[t]}; s-t, \bar{\nu}^{[t]}],$$

with $\bar{\mu}^{[t]} = (\mu_1, \cdots, \mu_{r-t})$, $\bar{\nu}^{[s]} = (\nu_1, \cdots, \nu_{s-t})$. Note that $\mu^{[t]}$ and $\nu^{[t]}$ are multiplicity vectors of length $r - t$ and $s - t$ for the exponents $p - \rho$ and $q - \rho$ respectively, where $\rho = \mu_{r-t+1} + \cdots + \mu_r = \nu_{s-t+1} + \cdots + \nu_s$. Since $\mu_{r-t} \neq \nu_{s-t}$, we can apply (5.19), with the parameters $h, p, q, r, s, \bar{\mu}, \bar{\nu}$ in (5.19) replaced by $h_t, p-\rho, q-\rho, r-t, s-t, \bar{\mu}^{[t]}, \bar{\nu}^{[t]}$ respectively. This leads to the estimate

$$\int G_{\bar{\alpha}}(x)\, dx \cong 0 \text{ for every } \bar{\alpha} \in \mathbb{M}'_{h_t}[r-t, \bar{\mu}^{[t]}; s-t, \bar{\nu}^{[t]}].$$

After summing over all the indices $\bar{\alpha}$ in the relevant collection $\mathbb{M}'_h$, the above relation yields that $\mathsf{G}'_{h_t}[r-t, \bar{\mu}^{[t]}; s-t, \bar{\nu}^{[t]}] \cong 0$. This in turn shows that the iterated sum in (5.21) is also $\cong 0$, since the number of summands is at most $h^t = O(m^p)$. This completes the proof for $\bar{\mu} \neq \bar{\nu}$.

On the other hand, if $\bar{\mu} = \bar{\nu}$, then iterating (5.20) $r = s$ times we find that

$$\mathsf{G}_h[r, \bar{\mu}; s, \bar{\mu}] \cong \mathsf{G}'_h[r, \bar{\mu}; s, \bar{\mu}] \cong \sum_{h_1=1}^{h-1} \cdots \sum_{h_{r-1}=1}^{h_{r-2}-1} \mathsf{G}'_{h_{r-1}}[1, \mu_1; 1, \mu_1]$$

$$= \sum_{h_1=1}^{h-1} \cdots \sum_{h_{r-1}=1}^{h_{r-2}-1} 1 = O(h^{r-1}).$$

At the penultimate step above, we have computed for any $h_{r-1} = l$,

$$\mathsf{G}'_l[1, \mu_1; 1, \mu_1] = \sum_{m_1 : h(m_1)=l} \int_{\mathbb{R}^n} \chi_{m_1} = \int_Q \sum_{m_1 : h(m_1)=l} \chi_{m_1} = 1.$$

The last step is a consequence of Lemma 4.2(b) with $N_0 = 1$. This completes the proof. $\square$

5.7. **Summing over subtrees: Proof of** (5.20).

Proof. Any $\bar{\alpha} = (\bar{m}, \bar{n}) \in \mathbb{M}^*_h[r, \bar{\mu}; s, \bar{\nu}]$ can be written as $\bar{\alpha} = (\bar{m}', m_r; \bar{n}', m_r)$, where

$$h(m_r) = h, \quad \bar{m}' = (m_1, \cdots, m_{r-1}), \quad \bar{n}' = (n_1, \cdots, n_{s-1}),$$

and $\bar{\alpha}' = (\bar{m}', \bar{n}')$ is a string of vertices lying on a ray in $\mathcal{T}_m$ and terminating in the vertex $||\bar{\alpha}'||_\infty = \max(m_{r-1}, n_{s-1})$ of height $< h$. As such, $\mathbb{M}^*_h$ can be partitioned as

$$\mathbb{M}^*_h[r, \bar{\mu}; s, \bar{\nu}] = \bigsqcup_{h_1=1}^{h-1} \left\{ \bar{\alpha} = (\bar{m}', m_r, \bar{n}', m_r) \Big| \begin{array}{l} \bar{\alpha}' = (\bar{m}', \bar{n}') \in \mathbb{M}'_{h_1}[r-1, \bar{\mu}'; s-1, \bar{\nu}'] \\ h(m_r) = h \end{array} \right\}.$$

This results in a corresponding decomposition for the sum representing G'_h:

(5.22)
$$\begin{aligned}
\mathsf{G}'_h[r, \bar{\mu}; s, \bar{\nu}] &\cong \mathsf{G}^*_h[r, \bar{\mu}; s, \bar{\nu}] \\
&= \sum_{h_1=1}^{h-1} \sum_{\bar{\alpha}', m_r} \int G_{\bar{\alpha}'}(x) \chi_{m_r}(x) \, dx \\
&= \sum_{h_1=1}^{h-1} \sum_{\bar{\alpha}'} \int G_{\bar{\alpha}'}(x) \sum_{m_r} \chi_{m_r}(x) \, dx \\
&= \sum_{h_1=1}^{h-1} \sum_{\bar{\alpha}'} \int G_{\bar{\alpha}'}(x) \chi_{E_{\bar{\alpha}'}}(x) \, dx \\
&= \sum_{h_1=1}^{h-1} \sum_{\bar{\alpha}'} \int G_{\bar{\alpha}'}(x) \, dx = \sum_{h_1=1}^{h-1} \mathsf{G}'_{h_1}[r-1, \bar{\mu}^{[1]}; s-1, \bar{\nu}^{[1]}].
\end{aligned}$$

In all the sums above, $\bar{\alpha}'$ ranges over $\mathbb{M}'_{h_1}[r-1, \bar{\mu}'; s-1, \bar{\nu}']$. For a given $\bar{\alpha}$, the summation index m_r ranges over descendants of $||\bar{\alpha}'||_\infty$ of height h in $\mathcal{T}_m$. This has been described in Figure 7. In the third equality, the summation in m_r follows from the property that $\{\chi_N\}$ is a tree system. In particular we have invoked Lemma 4.2(b) with $N_0 = ||\bar{\alpha}'||_\infty$, along with (5.13). $\square$
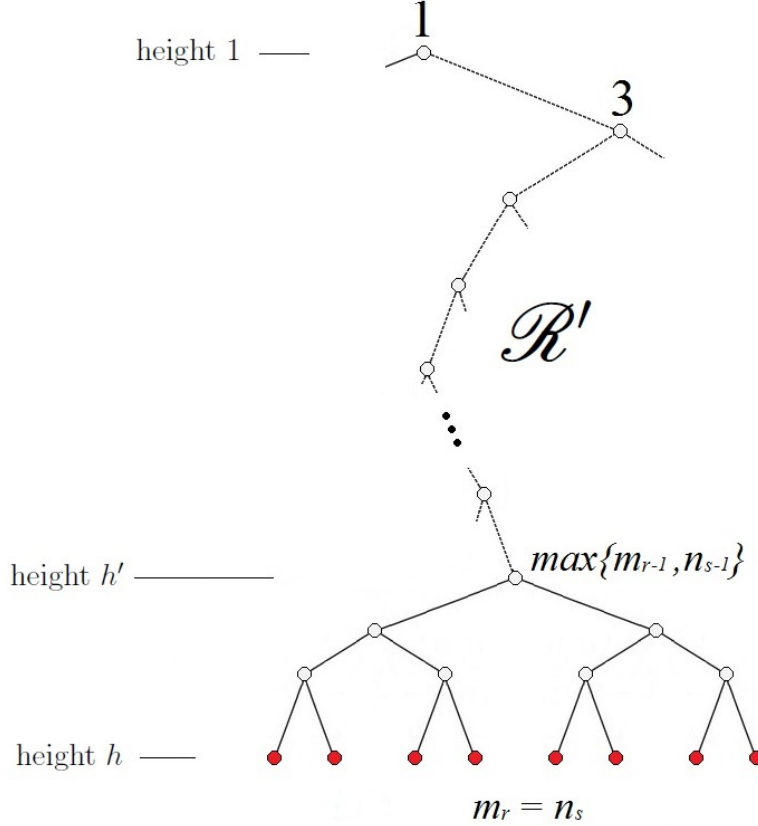
FIGURE 7. The process of summation in (5.22): the fixed vertices $m_1, \ldots m_{r-1}, n_1, \ldots n_{s-1}$ lie on the ray $\mathscr{R}'$, from the root of $\mathcal{T}_m$ to the vertex $\max \{m_{r-1}, n_{s-1}\}$. The innermost summation (in m_r) is over all vertices at height h of the subtree rooted at $\max(m_{r-1}, n_{s-1})$.

6. PROOF OF LEMMA 3.3

In this section we prove the geometric result used in the proof of the theorem in Section 3. Recall that $\Gamma_v := \{x \in \mathbb{R}^n : x \cdot v > 0\}$. We restate the lemma below for easier referencing.

Lemma 6.1. *Let U be a set of unit vectors in $\mathbb{R}^n$, all pointing in distinct directions. Assume that $\#U = M$ for some $M \geq 2$, and that all vectors $v \in U$ obey $v \cdot e_n > 0$, where $e_n = (0, \ldots 0, 1)$. Then there is an ordering $\{u_1, \ldots, u_M\}$ of vectors in U, and a collection of pairwise disjoint sectors $S_1, \ldots S_{M-1} \subset \mathbb{R}^n$ (see Definition 3.1), such that, up to sets of Lebesgue measure 0, we have for $l = 2, \ldots, M$*

$$(6.1) \qquad \Gamma_{u_l} \cap S_i = \begin{cases} S_i & \text{if } i < l, \\ \emptyset & \text{if } i \geq l. \end{cases}$$

Proof. For a unit vector $v \in \mathbb{R}^n$, we will use π_v to denote the hyperplane $\{x \in \mathbb{R}^n : x \cdot v = 0\}$. By a slight abuse of notation, we will also write $\Gamma_v^c = \{x \in \mathbb{R}^n : x \cdot v < 0\}$.

For $x' \in \mathbb{R}^{n-1}$, let $r_{x'}$ be the line

$$r_{x'} = \{(x', t) \in \mathbb{R}^n : t \in \mathbb{R}\}.$$

Since $v \cdot e_n > 0$ for all $v \in U$, the line $r_{x'}$ is not parallel to any of the corresponding hyperplanes π_v. Moreover, for all x' outside of an exceptional set of $(n-1)$-dimensional Lebesgue measure 0, it intersects these hyperplanes at distinct points. Fix one such point x', and let $A_i = (x', t_i)$ be the intersection points listed in the order of decreasing t so that $t_1 > t_2 > \cdots > t_M$. We then label the vectors in U as $u_1, \ldots, u_M$ so that

$$A_i \in \pi_{u_i}, \ i = 1, \ldots, M,$$

and define

$$S_i := \Gamma_{u_1}^c \cap \ldots \cap \Gamma_{u_i}^c \cap \Gamma_{u_{i+1}} \cap \ldots \cap \Gamma_{u_M}, \ i = 1, \ldots, M-1.$$

Then $S_i \subset \Gamma_{u_l}^c$ if $l \leq i$ and $S_i \subset \Gamma_{u_l}$ if $l > i$, so that we have (6.1). It is also clear from the definition that the sectors S_i are pairwise disjoint.

To see that they are non-empty, it suffices to check that $B_i \in S_i$ for $i = 1, \ldots, M-1$, where $B_i = (x', \tau_i)$ for some choice of scalars τ_i obeying $t_i > \tau_i > t_{i+1}$. Indeed, for any $1 \leq i \leq M-1$ and $1 \leq l \leq M$, we have

$$\vec{OB_i} \cdot u_l = (\vec{OA_l} + \vec{A_l B_i}) \cdot u_l = \vec{A_l B_i} \cdot u_l = (\tau_i - t_l)(e_n \cdot u_l),$$

$$\text{which is} \begin{cases} < 0 & \text{for } l \leq i \text{ since } \tau_i < t_l, \\ > 0 & \text{for } l > i \text{ since } \tau_i > t_l. \end{cases}$$

Thus $B_i \in \Gamma_l^c$ if $l \leq i$ and $B_i \in \Gamma_l$ if $l > i$, proving the claim. $\qquad\square$
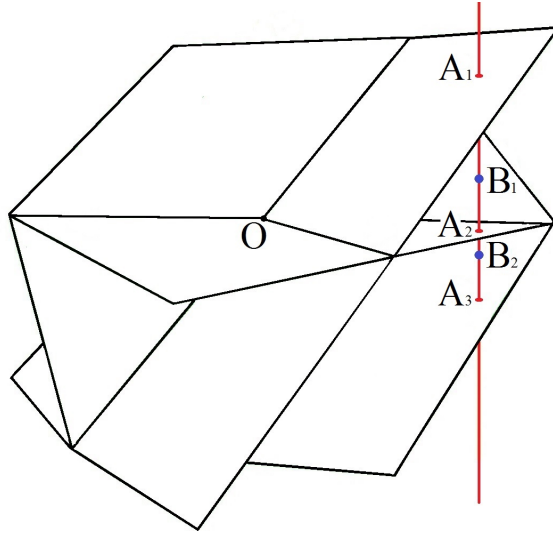

FIGURE 8. An example of the geometric construction in Lemma 3.3 with $n = 3$ and $M = 3$. The vertical line $r_{x'}$ intersects all three planes at distinct points.

Remark: We point out the main distinctions of Lemma 3.3 in general dimensions relative to its planar counterpart in [11]. In dimension two, the hyperplanes π_v are lines passing through the origin. Any conical sector is bounded by exactly two such lines. Thus M lines of the form π_v divide a half-plane into exactly $M+1$ sectors that admit an obvious ordering simply by moving in a clockwise direction. In $\mathbb{R}^n$, hyperplanes intersect in more complicated

ways. A conical sector may be bounded by a number of hyperplanes far greater than n. Furthermore, a collection of M vectors in U typically generates many more than M sectors, among which there is no natural "global" ordering. In Lemma 3.3, we choose, from the collection of all sectors, a subset of size $M - 1$, on which we impose a natural ordering, in terms of the height of the sector above a fixed point in the $\{x_n = 0\}$-hyperplane.

7. Acknowledgement

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Izabella Łaba

University of British Columbia, Vancouver, Canada.

Electronic address: `ilaba@math.ubc.ca`

Alessandro Marinelli

University of British Columbia, Vancouver, Canada.

Electronic address: `marine7@math.ubc.ca`

Malabika Pramanik

University of British Columbia, Vancouver, Canada.

Electronic address: `malabika@math.ubc.ca`