ON THE SHARPNESS OF MOCKENHAUPT'S RESTRICTION THEOREM

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ABSTRACT. We prove that the range of exponents in Mockenhaupt's restriction theorem for Salem sets [12], with the endpoint estimate due to Bak and Seeger [1], is optimal.
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1. INTRODUCTION

Using a Stein-Tomas type argument, Mockenhaupt [12] (see also Mitsis [11]) proved the following restriction theorem, with the endpoint due to Bak and Seeger [1].

Theorem 1. *Let μ be a compactly supported positive measure on $\mathbb{R}^n$ such that for some $\alpha, \beta \in (0, n)$ we have*

$$\mu(B(x,r)) \leq C_1 r^\alpha \text{ for all } x \in \mathbb{R}^n \text{ and } r > 0, \tag{1.1}$$

$$|\widehat{\mu}(\xi)| \leq C_2(1+|\xi|)^{-\beta/2} \text{ for all } \xi \in \mathbb{R}^n. \tag{1.2}$$

Then for all $p \geq p_{n,\alpha,\beta} := \frac{2(2n-2\alpha+\beta)}{\beta}$, there is a $C(p) > 0$ such that

$$\|\widehat{fd\mu}\|_{L^p(\mathbb{R}^n)} \leq C(p)\|f\|_{L^2(d\mu)} \tag{1.3}$$

for all $f \in L^2(d\mu)$. The equivalent dual form of this assertion is: For all $1 \leq p' \leq \frac{2(2n-2\alpha+\beta)}{4(n-\alpha)+\beta}$, there is a $C(p') > 0$ such that

$$\|\widehat{f}\|_{L^2(d\mu)} \leq C(p')\|f\|_{L^{p'}(\mathbb{R}^n)} \tag{1.4}$$

for all $f \in L^{p'}(\mathbb{R}^n)$.

When $\alpha = \beta = n-1$ and μ is the surface measure on the unit sphere S^{n-1} in $\mathbb{R}^n$, this is the classical Stein-Thomas theorem [16], [17], [14], [15]. The point of Theorem 1 is that similar estimates hold for less regular measures obeying (1.1) and (1.2), including fractal measures with α, β not necessarily integer.

It is well known (see e.g. [10], [18]) that if a measure μ is supported on a set of Hausdorff dimension $\alpha_0 < n$ and obeys (1.1) and (1.2), we must necessarily have $\alpha \leq \alpha_0$ and $\beta \leq \alpha_0$. The surface measure on the sphere provides an example with $\alpha = \beta = \alpha_0$. We do not know whether this is possible when α_0 is non-integer, but there are many constructions of measures supported on sets of fractional Hausdorff dimension α_0 for which (1.1) and (1.2) hold with α and β both arbitrarily close to α_0. Salem [13] constructed measures on $[0, 1]$ supported on sets of Hausdorf dimension $0 < \alpha < 1$, and obeying (1.1) with the same α, such that (1.2) holds for all $0 < \beta < \alpha$ with the constant C_2 depending on β. (The verification of (1.1) for Salem's construction is in [12].) Further examples are in [3], [4], [6], [7], [9].

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We are interested in the question of the sharpness of the range of p in Theorem 1. It is easy to see that if μ is a probability measure on $\mathbb{R}^n$ supported on a compact set of Hausdorff dimension $\alpha_0 < n$, then (1.3) cannot hold for any $p < 2n/\alpha_0$, even if the L^2 norm on the right side is replaced by the stronger L^∞ norm. Indeed, let $f \equiv 1$, so that $\widehat{fd\mu} = \widehat{\mu}$. The assumption on the support of μ implies that for any $\gamma > \alpha_0$ we have

$$I_\gamma(\mu) = \int_{|\xi|\geq 1} |\widehat{\mu}(\xi)|^2 \, |\xi|^{-(n-\gamma)} d\xi = \infty$$

(This is the usual energy integral, with the $|\xi| \leq 1$ region removed. See e.g. [10], [18].) On the other hand, by Hölder's inequality we have

$$I_\gamma(\mu) \leq \|\widehat{\mu}\|_p^2 \left(\int_{|\xi|\geq 1} |\xi|^{-(n-\gamma)\frac{p}{p-2}} \right)^{\frac{p-2}{p}},$$

and the last integral is finite for $p < 2n/\gamma$, so that $\|\widehat{\mu}\|_p = \infty$ for such p. The conclusion follows by letting $\gamma \to \alpha_0$.

In the most interesting case when α and β can be taken arbitrarily close to α_0, this leaves the intermediate range

$$\frac{2n}{\alpha_0} \leq p < \frac{4n - 2\alpha_0}{\alpha_0}. \tag{1.5}$$

In the case of the Tomas-Stein theorem, where μ is the surface measure on the unit sphere in $\mathbb{R}^n$ and $\alpha = \beta = n-1$, the estimate (1.3) is known to fail for all $p < \frac{4n-2\alpha}{\alpha} = \frac{2n+2}{n-1}$. This is seen from the so-called Knapp example, where (1.3) is tested on characteristic functions of small spherical caps (see e.g. [15], [18]). It has not been known whether similar examples exist for sets of fractional dimension. Mockenhaupt [12] stated that he could not exclude the possibility that for $n = 1$ and $\alpha_0 = \alpha \in (0,1)$, the estimate (1.3) could in fact hold for all $p > 2/\alpha$. Mitsis [11] and Bak and Seeger [1] did not try to address this question.

In this regard, we have the following result for $n = 1$.

Theorem 2. *For $\alpha \in (0,1)$ such that $\alpha = \frac{\log(t_0)}{\log(n_0)}$ for some $t_0, n_0 \in \mathbb{N}$, $n_0 \neq 1$, and for every $1 \leq p < \frac{4}{\alpha} - 2$, the following holds. There is a probability measure μ on $[0,1]$ supported on a set E of dimension α, and a sequence of functions $\{f_\ell\}_{j\in\mathbb{N}}$ on $[0,1]$ (characteristic functions of finite unions of intervals), such that*

- *μ obeys (1.1) with the given value of α,*
- *μ obeys (1.2) for every $\beta < \alpha$ (with C_2 depending on β),*
- *the restriction estimate (1.3) fails for the sequence $\{f_\ell\}$, i.e.*

$$\frac{\|\widehat{f_\ell d\mu}\|_{L^p(\mathbb{R})}}{\|f_\ell\|_{L^2(d\mu)}} \to \infty \quad as \quad \ell \to \infty. \tag{1.6}$$

The set of α in the assumptions of the theorem is dense in $(0,1)$. It is likely that the construction could be modified to yield such a measure and sequence of functions for every $\alpha \in (0,1)$, but this would not strengthen our conclusions significantly, considering that for a fixed p the relevant range of α is given by a strict inequality and that in any event we cannot produce a measure with $\alpha = \beta = \alpha_0$.

The Salem set E will be constructed via a randomized Cantor iteration. The main idea is that, while Salem sets are random overall, they may nonetheless contain much smaller sets that come close to being arithmetically structured. In our case, E will contain subsets $E \cap F_\ell$, where F_ℓ is a finite iteration of a smaller Cantor set with endpoints in a generalized arithmetic progression. The functions f_ℓ will then be characteristic functions of F_ℓ.

In a sense, this may be viewed as a one-dimensional analogue of Knapp's counterexample. The latter is based on the fact that an "almost flat" spherical cap is contained in the curved sphere, or equivalently, that the sphere is tangent to a flat hyperplane. Here, the set E may be thought of as random but nonetheless "tangent" to the arithmetically structured sets F_ℓ.

The construction of the Salem set E is similar to that in [9], but we have to be careful to make sure that the inclusion of the sets $E \cap F_\ell$ does not disturb the Fourier estimates. Our lower bound on $\|\widehat{f_\ell d\mu}\|_p$ relies on arithmetic arguments, specifically on counting solutions to linear equations in the set of endpoints of the Cantor intervals in the construction. Optimizing the parameters in the construction, we get Theorem 2.

If instead of Salem measures obeying (1.1) and (1.2) one considers more general measures on $\mathbb{R}$ supported on sets of Hausdorff dimension $\alpha_0 \in (0,1)$, then an example due to Chen [5] (based on the work of Körner [8]) shows that restriction estimates (1.3) for such measures can in fact hold for all $p \geq 2/\alpha_0$. (Körner's measures do not necessarily obey (1.1) and (1.2) with α, β near α_0, and it is not clear whether his construction can be modified to ensure these properties.)

It is still possible that *some* Salem sets do not contain structured subsets, and that the range of p in (1.3) can be improved for such sets. However, our result shows that Theorem 1 in its stated generality is optimal with regard to the range of p.

We also note that the same construction yields the following.

Theorem 3. *Let α be as in Theorem 2, and assume that the exponents $1 \leq p, q < \infty$ obey*

$$p < \frac{q(2-\alpha)}{\alpha(q-1)} \tag{1.7}$$

Then there is a measure μ on $[0,1]$ and a sequence of functions $\{f_\ell\}_{\ell\in\mathbb{N}}$, constructed as in the proof of Theorem 2, such that

$$\frac{\|\widehat{f_\ell d\mu}\|_{L^p(\mathbb{R}^n)}}{\|f_\ell\|_{L^q(d\mu)}} \to \infty \;\; as \;\; \ell \to \infty. \tag{1.8}$$

2. The construction of μ

Let N_0 and t_0 be integers such that $1 < t_0 < N_0$, and let $\alpha = \log t_0 / \log N_0$. Let also $N = N_0^{2n_0}$ and $t = t_0^{2n_0}$, where n_0 is a large integer to be chosen later. Observe that $\log t / \log N = \alpha$ regardless of the value of n_0, so that we may freely assume that n_0 is large enough while keeping α fixed. For short, we will write $[N] = \{0, 1, \ldots, N-1\}$.

We use C, C', etc. to denote constants that may change from line to line. Whenever such constants depend on n_0 or on any of the running parameters j, k, ℓ, m, we will indicate this explicitly by writing, e.g., $C(n_0)$; all other constants may depend on α, but are independent of n_0, j, k, ℓ, m.

We will construct μ and f_ℓ simultaneously via a sequence of Cantor iterations. We will have a sequence of sets $A_0, A_1, A_2, \ldots$ satisfying

$$A_0 = \{0\}\,,$$
$$A_{j+1} = \bigcup_{a\in A_j} (a + A_{j+1,a}),$$
$$A_{j+1,a} \subset N^{-(j+1)}[N]$$
$$|A_{j+1,a}| = t$$

Note that $A_j \subset N^{-j}\mathbb{Z}$ and $|A_j| = t^j$. The freedom in the construction comes in how we choose the subsets $A_{j+1,a} \subset N^{-(j+1)}[N]$; we can make separate choices for each j and each $a \in A_j$.

Given such a sequence A_j, we define

$$E_j = \bigcup_{a\in A_j} a + [0, N^{-j}], \qquad E = \bigcap_{j=1}^{\infty} E_j. \tag{2.1}$$

Since $E_1 \supset E_2 \supset \cdots$, E is a closed non-empty set.

There is a natural probability measure μ on E, defined as the weak limit of the absolutely continuous measures μ_j with densities

$$\frac{d\mu_j}{dx} = \sum_{a\in A_j} t^{-j} N^j \mathbf{1}_{[a,a+N^{-j}]}. \tag{2.2}$$

Lemma 4. *For any choice of A_j as above, E has Hausdorff dimension α, and μ obeys*

$$\mu([x, x+\epsilon]) \le C_1(n_0)\epsilon^\alpha \textit{ for all } \epsilon > 0.$$

Proof. This is standard. See, for example, Lemma 6.1 in [9]. □

We will also construct sequences of sets $P_j \subset A_j$ and $F_j \subset E_j$ so that:

- $P_0 = \{0\}$
- $P_{j+1} = \bigcup_{a\in P_j} (a + N^{-(j+1)}P)$ for $j = 0, 1, 2, \ldots$, where $P \subset \{0, 1, \ldots, N-1\}$ is an arithmetic progression of length $t^{1/2} = t_0^{n_0}$
- $F_j = \bigcup_{a\in P_j} a + [0, N^{-j})$.

Note that $|P_j| = t^{j/2}$. We also define

$$f_\ell = \mathbf{1}_{F_\ell}.$$

The main result of this section is the following.

Proposition 5. *Assume that n_0 is sufficiently large. There is a choice of A_j, $j = 1, 2, \ldots$, with the above properties such that for every $0 < \beta < \alpha$ we have*

$$|\hat{\mu}(k)| \le C(\beta, n_0)|k|^{-\beta/2} \qquad (k \in \mathbb{Z}\setminus\{0\}), \tag{2.3}$$

$$|\widehat{f_\ell\mu_j}(k)| \le C(\beta, \ell, n_0)|k|^{-\beta/2} \qquad (k \in \mathbb{Z}\setminus\{0\}, j \ge \ell), \tag{2.4}$$

Proof. Our starting point is the construction of Salem sets in [9], Section 6. We will modify it to make A_j contain the structured sets P_j while also preserving the Fourier estimates (2.3), (2.4). We will proceed by induction. Define $A_0 = \{0\}$, and let $A_1 \subset N^{-1}[N]$ be an arbitrary set of cardinality t so that $P_1 \subset A_1$. Assuming that $j \geq 1$ and that A_j is given so that $P_j \subset A_j$, we define A_{j+1} by constructing $A_{j+1,a}$ for each $a \in A_j$.

If $A \subset \mathbb{R}$ is a finite set, we will write for $k \in \mathbb{Z}$

$$S_A(k) = \sum_{a \in A} e^{-2\pi i a k}.$$

The outline is as follows. We first construct a set $B_{j+1} \subset N^{-(j+1)}[N]$ so as to minimize the differences

$$\left| \frac{1}{t} S_{B_{j+1}}(k) - \frac{1}{N} S_{N^{-(j+1)}[N]}(k) \right| \tag{2.5}$$

for $k \in \mathbb{Z}$, subject to the constraint that $|B_{j+1}| = t$. Moreover, we will want (2.5) to be similarly small if B_{j+1} is replaced by any of its "rotated" copies $B_{j+1,x}$ with $x \in [N]$ (the terminology will be explained shortly). These sets will serve as our initial candidates for $A_{j+1,a}$. Next, we choose the "rotations" $x(a)$ for $a \in A_j$ so as to minimize the Fourier coefficients of the next generation Cantor sets with $B_{j+1,x(a)}$ used in place of $A_{j+1,a}$.

Finally, recall that we had $P_j \subset A_j$. For each $a \in P_j$, we add $N^{-(j+1)}P$ to $B_{j+1,x(a)}$, then subtract a matching number of elements of $B_{j+1,x(a)}$ that are not in $N^{-(j+1)}P$, so that the resulting set has cardinality t again. This will be $A_{j+1,a}$ for $a \in P_j$. For $a \in A_j \setminus P_j$, we simply let $A_{j+1,a} = B_{j+1,x(a)}$. We will prove that these modifications can be made without destroying the Fourier estimates.

We now turn to the details. As in [9], we will need Bernstein's inequality (see e.g. [2]).

Lemma 6 (Bernstein's inequality)**.** *Let $X_1, \dots, X_n$ be independent complex-valued random variables with $|X_j| \leq 1$, $\mathbb{E}X_i = 0$, and $\mathbb{E}|X_j|^2 = \sigma_j^2$. Let $\sigma > 0$ be such that $\sigma^2 \geq \sum_{j=1}^n \sigma_j^2$ and $\sigma^2 \geq 6n\lambda$. Then*

$$\mathbb{P}\left(\left| \sum_{j=1}^n X_j \right| \geq n\lambda \right) \leq 4 \exp\left(-\frac{n^2\lambda^2}{8\sigma^2} \right).$$

Define $\eta_j > 0$ by

$$\eta_j^2 = 192 t^{-1} \ln(8N^{j+2}). \tag{2.6}$$

Lemma 7. *There is a set $B_{j+1} \subset N^{-(j+1)}[N]$ with $|B_{j+1}| = t$ such that*

$$\left| \frac{S_{B_{j+1,x}}(k)}{t} - \frac{S_{N^{-(j+1)}[N]}(k)}{N} \right| \leq \eta_j \tag{2.7}$$

for all $k \in \mathbb{Z}$ and $x \in \{0, 1, \dots, N-1\}$. Here

$$B_{j+1,x} = \left\{ \frac{(x+y) \pmod N}{N^{j+1}} : \frac{y}{N^{j+1}} \in B_{j+1} \right\}.$$

Proof. This is Lemma 6.2 of [9]; we include the proof because it is short and provides a good warm-up for the main argument.

If j is large enough so that $\eta_j \geq 2$, then we may choose B_{j+1} to be an arbitrary subset of $N^{-(j+1)}[N]$ of cardinality t. Then (2.7) holds trivially, since each term on the left side of (2.7) is bounded by 1 in absolute value. Assume therefore that $\eta_j \leq 2$.

Let $B_{j+1} \subset N^{-(j+1)}[N]$ be a random set constructed by stipulating that for each $b \in N^{-(j+1)}[N]$ the probability that $b \in B_{j+1}$ is $p = t/N$.

Fix $k \in \mathbb{Z}$ and $x \in [N]$. For each $b \in N^{-(j+1)}[N]$, define the random variable $X_b(k,x) = (\mathbf{1}_{B_{j+1,x}}(b) - p)e^{-2\pi ibk}$. The $X_b(k,x)$'s satisfy $\mathbb{E}_b X_b(k,x) = 0$ and $\mathbb{E}_b|X_b(k,x)|^2 = p(1-p)$. Set $\sigma^2 = 6t$, $n = N$, and $\lambda = \eta_j p/2$. Then $\sigma^2 \geq \sum_{b\in N^{-(j+1)}[N]} \mathbb{E}_b|X_b(k,x)|^2$, and $\sigma^2 \geq 6n\lambda = 3\eta_j t$.

We apply Lemma 6 to the $X_b(k,x)$'s. Since

$$\frac{S_{B_{j+1,x}}(k)}{t} - \frac{S_{N^{-(j+1)}[N]}(k)}{N} = t^{-1} \sum_{b\in N^{-(j+1)}[N]} X_b(k,x),$$

and

$$4\exp\left(-\frac{n^2\lambda^2}{8\sigma^2}\right) = 4\exp\left(-\ln(8N^{j+2})\right) = \frac{1}{2N^{j+2}},$$

Lemma 6 gives

$$\mathbb{P}\left(\left|\frac{S_{B_{j+1,x}}(k)}{t} - \frac{S_{N^{-(j+1)}[N]}(k)}{N}\right| \geq \frac{\eta_j}{2}\right) = \frac{1}{2N^{j+2}} \tag{2.8}$$

for fixed $k \in \mathbb{Z}$ and $x \in [N]$. Since $S_{B_{j+1,x}}(k)$ and $S_{N^{-(j+1)}[N]}(k)$ are periodic with period N^{j+1}, it suffices to consider $k \in \{0, 1, \dots, N^{j+1} - 1\}$. Thus the probability that the event in (2.8) occurs for some $k \in \mathbb{Z}$ and $x \in \{0, 1, \dots, N-1\}$ is bounded by $1/2$.

Hence with positive probability we have

$$\left|\frac{S_{B_{j+1,x}}(k)}{t} - \frac{S_{N^{-(j+1)}[N]}(k)}{N}\right| \leq \frac{\eta_j}{2} \tag{2.9}$$

for all $k \in \mathbb{Z}$ and $x \in [N]$. When $k = 0$ and $x = 0$, (2.9) says $||B_{j+1}| - t| \leq \eta_j t/2$. Therefore, by either adjoining to B_{j+1} or removing from it at most $\eta_j t/2$ elements, we get a set of cardinality exactly t obeying (2.7) for all k, x as above. $\square$

The main step in the proof of Proposition 5 is the following lemma.

Lemma 8. *There is a choice of the rotations $x(a)$, $a \in A_j$, such that*

$$|\widehat{\mu_{j+1}}(k) - \widehat{\mu_j}(k)| \leq C \min\left(1, \frac{N^{j+1}}{|k|}\right) t^{-(j+1)/2} \ln(8N^{j+1}). \tag{2.10}$$

for all $k \in \mathbb{Z}$, $j \geq 1$, and

$$\left|\widehat{f_\ell\mu_{j+1}}(k) - \widehat{f_\ell\mu_j}(k)\right| \leq C \min\left(1, \frac{N^{j+1}}{|k|}\right) t^{-(j+1)/2} \ln(8N^{j+1}). \tag{2.11}$$

for all $k \in \mathbb{Z}$, $j \geq 2$, and $\ell \in \{1, \dots, j\}$.

Proof. **Step 1.** Consider the random variables

$$\chi_a(k) = e^{-2\pi ika}\left(\frac{S_{B_{j+1,x(a)}}(k)}{t} - \frac{S_{N^{-(j+1)}[N]}(k)}{N}\right), \ \ a \in A_j, \ k \in \mathbb{Z},$$

where for each $a \in A_j$ we choose $x(a)$ (the same for all k) independently and uniformly at random from the set $[N]$. Let c be a large constant. We claim that there is a choice of $x(a)$ such that

$$\left| t^{-j} \sum_{a \in A_j} \chi_a(k) \right| < \lambda_j := ct^{-(j+1)/2} \ln(8N^{j+1}) \tag{2.12}$$

for all $k \in \mathbb{Z}$ and

$$\left| t^{-j+\ell/2} \sum_{a \in F_\ell \cap A_j} \chi_a(k) \right| < \lambda_{j,\ell} := ct^{-\frac{j+1}{2}+\frac{\ell}{4}} \ln(8N^{j+1}) \tag{2.13}$$

for all $k \in \mathbb{Z}$ and all $\ell \in \{1, \ldots, j\}$.

Consider the following events:

- $\mathcal{E}$ is the event that $\left| t^{-j} \sum_{a \in A_j} \chi_a(k) \right| \geq \lambda_j$ for some $k \in \mathbb{Z}$,
- $\mathcal{E}_\ell$ is the event that $\left| t^{-j+\ell/2} \sum_{a \in F_\ell \cap A_j} \chi_a(k) \right| \geq \lambda_{j,\ell}$ for some $k \in \mathbb{Z}$.

We will prove that $\mathbb{P}(\mathcal{E}) < 1/2$ and $\mathbb{P}(\mathcal{E}_\ell) < 1/(2j)$ for $\ell = 1, 2, \ldots, j$. Since the failure of $\mathcal{E}$ implies (2.12), and the failure of all $\mathcal{E}_\ell$ with $\ell = 1, 2, \ldots, j$ implies (2.13), there must be a choice of $x(a)$ for which both (2.12) and (2.13) hold.

We begin with $\mathcal{E}$. By periodicity, it suffices to consider $k \in [N^{j+1}]$. The random variables $\chi_a(k)$, $a \in A_j$, are independent and have expectation $\mathbb{E}\chi_a(k) = 0$. By Lemma 7, $|\chi_a(k)| \leq \eta_j$. With $n = t^j$ and $\sigma^2 = cn\eta_j^2 = 192ct^{j-1} \ln(8N^{j+2})$, we have $\sigma^2 \geq \sum_{a \in A_j} \mathbb{E}|\chi_a(k)|^2$ and $\sigma^2 \geq 6n\lambda_j$. Therefore, by Lemma 6, we have for each fixed k

$$\mathbb{P}\left(\left| t^{-j} \sum_{a \in A_j} \chi_a(k) \right| \geq \lambda_j \right) \leq 4 \exp\left(-\frac{\lambda_j^2\, t^{2j}}{8\sigma^2} \right).$$

Hence $\mathcal{E}$ has probability at most $4N^{j+1} \exp\left(-\lambda_j^2\, t^{2j}/8\sigma^2 \right)$, which is less than $1/2$ if $c \geq 3072$.

Next, we turn to $\mathcal{E}_\ell$. Again, let $k \in [N^{j+1}]$. We apply Bernstein's inequality as before, but this time with $n = |F_\ell \cap A_j| = t^{\ell/2} t^{j-\ell} = t^{j-\ell/2}$ and $\sigma^2 = cn\eta_j^2 = 192ct^{j-\ell/2-1} \ln(8N^{j+2})$. We get that

$$\mathbb{P}\left(\left| t^{-j+\ell/2} \sum_{a \in F_\ell \cap A_j} \chi_a(k) \right| \geq \lambda_{j,\ell} \right) \leq 4 \exp\left(-\frac{\lambda_{j,\ell}^2\, t^{2j-\ell}}{8\sigma^2} \right).$$

Hence $\mathcal{E}_\ell$ has probability at most $4N^{j+1} \exp\left(-\lambda_{j,\ell}^2\, t^{2j-\ell}/8\sigma^2 \right)$, which is less than $1/2j$ if $c \geq 6144$.

Step 2. Define A_{j+1} as follows. Recall that $P_j \subset A_j$. For each $a \in P_j$, construct $A_{j+1,a}$ by adjoining $N^{-(j+1)}P$ to $B_{j+1,x(a)}$ with $x(a)$ chosen as in Step 1, then subtract a matching number of elements of $B_{j+1,x(a)}$ that are not in $N^{-(j+1)}P$, so that $N^{-(j+1)}P \subset A_{j+1,a}$ and $|A_{j+1,a}| = t$. For $a \in A_j \setminus P_j$, we let $A_{j+1,a} = B_{j+1,x(a)}$. We claim that

$$\left| \frac{S_{A_{j+1}}(k)}{t^{j+1}} - \sum_{a \in A_j} e^{-2\pi i k a}\ \frac{S_{N^{-(j+1)}[N]}(k)}{t^j N} \right| \leq 2ct^{-(j+1)/2} \ln(8N^{j+1}), \tag{2.14}$$

$$\left| \frac{S_{A_{j+1}\cap F_\ell}(k)}{t^{j+1}} - \sum_{a\in A_j\cap F_\ell} e^{-2\pi i k a}\ \frac{S_{N^{-(j+1)}[N]}(k)}{t^j N} \right| \le 2ct^{-(j+1)/2}\ln(8N^{j+1}). \tag{2.15}$$

To see this, first let $\tilde{A}_{j+1} = \bigcup_{a\in A_j} B_{j+1,x(a)}$. Then by (2.12)

$$\left| \frac{S_{\tilde{A}_{j+1}}(k)}{t^{j+1}} - \sum_{a\in A_j} e^{-2\pi i k a}\ \frac{S_{N^{-(j+1)}[N]}(k)}{t^j N} \right| = \left| t^{-j}\sum_{a\in A_j}\chi_a(k)\right| < \lambda_j.$$

Since A_{j+1} differs from $\tilde{A}_{j+1}$ by at most $t^{(j+1)/2}$ elements, we have

$$\left| \frac{S_{\tilde{A}_{j+1}}(k)}{t^{j+1}} - \frac{S_{A_{j+1}}(k)}{t^{j+1}} \right| \le t^{-(j+1)/2}.$$

and (2.14) follows.

Similarly, by (2.13)

$$\begin{aligned}\left| \frac{S_{\tilde{A}_{j+1}\cap F_\ell}(k)}{t^{j+1}} - \sum_{a\in A_j\cap F_\ell} e^{-2\pi i k a}\ \frac{S_{N^{-(j+1)}[N]}(k)}{t^j N} \right| &= \left| t^{-j}\sum_{a\in A_j\cap F_\ell}\chi_a(k)\right| \\ &< t^{-\ell/2}\lambda_{j,\ell} = ct^{-\frac{j+1}{2}-\frac{\ell}{4}}\ln(8N^{j+1})\end{aligned}$$

Since $A_{j+1}\cap F_\ell$ differs from $\tilde{A}_{j+1}\cap F_\ell$ by at most $t^{(j+1)/2}$ elements, the left side again differs from the left side of (2.15) by at most $t^{-(j+1)/2}$, so that (2.15) follows.

Step 3. We will first show that (2.14) implies (2.10). We have

$$\begin{aligned}\widehat{\mu_j}(k) &= N^j t^{-j}\sum_{a\in A_j}\int_a^{a+N^{-j}} e^{-2\pi i k x}dx \\ &= \frac{1-e^{-2\pi i k/N^j}}{2\pi i k/N^j} t^{-j} S_{A_j}(k) \\ &= \frac{1-e^{-2\pi i k/N^{j+1}}}{2\pi i k/N^{j+1}} t^{-j}\sum_{a\in A_j} e^{-2\pi i k a}\frac{S_{N^{-(j+1)}[N]}(k)}{N},\end{aligned}$$

and

$$\widehat{\mu_{j+1}}(k) = \frac{1-e^{-2\pi i k/N^{j+1}}}{2\pi i k/N^{j+1}} t^{-(j+1)} S_{A_{j+1}}(k)$$

Therefore,

$$\begin{aligned}|\widehat{\mu_{j+1}}(k) - \widehat{\mu_j}(k)| &= \left|\frac{1-e^{-2\pi i k/N^{j+1}}}{2\pi i k/N^{j+1}}\right| \left| \frac{S_{A_{j+1}}(k)}{t^{j+1}} - \sum_{a\in A_j} e^{-2\pi i k a}\ \frac{S_{N^{-(j+1)}[N]}(k)}{t^j N} \right| \\ &\le 2ct^{-(j+1)/2}\ln(8N^{j+2}) \left|\frac{1-e^{-2\pi i k/N^{j+1}}}{2\pi i k/N^{j+1}}\right|\end{aligned}$$

Estimating the last factor by $\min(1, N^{j+1}/\pi|k|)$, we get (2.10).

Next, we show (2.15) implies (2.11). Let $\ell \in \{1, \cdots, j\}$. We have

$$\widehat{f_\ell d\mu_j}(k) = \frac{1 - e^{-2\pi i k/N^{j+1}}}{2\pi i k/N^{j+1}} \frac{1}{t^j N} \sum_{a \in A_j \cap F_\ell} e^{-2\pi i k a} S_{N^{-(j+1)}[N]}(k)$$

and

$$\widehat{f_\ell d\mu_{j+1}}(k) = \frac{1 - e^{-2\pi i k/N^{j+1}}}{2\pi i k/N^{j+1}} t^{-(j+1)} S_{A_{j+1} \cap F_\ell}(k).$$

Then (2.11) follows as above, using (2.15) instead of (2.14).

□

Lemma 9 (cf. [9], Lemma 6.5)**.** *Assume that n_0 is large enough. For every $0 < \beta < \alpha$, there is a constant $C(n_0\beta)$ such that*

$$\sum_{j=0}^{\infty} \min\left(1, \frac{N^{j+1}}{|k|}\right) t^{-(j+1)/2} \ln(8N^{j+1}) \leq C(n_0, \beta)|k|^{-\beta/2}$$

for all $k \in \mathbb{Z}$, $k \neq 0$.

Proof. Split the sum as $\sum_{j \leq \frac{\ln |k|}{\ln N}} + \sum_{j > \frac{\ln |k|}{\ln N}}$ and estimate each term separately. For details, see the proof of Lemma 6.5 of [9]. □

We can now conclude the proof of Proposition 5. Since μ_j converges to μ weakly, $\hat{\mu}_j$ converges to $\hat{\mu}$ pointwise. Hence

$$|\widehat{\mu}(k)| \leq |\widehat{\mu_1}(k)| + \sum_{j=1}^{\infty} |\widehat{\mu_{j+1}}(k) - \widehat{\mu_j}(k)|.$$

The sum is bounded by $C(n_0, \beta)|k|^{-\beta/2}$, by Lemmas 8 and 9, and we have

$$|\hat{\mu_1}(k)| = \left| \frac{1 - e^{-2\pi i k/N}}{2\pi i k/N} \frac{1}{t} \sum_{a \in A_1} e^{-2\pi i a k} \right| \leq \frac{C(n_0)}{|k|}.$$

This proves (2.3).

To prove (2.4), we first note the inequality

$$|\widehat{f_\ell d\mu_h}(k)| = \left| \frac{1 - e^{-2\pi i k/N^h}}{2\pi i k/N^h} t^{-h} S_{A_h \cap F_\ell}(k) \right| \leq \frac{N^h t^{-h}}{\pi |k|} |A_h \cap F_\ell| = \frac{N^h t^{-\ell/2}}{\pi |k|}. \tag{2.16}$$

Then (2.4) is immediate in case $j = \ell$. If $j > \ell$, we write

$$|\widehat{f_\ell d\mu_j}(k)| \leq |\widehat{f_\ell d\mu_\ell}(k)| + |\widehat{f_\ell d\mu_{\ell+1}}(k) - \widehat{f_\ell d\mu_\ell}(k)| + \sum_{i=\ell+1}^{j-1} |\widehat{f_\ell d\mu_{i+1}}(k) - \widehat{f_\ell d\mu_i}(k)|.$$

Lemmas 8 and 9 imply the sum is bounded by $C(n_0, \beta)|k|^{-\beta/2}$. For the remaining terms, we use (2.16).

□

3. The estimates on f_ℓ

We start with the easy part.

Lemma 10. *For all $1 \le q < \infty$, we have $\|f_\ell\|^q_{L^q(d\mu)} = \mu(F_\ell) = t^{-\ell/2}$.*

Theorem 2 will follow from this and Proposition 11 below.

Proposition 11. *Fix $r \in \mathbb{N}$ with $r > \frac{1}{\alpha}$ and assume that n_0 is large enough (depending on r). Let $1 \le p \le 2r$. Then for all ℓ sufficiently large we have*

$$\left\| \widehat{f_\ell d\mu} \right\|^p_{L^p(\mathbb{R})} \ge C(r) \frac{N^\ell r^{-\ell-1}}{t^{\ell(p+1)/2}}. \tag{3.1}$$

Proof of Theorems 2 and 3, given Proposition 11. Fix r large enough so that $r > 1/\alpha$ and $2r \ge \frac{q(2-\alpha)}{\alpha(q-1)}$. Applying Proposition 11, we see that (3.1) holds for all p as in (1.7). Hence

$$\frac{\|\widehat{f_\ell d\mu}\|_{L^p(\mathbb{R})}}{\|f_\ell\|_{L^q(d\mu)}} \ge C(r) \left(\frac{N^\ell r^{-\ell-1}}{t^{\ell(p+1)/2}} \right)^{1/p} t^{\ell/2q}.$$

After some algebra, this is seen to go to infinity provided that (1.7) holds and that n_0 is large enough depending on p.

It remains to prove Proposition 11. This will occupy the rest of this section, and will be done in several steps. If $Y \subset \mathbb{R}$ is a finite set and $r \in \mathbb{N}$, we will write

$$M_Y = \# \left\{ (a_1, \dots, a_{2r}) \in Y^{2r} : \ \textstyle\sum_{i=1}^r a_i = \sum_{i=r+1}^{2r} a_i \right\}$$

Lemma 12. *For every $j, \ell, r \in \mathbb{N}$ such that $j \ge \ell$,*

$$M_{F_\ell \cap A_j} \ge r^{-\ell-1} t^{(2r-1)\ell/2} \left(\frac{t^{2r}}{N} \right)^{j-\ell}. \tag{3.2}$$

Proof. Throughout the proof, the parameters j, ℓ will be kept fixed. Let

$$Y = A_j \cap F_\ell, \quad |Y| = t^{\ell/2} t^{j-\ell}$$

and

$$Z = \{a_1 + \dots + a_r : \ a_1, \dots, a_r \in Y\}$$

We claim that

$$|Z| \le (rt^{1/2})^\ell \, r N^{j-\ell}. \tag{3.3}$$

Indeed, each $y \in Y$ has a unique digit representation

$$y = \sum_{k=1}^{\ell} y^{(k)} N^{-k} + y^{(\ell+1)} N^{-j}$$

where $y^{(k)} \in P$ for $k = 1, \dots, \ell$ and $y^{(\ell+1)} \in [N^{j-\ell}]$. We may assume that $P = \{x, x+d, \dots, x+(t^{1/2}-1)d\}$. Then each $z \in Z$ can be written (not necessarily uniquely) as

$$z = \sum_{k=1}^{\ell} z^{(k)} N^{-k} + z^{(\ell+1)} N^{-j}$$

where $z^{(\ell+1)} \in \{0, 1, \ldots, r(N^{j-\ell} - 1)\}$ and

$$z^{(k)} \in P' := \{rx, rx + d, \ldots, rx + r(t^{1/2} - 1)d\}$$

for $k = 1, \ldots, \ell$. Since $|\{0, 1, \ldots, r(N^{j-\ell} - 1)\}| \leq rN^{j-\ell}$ and $|P'| \leq rt^{1/2}$, (3.3) follows.

We now prove (3.2). For $z \in N^{-j}\mathbb{Z}$, let

$$g(z) = \#\left\{(y_1, \ldots, y_r) \in Y^r : \textstyle\sum_{i=1}^r y_i = z\right\}$$

Then $\|g\|_{\ell^1} = |Y|^r$, $\|g\|_{\ell^2}^2 = M_Y$, and g is supported on Z. By Hölder's inequality, $\|g\|_{\ell^1} \leq \|g\|_{\ell^2} |Z|^{1/2}$, so that

$$M_Y \geq \frac{\|g\|_{\ell^1}^2}{|Z|} \geq \frac{(t^{\ell/2} t^{j-\ell})^{2r}}{(rt^{1/2})^\ell \, rN^{j-\ell}}$$

as claimed.

□

The next lemma is Lemma 9.A.4 of [18]. We will use it in the proof of Lemma 14.

Lemma 13. *Let m be a measure on the torus $\mathbb{T} = \mathbb{R}/\mathbb{Z}$, and let ϕ be a Schwartz function on $\mathbb{R}$. Define a measure m' on $\mathbb{R}$ by*

$$dm'(x) = \phi(x) dm(\{x\}),$$

where $\{x\}$ is the fractional part of x. Then for all $\xi \in \mathbb{R}$,

$$\widehat{m'}(\xi) = \sum_{k \in \mathbb{Z}} \widehat{m}(k) \widehat{\phi}(\xi - k).$$

Moreover, if there are $C > 0$ and $\alpha > 0$ such that

$$|\widehat{m}(k)| \leq C(1 + |k|)^{-\alpha} \quad \text{for all } k \in \mathbb{Z},$$

then there is a $C' > 0$ such that

$$|\widehat{m'}(\xi)| \leq C'(1 + |\xi|)^{-\alpha} \quad \text{for all } \xi \in \mathbb{R}.$$

Lemma 14. *Let $\ell, r \in \mathbb{N}$ with $r > \frac{1}{\alpha}$. Then*

$$\left\| \widehat{f_\ell d\mu} \right\|_{L^{2r}(\mathbb{R})}^{2r} \geq C_{2r} \frac{N^\ell r^{-\ell-1}}{t^{\ell(2r+1)/2}}, \tag{3.4}$$

where

$$C_{2r} = \int_\infty^{-\infty} \left(\frac{\sin(\pi x)}{\pi x} \right)^{2r} dx \in (0, \infty).$$

Proof. By Proposition 5, for every $0 < \beta < \alpha$ we have

$$|\widehat{f_\ell d\mu_j}(k)| \leq C|k|^{-\beta/2}$$

for $k \in \mathbb{Z} \setminus \{0\}$ and $j \geq \ell$. By Lemma 13, this inequality extends to

$$|\widehat{f_\ell d\mu_j}(\xi)| \leq C|\xi|^{-\beta/2}$$

for $|\xi| \geq 1$ and $j \geq \ell$. Fix $\beta \in (0, \alpha)$ such that $r > 1/\beta > 1/\alpha$, and let $g(\xi) := \min\left(1, C|\xi|^{-\beta/2}\right)$. Assume $C > 1$ without loss of generality. We have $\left|\widehat{f_\ell d\mu_j}\right| \leq g$ and $g \in L^{2r}(\mathbb{R})$. By a straightforward application of the portmanteau theorem on the weak convergence of measures (cf. [2]), the fact that $\mu_j \to \mu$ weakly implies we have $\widehat{f_\ell d\mu_j} \to \widehat{f_\ell d\mu}$ pointwise. So, by the dominated convergence theorem, $\left\|\widehat{f_\ell d\mu_j}\right\|_{2r} \to \left\|\widehat{f_\ell d\mu}\right\|_{2r}$. Therefore, it will suffice to prove that

$$\left\|\widehat{f_\ell d\mu_j}\right\|_{2r}^{2r} \geq C_{2r} \frac{N^\ell r^{-\ell}}{t^{\ell(2r+1)/2}}$$

for $j \geq \ell$.

By (2.2) we have

$$f_\ell d\mu_j = t^{-j} N^j \sum_{b \in P_\ell} \sum_{a \in A_j \cap [b, b+N^{-\ell}]} \mathbf{1}_{[a, a+N^{-j}]} dx$$

so that

$$\begin{aligned}
\widehat{f_\ell d\mu_j}(\xi) &= \frac{1 - e^{-2\pi i \xi / N^j}}{2\pi i \xi / N^j} t^{-j} \sum_{b \in P_\ell} \sum_{a \in A_j \cap [b, b+N^{-\ell}]} e^{-2\pi i a \xi} \\
&= e^{-\pi i \xi / N^j} \operatorname{sinc}(\xi / N^j)\, t^{-j} \sum_{a \in F_\ell \cap A_j} e^{-2\pi i a \xi},
\end{aligned}$$

where $\operatorname{sinc}(x) = \sin(\pi x)/(\pi x)$. Therefore

$$\begin{aligned}
\left\|\widehat{f_\ell d\mu_j}\right\|_{2r}^{2r} &= t^{-2rj} \int_{-\infty}^{\infty} \operatorname{sinc}^{2r}(\xi / N^j) \left| \sum_{a \in F_\ell \cap A_j} e^{-2\pi i a \xi} \right|^{2r} d\xi \\
&= \frac{N^j}{t^{2rj}} \int_{-\infty}^{\infty} \operatorname{sinc}^{2r}(\eta) \left| \sum_{a \in N^j (F_\ell \cap A_j)} e^{-2\pi i a \eta} \right|^{2r} d\eta \\
&= \frac{N^j}{t^{2rj}} \int_{-\infty}^{\infty} \operatorname{sinc}^{2r}(\eta) \sum_{a_1, \ldots, a_{2r} \in N^j (F_\ell \cap A_j)} e^{-2\pi i \eta \sum_{n=1}^{r} (a_n - a_{n+r})} \\
&= \frac{N^j}{t^{2rj}} \sum_{a_1, \ldots, a_{2r} \in N^j (F_\ell \cap A_j)} \widehat{\operatorname{sinc}^{2r}} \left(\sum_{n=1}^{r} (a_n - a_{n+r}) \right).
\end{aligned}$$

But

$$\widehat{\operatorname{sinc}^{2r}} = *_{i=1}^{2r} \widehat{\operatorname{sinc}} = *_{i=1}^{2r} \mathbf{1}_{[-1/2, 1/2]} \geq 0.$$

So

$$\left\|\widehat{f_\ell d\mu_j}\right\|_{2r}^{2r} \geq \frac{N^j}{t^{2rj}} \ \widehat{\operatorname{sinc}^{2r}}(0) M_{N^j (F_\ell \cap A_j)}.$$

Appealing to Lemma 12 completes the proof. □

We can now prove Proposition 11.

Proof of Proposition 11. Fix $r \in \mathbb{N}$ so that $r > 1/\alpha$. By Lemma 14, (3.1) holds with $p = 2r$, provided that n_0 is large enough. It suffices to prove that it also holds for all p such that $1 \leq p < 2r$.

Let ϕ be a function in $L^\infty(\mathbb{R})$, then for $1 \leq p < 2r$ we have

$$\|\phi\|_{2r}^{2r} = \int |\phi|^{2r} = \int |\phi|^p \, |\phi|^{2r-p} \leq \|\phi\|_p^p \, \|\phi\|_\infty^{2r-p}.$$

We apply this with $\phi = \widehat{f_\ell d\mu}$. We have $\|\widehat{f_\ell d\mu}\|_\infty \le \mu(F_\ell) = t^{-l/2}$, so that

$$\|\widehat{f_\ell d\mu}\|_p^p \ge C \frac{N^\ell r^{-\ell-1}}{t^{\ell(2r+1)/2}} \cdot (t^{\ell/2})^{2r-p} = C \frac{N^\ell r^{-\ell-1}}{t^{\ell(p+1)/2}}$$

as claimed.

□

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