THE COVEN-MEYEROWITZ TILING CONDITIONS FOR 3 ODD PRIME FACTORS

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Abstract. It is well known that if a finite set $A \subset \mathbb{Z}$ tiles the integers by translations, then the translation set must be periodic, so that the tiling is equivalent to a factorization $A \oplus B = \mathbb{Z}_M$ of a finite cyclic group. We are interested in characterizing all finite sets $A \subset \mathbb{Z}$ that have this property. Coven and Meyerowitz [3] proposed conditions (T1), (T2) that are sufficient for A to tile, and necessary when the cardinality of A has at most two distinct prime factors. They also proved that (T1) holds for all finite tiles, regardless of size. It is not known whether (T2) must hold for all tilings with no restrictions on the number of prime factors of $|A|$.

We prove that the Coven-Meyerowitz tiling condition (T2) holds for all integer tilings of period $M = (p_i p_j p_k)^2$, where p_i, p_j, p_k are distinct odd primes. The proof also provides a classification of all such tilings.

Contents

Date: July 15, 2021.

2020 *Mathematics Subject Classification.* Primary 05B45, 11B75, 20K01. Secondary 11C08, 43A75, 51D20, 52C22.

1. Introduction

We say that a set $A \subset \mathbb{Z}$ *tiles the integers by translations* if there is a set $T \subset \mathbb{Z}$ such that every integer n can be represented uniquely as $n = a + t$ with $a \in A$ and $t \in T$. Throughout this article, we assume that A is finite. It is well known (see [34]) that any tiling of $\mathbb{Z}$ by a finite set A must be periodic, i.e. $T = B \oplus M\mathbb{Z}$ for some finite set $B \subset \mathbb{Z}$ such that $|A|\,|B| = M$. Equivalently, $A \oplus B$ is a factorization of the cyclic group $\mathbb{Z}_M$.

We are interested in determining which finite sets $A \subset \mathbb{Z}$ have this property, and, in particular, in a characterization proposed by Coven and Meyerowitz [3]. In order to state their conditions, we need to introduce some notation. By translational invariance, we may assume that $A, B \subset \{0, 1, \dots\}$ and that $0 \in A \cap B$. The *characteristic polynomials* (also known as *mask polynomials*) of A and B are

$$A(X) = \sum_{a \in A} X^a, \ B(x) = \sum_{b \in B} X^b.$$

Then the tiling condition $A \oplus B = \mathbb{Z}_M$ is equivalent to

$$A(X)B(X) = 1 + X + \cdots + X^{M-1} \mod (X^M - 1). \tag{1.1}$$

Let $\Phi_s(X)$ be the s-th cyclotomic polynomial, i.e., the unique monic, irreducible polynomial whose roots are the primitive s-th roots of unity. Alternatively, Φ_s can be defined inductively via the identity

$$X^n - 1 = \prod_{s|n} \Phi_s(X). \tag{1.2}$$

In particular, (1.1) is equivalent to

$$|A||B| = M \text{ and } \Phi_s(X) \mid A(X)B(X) \text{ for all } s|M,\ s \neq 1. \tag{1.3}$$

Since Φ_s are irreducible, each $\Phi_s(X)$ with $s|M$ must divide at least one of $A(X)$ and $B(X)$.

The following result is due to Coven and Meyerowitz [3].

Theorem 1.1. [3] *Let S_A be the set of prime powers p^α such that $\Phi_{p^\alpha}(X)$ divides $A(X)$. Consider the following conditions.*

(T1) $A(1) = \prod_{s\in S_A} \Phi_s(1)$,

(T2) if $s_1, \ldots, s_k \in S_A$ are powers of different primes, then $\Phi_{s_1 \ldots s_k}(X)$ divides $A(X)$.

Then:

- *if A satisfies (T1), (T2), then A tiles $\mathbb{Z}$;*
- *if A tiles $\mathbb{Z}$ then (T1) holds;*
- *if A tiles $\mathbb{Z}$ and $|A|$ has at most two distinct prime factors, then (T2) holds.*

While (T1) is relatively easy to prove, (T2) turns out to be much deeper and more difficult. Coven and Meyerowitz [3] proved that if A satisfies (T2), then $A \oplus B^\flat = \mathbb{Z}_M$, where $M = \mathrm{lcm}(S_A)$ and $B^\flat$ is an explicit, highly structured "standard" tiling complement (defined here in Section 2.4). We prove in [24] that having a tiling complement of this type is in fact equivalent to (T2). In this formulation, (T2) bears some resemblance to questions on replacement of factors in theory of factorizations of abelian groups (see [44] for an overview of the latter).

The proof of Theorem 1.1 in [3] is based on an inductive argument. Coven and Meyerowitz use a theorem of Tijdeman [47] to prove that if A tiles the integers, then it also tiles $\mathbb{Z}_M$ for some M which has the same prime factors as $|A|$. Hence, if $|A|$ has at most two distinct prime factors, we may assume that so does M. The authors then use Sands's theorem [37], which states that, in any tiling $A \oplus B = \mathbb{Z}_M$ with M divisible by at most 2 primes, at least one of A and B must be contained in $p\mathbb{Z}_M$ for some prime $p|M$. Coven and Meyerowitz use this to decompose the given tiling into tilings of smaller groups while keeping track of the (T2) property. We also note that if $|A|$ is a prime power, then the Coven-Meyerowitz characterization simplifies further since (T2) is vacuous; in this case, the result had been proved earlier by Newman [34].

The Coven-Meyerowitz proof does not extend to the general case. Sands's factor replacement theorem is false if M has three or more prime factors, with counterexamples in [43], [25]. On the other hand, we prove in [24, Corollary 6.2] (using a relatively minor modification of the argument in [3]) that if $A \oplus B = \mathbb{Z}_M$, and if $|A|$ and $|B|$ share at most two distinct prime factors, then both A and B satisfy (T2). (See also [46], [39].) Thus the simplest case that is not covered by these methods is when $|A| = |B| = p_i p_j p_k$, where p_i, p_j, p_k are distinct primes.

Our main result is the following theorem.

Theorem 1.2. *Let $M = p_i^2 p_j^2 p_k^2$, where p_i, p_j, p_k are distinct odd primes. Assume that $A \oplus B = \mathbb{Z}_M$, with $|A| = |B| = p_i p_j p_k$. Then both A and B satisfy (T2).*

We also obtain a classification of all tilings $A \oplus B = \mathbb{Z}_M$, where $M = p_i^2 p_j^2 p_k^2$. Our main results in that regard are Theorems 3.1 and 3.2. Since those theorems require some notation and definitions, we postpone their statements until Section 3.

The proof of Theorem 1.2 relies on the methods and concepts introduced in [24]. In order to keep this article reasonably self-contained modulo results that can be used as black boxes, we provide a summary of the concepts and results that we will need here, specialized to the 3-prime setting, in Section 2. We then state our classification results in Section 3.1. In Section 3.2, we discuss our strategy and the main new ideas of the proof. The rest of the paper is devoted to the proof of Theorems 1.2, 3.1, and 3.2.

Since [3], there has been essentially no progress on proving (T2), except for a few special cases that either assume particular structure of the tiling (see [22], [4]) or are covered by the methods of [3] as in [24, Corollary 6.2] (see [46], [39]). However, there has been recent work on other questions related to tiling. For instance, Bhattacharya [1] has established the periodic tiling conjecture in $\mathbb{Z}^2$, with a quantitative version due to Greenfeld and Tao [13]. There has also been interesting work on tilings of the real line by a function (see [19] for a survey and some open questions).

The Coven-Meyerowitz tiling conditions have implications for the ongoing work on Fuglede's spectral set conjecture [10]. Fuglede conjectured that a set $\Omega \subset \mathbb{R}^n$ of positive n-dimensional Lebesgue measure tiles $\mathbb{R}^n$ by translations if and only if it is *spectral*, in the sense that the space $L^2(\Omega)$ admits an orthogonal basis of exponential functions. While the conjecture has been disproved in its full generality in dimensions 3 and higher [45], [20], [21], [8], [33], [9], significant connections between tiling and spectrality do exist (see [5] for an overview of the problem in dimension 1), and there is a large body of work investigating such connections from many points of view. In higher dimensions, the conjecture has been proved for convex sets in $\mathbb{R}^n$, by Iosevich, Katz and Tao [14] for $n = 2$, Greenfeld and Lev [12] for $n = 3$, and by Lev and Matolsci [29] for general n. There have been many recent results on special cases of the finite abelian group analogue of the conjecture [15], [31], [6], [16], [17], [18], [30], [39], [40], [41], [7].

If (T2) could be proved for all finite integer tiles, this would imply the "tiling implies spectrum" part of Fuglede's spectral set conjecture for all compact tiles in dimension 1, as well as for all cyclic groups $\mathbb{Z}_M$. This follows from the results of [26], [27], [23]. Proving (T2) for specific tiling problems does not resolve the full conjecture, but it does imply that the conjecture holds in those settings. In that regard, our Theorem 1.2 combined with [23, Theorem 1.5] and [24, Corollary 6.2] has the following immediate corollary.

Corollary 1.3. *Let $M = p_i^2 p_j^2 p_k^2$.*

(i) The "tiling implies spectrum" part of Fuglede's spectral set conjecture holds for the cyclic group $\mathbb{Z}_M$. In other words, if $A \subset \mathbb{Z}_M$ tiles $\mathbb{Z}_M$ by translations, then it is spectral.

(ii) Let $A \subset \mathbb{Z}$ be a finite set such that A mod M tiles $\mathbb{Z}_M$, and let $F = \bigcup_{a\in A}[a, a+1]$, so that F tiles $\mathbb{R}$ by translations. Then F is spectral.

Indeed, let M and A be as in Corollary 1.3. If $|A| \neq p_i p_j p_k$, then both A and B satisfy (T2) by [24, Corollary 6.2]. If on the other hand $|A| = p_i p_j p_k$, then both A and B satisfy (T2) by Theorem 1.2. In both cases, spectrality follows from [23, Theorem 1.5]. (While Theorem 1.5 in [23] is stated for unions of finite intervals as in (ii), the same argument applies in the finite group setting. See e.g. [5].)

2. Notation and preliminaries

This section summarizes the relevant definitions and results of [24], specialized to the 3-prime case. All material due to other authors is indicated explicitly as such.

2.1. **Multisets and mask polynomials.** Throughout this paper, we will assume that $M = p_i^{n_i} p_j^{n_j} p_k^{n_k}$, where p_i, p_j, p_k are distinct primes and $n_i, n_j, n_k \in \mathbb{N}$. The indices $\{i, j, k\}$ can be thought of as a permutation of $\{1, 2, 3\}$; however, we will always use i, j, k for this purpose, freeing up numerical subscripts for other uses. While the full proof of Theorem 1.2 requires that $n_i = n_j = n_k = 2$ and that $p_i, p_j, p_k \neq 2$, many of our intermediate results are valid under weaker assumptions as indicated.

We will always work in either $\mathbb{Z}_M$ or in $\mathbb{Z}_N$ for some $N|M$. We use $A(X)$, $B(X)$, etc. to denote polynomials modulo $X^M - 1$ with integer coefficients. Each such polynomial $A(X) = \sum_{a \in \mathbb{Z}_M} w_A(a) X^a$ is associated with a weighted multiset in $\mathbb{Z}_M$, which we will also denote by A, with weights $w_A(x)$ assigned to each $x \in \mathbb{Z}_M$. (If the coefficient of X^x in $A(X)$ is 0, we set $w_A(x) = 0$.) In particular, if A has $\{0, 1\}$ coefficients, then w_A is the characteristic function of a set $A \subset \mathbb{Z}_M$. We will use $\mathcal{M}(\mathbb{Z}_M)$ to denote the family of all weighted multisets in $\mathbb{Z}_M$, and reserve the notation $A \subset \mathbb{Z}_M$ for sets.

If $N|M$, then any $A \in \mathcal{M}(\mathbb{Z}_M)$ induces a weighted multiset A mod N in $\mathbb{Z}_N$, with the corresponding mask polynomial $A(X)$ mod $(X^N - 1)$ and induced weights

$$w_A^N(x) = \sum_{x' \in \mathbb{Z}_M : x' \equiv x \bmod N} w_A(x'), \quad x \in \mathbb{Z}_N. \tag{2.1}$$

We will continue to write A and $A(X)$ for A mod N and $A(X)$ mod $X^N - 1$, respectively, while working in $\mathbb{Z}_N$.

If $A, B \in \mathcal{M}(\mathbb{Z}_M)$, we will use $A + B$ to indicate the weighted multiset corresponding to the mask polynomial $(A + B)(X) = A(X) + B(X)$, with the weight function $w_{A+B}(x) = w_A(x) + w_B(x)$. We use the convolution notation $A * B$ to denote the weighted sumset of A and B, so that $(A * B)(X) = A(X)B(X)$ and

$$w_{A*B}(x) = (w_A * w_B)(x) = \sum_{y \in \mathbb{Z}_M} w_A(x - y) w_B(y).$$

If one of the sets is a singleton, say $A = \{x\}$, we will simplify the notation and write $x * B = \{x\} * B$. The direct sum notation $A \oplus B$ is reserved for tilings, i.e., $A \oplus B = \mathbb{Z}_M$ means that $A, B \subset \mathbb{Z}_M$ are both sets and $A(X)B(X) = \frac{X^M - 1}{X - 1} \bmod X^M - 1$. We will not use derivatives of polynomials in this paper, hence notation such as A', A'', etc., will be used to denote auxiliary multisets and polynomials rather than derivatives.

2.2. **Array coordinates and geometric representation.** For $\nu \in \{i, j, k\}$, define $M_\nu := M/p_\nu^{n_\nu} = \prod_{\kappa \neq \nu} p_\kappa^{n_\kappa}$. Then each $x \in \mathbb{Z}_M$ can be written uniquely as

$$x = \sum_{\nu \in \{i,j,k\}} \pi_\nu(x) M_\nu, \quad \pi_\nu(x) \in \mathbb{Z}_{p_\nu^{n_\nu}}.$$

This sets up an isomorphism $\mathbb{Z}_M = \mathbb{Z}_{p_i^{n_i}} \oplus \mathbb{Z}_{p_j^{n_j}} \oplus \mathbb{Z}_{p_k^{n_k}}$, and identifies each element $x \in \mathbb{Z}_M$ with an element of a 3-dimensional lattice with coordinates $(\pi_i(x), \pi_j(x), \pi_k(x))$. The tiling $A \oplus B = \mathbb{Z}_M$ can then be interpreted as a tiling of that lattice.

For $D|M$, a *D-grid* in $\mathbb{Z}_M$ is a set of the form

$$\Lambda(x, D) := x + D\mathbb{Z}_M = \{x' \in \mathbb{Z}_M : \ D|(x - x')\}$$

for some $x \in \mathbb{Z}_M$. A few special cases have a geometric interpretation of interest. A *line* through $x \in \mathbb{Z}_M$ in the p_ν direction is the set $\ell_\nu(x) := \Lambda(x, M_\nu)$, and a *plane* through $x \in \mathbb{Z}_M$ perpendicular to the p_ν direction, on the scale $M_\nu p_\nu^{\alpha_\nu}$, is the set $\Pi(x, p_\nu^{\alpha_\nu}) := \Lambda(x, p_\nu^{\alpha_\nu})$.

An *M-fiber in the p_ν direction* is a set of the form $x * F_\nu$, where $x \in \mathbb{Z}_M$ and

$$F_\nu = \{0, M/p_\nu, 2M/p_\nu, \dots, (p_\nu - 1)M/p_\nu\}. \tag{2.2}$$

Thus $x * F_\nu = \Lambda(x, M/p_\nu)$. A set $A \subset \mathbb{Z}_M$ is *M-fibered in the p_ν direction* if there is a subset $A' \subset A$ such that $A = A' * F_\nu$.

If $N = p_i^{\alpha_1} p_j^{\alpha_j} p_k^{\alpha_k}$ is a divisor of M, with $0 \le \alpha_\nu \le n_\nu$, we let

$$D(N) := p_i^{\gamma_i} p_j^{\gamma_j} p_k^{\gamma_k}, \quad \text{where } \gamma_\nu = \max(0, \alpha_\nu - 1) \text{ for } \nu \in \{i, j, k\}.$$

We will also write $N_\nu = M/p_\nu$ for $\nu \in \{i, j, k\}$.

2.3. **Divisor set and divisor exclusion.** For $N|M$ and $A \subset \mathbb{Z}_M$, we define

$$\mathrm{Div}_N(A) := \{(a - a', N) : \ a, a' \in A\} \tag{2.3}$$

When $N = M$, we will omit the subscript and write $\mathrm{Div}(A) = \mathrm{Div}_M(A)$. Informally, we will refer to the elements of $\mathrm{Div}(A)$ as the *divisors of A* or *differences in A*. A theorem due to Sands [37] states that $A \oplus B = \mathbb{Z}_M$ if and only if $A, B \subset \mathbb{Z}_M$ are sets such that $|A|\,|B| = M$ and

$$\mathrm{Div}(A) \cap \mathrm{Div}(B) = \{M\}. \tag{2.4}$$

We will refer to this as *divisor exclusion*.

In cases when we need to indicate where a particular divisor of A must occur, we will use the following notation for localized divisor sets. If $A, A_1, A_2 \subset \mathbb{Z}_M$ and $a_0 \in \mathbb{Z}_M$, we will write

$$\begin{aligned} \mathrm{Div}_N(A_1, A_2) &:= \{(a_1 - a_2, N) : \ a_1 \in A_1, a_2 \in A_2\}, \\ \mathrm{Div}_N(A, a_0) = \mathrm{Div}_N(a_0, A) &:= \mathrm{Div}_N(A, \{a_0\}) = \{(a - a_0, N) : \ a \in A\}. \end{aligned} \tag{2.5}$$

For example, if $A \oplus B = \mathbb{Z}_M$, we will often need to consider $\mathrm{Div}(A_1, A_2)$, where A_1 and A_2 are restrictions of A to geometric structures such as planes or lines.

2.4. **Standard tiling complements.** Suppose that $A \oplus B = \mathbb{Z}_M$, and let

$$\mathfrak{A}_\nu(A) = \left\{\alpha_\nu \in \{1, 2, \dots, n_\nu\} : \ \Phi_{p_\nu^{\alpha_\nu}}(X) | A(X)\right\}$$

The *standard tiling complement* $A^\flat$ is defined via its mask polynomial

$$A^\flat(X) = \prod_{\nu \in \{i,j,k\}} \prod_{\alpha_\nu \in \mathfrak{A}_\nu(A)} \left(1 + X^{M_\nu p_\nu^{\alpha_\nu - 1}} + \dots + X^{(p_\nu - 1) M_\nu p_\nu^{\alpha_\nu - 1}}\right). \tag{2.6}$$

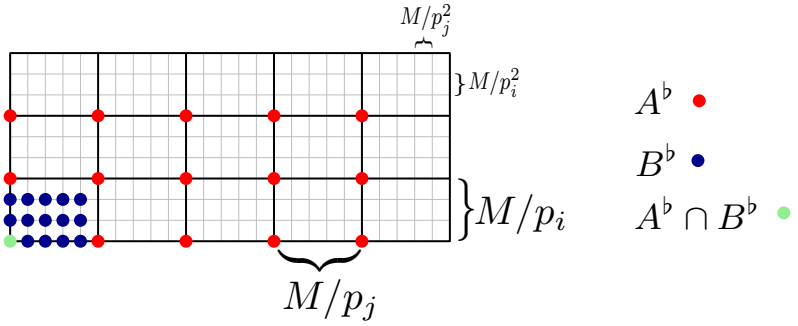

FIGURE 1. The standard sets $A^\flat, B^\flat \subset \mathbb{Z}_{p_i^2 p_j^2}$ with $p_i = 3, p_j = 5$ and $\Phi_{p_i^2}\Phi_{p_j^2}|A, \Phi_{p_i}\Phi_{p_j}|B$.

Then $A^\flat(X)$ satisfies (T2) and has the same prime power cyclotomic divisors as $A(X)$. For each prime power $s|M$, Φ_s divides exactly one of A and B [3], hence $A^\flat$ is also uniquely determined by M and B. Coven and Meyerowitz proved in [3] that if a finite tile satisfies (T2), it has a standard tiling complement. We prove the converse in [24].

Proposition 2.1. *Let $A \oplus B = \mathbb{Z}_M$. Then $A^\flat \oplus B = \mathbb{Z}_M$ if and only if B satisfies (T2).*

We say that the tilings $A \oplus B = \mathbb{Z}_M$ and $A' \oplus B = \mathbb{Z}_M$ are *T2-equivalent* if

$$A \text{ satisfies (T2)} \ \Leftrightarrow A' \text{ satisfies (T2)}. \tag{2.7}$$

Since A and A' tile the same group $\mathbb{Z}_M$ with the same tiling complement B, they must have the same cardinality and the same prime power cyclotomic divisors. We will sometimes say simply that A is T2-equivalent to A' if both M and B are clear from context. Usually, A' will derived from A using certain permitted manipulations such as fiber shifts (Lemma 2.9). In particular, if we can prove that either A or B in a given tiling is T2-equivalent to a standard tiling complement, this resolves the problem completely in that case.

Corollary 2.2. *Suppose that the tiling $A \oplus B = \mathbb{Z}_M$ is T2-equivalent to the tiling $A^\flat \oplus B = \mathbb{Z}_M$. Then A and B satisfy (T2).*

2.5. **Box product.** Let $A \subset \mathbb{Z}_M$ and $N|M$. For $x \in \mathbb{Z}_M$, define

$$\mathbb{A}_m^N[x] = \#\{a \in A : \ (x-a, N) = m\}.$$

We may think of $A_m^N[x]$, with x fixed and m ranging over the divisors of N, as the entries of the *N-box* $\mathbb{A}^N[x] = (\mathbb{A}_m^N[x])_{m|N}$ [24]. If $N = M$, we will usually omit the superscript and write $\mathbb{A}_m^M[x] = \mathbb{A}_m[x]$. If $C \subset \mathbb{Z}_M$, we write $\mathbb{A}_m^N[C] := \sum_{c\in C} \mathbb{A}_m^N[c]$. Furthermore, if $X \subset \mathbb{Z}_M$ and $x \in \mathbb{Z}_M$, define

$$\mathbb{A}_m^N[x|X] = \#\{a \in A \cap X : \ (x-a, N) = m\}.$$

If $A, B \subset \mathbb{Z}_M$, we define the *box product* of the associated M-boxes as

$$\langle \mathbb{A}[x], \mathbb{B}[y] \rangle = \sum_{m|M} \frac{1}{\phi(M/m)} \mathbb{A}_m[x]\mathbb{B}_m[y]. \tag{2.8}$$

Here ϕ is the Euler totient function: if $n = \prod_{\iota=1}^L q_\iota^{r_\iota}$, where $q_1, \ldots, q_L$ are distinct primes and $r_\iota \in \mathbb{N}$, then $\phi(n) = \prod_{\iota=1}^L (q_\iota - 1) q_\iota^{r_\iota - 1}$.

Theorem 2.3. *([24]; following* [11, Theorem 1]*) If $A \oplus B = \mathbb{Z}_M$ is a tiling, then*

$$\langle \mathbb{A}^M[x], \mathbb{B}^M[y] \rangle = 1 \quad \forall x, y \in \mathbb{Z}_M. \tag{2.9}$$

2.6. **Cuboids.**

Definition 2.4. *(i) A* cuboid type $\mathcal{T}$ *on $\mathbb{Z}_N$ is an ordered triple $\mathcal{T} = (N, \vec{\delta}, T)$, where:*

- *$N = \prod_{\nu\in\{i,j,k\}} p_\nu^{n_\nu - \alpha_\nu}$ is a divisor of M, with $0 \le \alpha_\nu \le n_\nu$ for each ν,*
- *$\vec{\delta} = (\delta_i, \delta_j, \delta_k)$, with $0 \le \delta_\nu \le n_\nu - \alpha_\nu$,*
- *the* template *T is a nonempty subset of $\mathbb{Z}_N$.*

(ii) A cuboid *Δ of type $\mathcal{T}$ is a weighted multiset corresponding to a mask polynomial of the form*

$$\Delta(X) = X^c \prod_{\nu\in\mathfrak{J}} (1 - X^{d_\nu}), \tag{2.10}$$

where $\mathfrak{J} = \mathfrak{J}_{\vec{\delta}} := \{\nu : \delta_\nu \neq 0\}$, *and* c, d_ν *are elements of* $\mathbb{Z}_M$ *such that* $(d_\nu, N) = N/p_\nu^{\delta_\nu}$ *for* $\nu \in \{i, j, k\}$. *The* vertices *of* Δ *are the points*

$$x_{\vec{\epsilon}} = c + \sum_{\nu \in \mathfrak{J}} \epsilon_\nu d_\nu : \ \vec{\epsilon} = (\epsilon_\nu)_{\nu \in \mathfrak{J}} \in \{0, 1\}^{|\mathfrak{J}|}, \tag{2.11}$$

with weights $w_\Delta(x_{\vec{\epsilon}}) = (-1)^{\sum_{\nu \in \mathfrak{J}} \epsilon_\nu}$.

(iii) Let $A \in \mathcal{M}(\mathbb{Z}_N)$, *and let* Δ *be a cuboid of type* $\mathcal{T}$. *Define*

$$\mathbb{A}^{\mathcal{T}}[\Delta] = \mathbb{A}_N^N[\Delta * T] = \sum_{\vec{\epsilon} \in \{0,1\}^k} w_\Delta(x_{\vec{\epsilon}}) \mathbb{A}_N^N[x_{\vec{\epsilon}} * T],$$

where we recall that $x * T = \{x + t : \ t \in T\}$, *so that*

$$\mathbb{A}_N^N[x_{\vec{\epsilon}} * T] := \sum_{t \in T} \mathbb{A}_N^N[x_{\vec{\epsilon}} + t].$$

For consistency, we will also write $\mathbb{A}^{\mathcal{T}}[x] = \mathbb{A}_N^N[x * T]$ for $x \in \mathbb{Z}_M$.

An important special case is as follows: for $N|M$, an N-*cuboid* is a cuboid of type $\mathcal{T} = (N, \vec{\delta}, T)$, where $N|M$, $T(X) = 1$, and $\delta_\nu = 1$ for all ν such that $p_\nu|N$. Thus, N-cuboids have the form

$$\Delta(X) = X^c \prod_{p_\nu | N} (1 - X^{d_\nu}), \tag{2.12}$$

with $(d_\nu, N) = N/p_\nu$ for all ν such that $p_\nu|N$. We reserve the term "N-cuboid", without cuboid type explicitly indicated, to refer to cuboids as in (2.12); for cuboids of any other type, we will always specify $\mathcal{T}$.

Cuboids provide useful criteria to determine cyclotomic divisibility properties of mask polynomials. We say that a multiset $A \in \mathcal{M}(\mathbb{Z}_M)$ *is* $\mathcal{T}$-*null* if for every cuboid Δ of type $\mathcal{T}$,

$$\mathbb{A}^{\mathcal{T}}[\Delta] = 0. \tag{2.13}$$

For $A \in \mathcal{M}(\mathbb{Z}_N)$, we have $\Phi_N(X)|A(X)$ if and only if $\mathbb{A}_N^N[\Delta] = 0$ for every N-cuboid Δ. This has been known and used previously in the literature, see e.g. [42, Section 3], or [16, Section 3]. In particular, for any $N|M$, Φ_N divides A if and only if it divides the mask polynomial of $A \cap \Lambda(x, D(N))$ for every $x \in \mathbb{Z}_M$.

More generally, if for every $m|N$ the polynomial $\Phi_m(X)$ divides at least one of $A(X)$, $T(X)$, or $\Delta(X)$ for every Δ of type $\mathcal{T}(N, \vec{\delta}, T)$, then A is $\mathcal{T}$-null. We use such cuboid types to test for divisibility by combinations of cyclotomic polynomials. For example, we have the following.

- Assume that $n_i \geq 2$, and let $\mathcal{T} = (M, \vec{\delta}, 1)$, with $\delta_i = 2$ and $\delta_j = \delta_k = 1$. Then
$$\Phi_M \Phi_{M/p_i} | A \Leftrightarrow A \text{ is } \mathcal{T}\text{-null}.$$
- Assume that $n_i = 2$, and let $\mathcal{T} = (M, \vec{\delta}, T)$, where $\delta_i = 0$, $\delta_j = \delta_k = 1$, and
$$T(X) = \frac{X^{M/p_i} - 1}{X^{M/p_i^2} - 1} = 1 + X^{M/p_i^2} + \cdots + X^{(p_i - 1)M/p_i^2}.$$
If $\Phi_M \Phi_{M/p_i^2} | A$, then A is $\mathcal{T}$-null.

2.7. **Tiling reductions.** The general formulations of the tiling reductions below are provided in Theorems 6.1, 6.5, and Corollary 6.6 in [24]. The additional assumption that $p_\nu \parallel |B|$ in Theorem 2.5 ensures that in any tiling $A' \oplus B' = \mathbb{Z}_{N_\nu}$ with $|A'| = |A|$ and $|B'| = |B|/p_\nu$, $|A'|$ and $|B'|$ have only two common factors. Hence the assumption (ii) of [24, Theorem 6.1] is satisfied by [24, Corollary 6.2], and we deduce that A and B both satisfy (T2). The assumption that $p_\nu \parallel |A|$ in Corollary 2.7 serves the same purpose.

Theorem 2.5. (Subgroup reduction) [3, Lemma 2.5] *Let $M = p_i^{n_i} p_j^{n_j} p_k^{n_k}$. Assume that $A \oplus B = \mathbb{Z}_M$, and that $A \subset p_\nu \mathbb{Z}_M$ for some $\nu \in \{i, j, k\}$ such that $p_\nu \parallel |B|$. Then A and B satisfy (T2).*

Theorem 2.6. *Assume that $A \oplus B = \mathbb{Z}_M$ and $\Phi_{p_\nu^{n_\nu}} | A$ for some $\nu \in \{i, j, k\}$. Define*

$$A_{p_\nu} = \{a \in A : \ 0 \le \pi_\nu(a) \le p_\nu^{n_\nu - 1} - 1\}. \tag{2.14}$$

Then the following are equivalent:

(i) For any translate A' of A we have $A'_{p_\nu} \oplus B = \mathbb{Z}_{M/p_\nu}$.

(ii) For every d such that $p_\nu^{n_\nu} | d | M$, at least one of the following holds:

$$\Phi_d | A, \tag{2.15}$$

$$\Phi_{d/p_\nu} \Phi_{d/p_\nu^2} \dots \Phi_{d/p_\nu^{n_\nu}} \mid B. \tag{2.16}$$

Corollary 2.7. (Slab reduction) *Let $M = p_i^{n_i} p_j^{n_j} p_k^{n_k}$. Assume that $A \oplus B = \mathbb{Z}_M$, and that there exists a $\nu \in \{i, j, k\}$ such that $\Phi_{p_\nu^2} | A$, $p_\nu \parallel |A|$, and A, B obey the condition (ii) of Theorem 2.6. (In particular, this holds if A is M-fibered in one of the p_i, p_j, p_k directions.) Then A and B satisfy (T2).*

2.8. **Saturating sets.** Let $A \oplus B = \mathbb{Z}_M$, and $x, y \in \mathbb{Z}_M$. Define

$$A_{x,y} := \{a \in A : \ (x - a, M) = (y - b, M) \text{ for some } b \in B\},$$

$$A_x := \{a \in A : (x - a, M) \in \mathrm{Div}(B)\} = \bigcup_{b \in B} A_{x,b}.$$

We will refer to A_x as the *saturating set* for x. The sets $B_{y,x}$ and B_y are defined similarly, with A and B interchanged.

By divisor exclusion, $A_a = \{a\}$ for all $a \in A$. For $x \in \mathbb{Z}_M \setminus A$, saturating spaces are more robust, but are still subject to geometric constraints based on divisor exclusion. For $x, x' \in \mathbb{Z}_M$ such that $(x - x', M) = p_i^{\alpha_i} p_j^{\alpha_j} p_k^{\alpha_k}$, with $0 \le \alpha_\nu \le n_\nu$, define

$$\begin{aligned} \mathrm{Span}(x, x') &= \bigcup_{\nu : \alpha_\nu < n_\nu} \Pi(x, p_\nu^{\alpha_\nu + 1}), \\ \mathrm{Bispan}(x, x') &= \mathrm{Span}(x, x') \cup \mathrm{Span}(x', x). \end{aligned} \tag{2.17}$$

Then for any $x, x', y \in \mathbb{Z}_M$, we have

$$A_{x',y} \subset A_{x,y} \cup \mathrm{Bispan}(x, x'), \tag{2.18}$$

and in particular,

$$A_x \subset \bigcap_{a \in A} \mathrm{Bispan}(x, a). \tag{2.19}$$

For example, if $x \in \mathbb{Z}_M \setminus A$ satisfies $(x-a, M) = M/p_i$ for some $a \in A$, then

$$A_x \subset \text{Bispan}(x, a) = \Pi(x, p_i^{n_i}) \cup \Pi(a, p_i^{n_i}).$$

In the sequel, whenever we evaluate saturating sets, we will always start with geometric restrictions based on (2.19).

2.9. **Fibers and cofibered structures.** The following is a simplified version of the definitions and results of [24, Section 8], restricted to $M = p_i^2 p_j^2 p_k^2$. Most of this article can be read with just these definitions, if the reader is willing to substitute $n_i = n_j = n_k = 2$ in all arguments. On those few occasions when the more general versions are necessary even in that case, we have to refer the reader to [24, Section 8].

Let $N|M$, $c \in \mathbb{N}$, and $\nu \in \{i, j, k\}$. An *N-fiber in the p_ν direction* with multiplicity c is a set $F \subset \mathbb{Z}_M$ such that $F \bmod N$ has the mask polynomial

$$F(X) \equiv cX^a(1 + X^{N/p_\nu} + X^{2N/p_\nu} + \cdots + X^{(p_\nu-1)N/p_\nu}) \mod (X^N - 1)$$

for some $a \in \mathbb{Z}_M$. We will say sometimes that F *passes through* a, or *is rooted at* a. A set $A \subset \mathbb{Z}_M$ is *N-fibered in the p_ν direction* if it can be written as a union of disjoint N-fibers in the p_ν direction, all with the same multiplicity.

Fiber chains in the p_ν direction are translates of sets that tile $(M/p_\nu^\gamma)\mathbb{Z}_M$ for some γ with $1 \le \gamma \le n_i$. For $M = p_i^2 p_j^2 p_k^2$, the only fiber chains of interest that are not fibers on some scale are multisets $\tilde{F}$ with mask polynomials

$$\tilde{F}(X) = cX^a(1 + X^{M_\nu} + X^{2M_\nu} + \cdots + X^{(p_\nu^2-1)M_\nu}), \quad \nu \in \{i, j, k\},\ a \in \mathbb{Z}_M,\ c \in \mathbb{N}.$$

If $F \subset \mathbb{Z}_M$ is an M-fiber in the p_ν direction, we say that an element $x \in \mathbb{Z}_M$ is at *distance* m from F if $m|M$ is the maximal divisor such that $(z - x, M) = m$ for some $z \in F$. It is easy to see that such m exists.

Let $A \oplus B = \mathbb{Z}_M$ be a tiling. We will often be interested in finding "complementary" fibers and fibered structures in A and B, in the following sense.

Definition 2.8 (**Cofibers and cofibered structures**). *Let $A, B \subset \mathbb{Z}_M$, with $M = p_i^2 p_j^2 p_k^2$, and let $\nu \in \{i, j, k\}$.*

(i) We say that $F \subset A, G \subset B$ are $(1,2)$-cofibers *in the p_ν direction if F is an M-fiber and G is an M/p_ν-fiber, both in the p_ν direction.*

(ii) We say that the pair (A, B) has a (1,2)-cofibered structure *in the p_ν direction if*

- *B is M/p_ν-fibered in the p_ν direction,*
- *A contains at least one "complementary" M-fiber $F \subset A$ in the p_ν direction, which we will call a* cofiber *for this structure.*

The advantage of cofibered structure is that it permits fiber shifts as described below. In many cases, we will be able to use this to reduce the given tiling to a simpler one.

Lemma 2.9 (**Fiber-Shifting Lemma**). *Let $A \oplus B = \mathbb{Z}_M$. Assume that the pair (A, B) has a $(1,2)$-cofibered structure, with a cofiber $F \subset A$. Let A' be the set obtained from A by shifting F to a point $x \in \mathbb{Z}_M$ at a distance M/p_i^2 from it. Then $A' \oplus B = \mathbb{Z}_M$, and A is T2-equivalent to A'.*

In order to identify cofibered structures in (A, B), we will use saturating sets, via the following lemma.

Lemma 2.10. *Assume that $A \oplus B = \mathbb{Z}_M$ is a tiling, with $M = p_i^2 p_j^2 p_k^2$. Suppose that $x \in \mathbb{Z}_M \setminus A$, $b \in B$, $M/p_\nu \in \mathrm{Div}(A)$, and $A_{x,b} \subset \ell_\nu(x)$ for some $\nu \in \{i,j,k\}$ and $b \in B$. Then*

$$\mathbb{A}^M_{M/p_\nu^2}[x]\mathbb{B}^M_{M/p_\nu^2}[b] = \phi(p_\nu^2). \tag{2.20}$$

with the product saturated by a $(1,2)$-cofiber pair (F,G) such that $F \subset A$ is at distance M/p_i^2 from x and $G \subset B$ is rooted at b. In particular, if $A_x \subset \ell_\nu(x)$, then the pair (A,B) has a $(1,2)$-cofibered structure.

3. Classification results

3.1. **Classification results.** We are now ready to state our classification results and provide a more detailed outline of the proof. Let $A \oplus B = \mathbb{Z}_M$, where $M = p_i^{n_i} p_j^{n_j} p_k^{n_k}$. By (1.3), we have $\Phi_s(X) \mid A(X)B(X)$ for all $s|M$ such that $s \neq 1$. In particular, Φ_M divides at least one of $A(X)$ and $B(X)$. Without loss of generality, we may assume that $\Phi_M|A$.

We have $\Phi_M|A$ if and only if Φ_M divides $A \cap \Lambda(x, D(M))$ for every $x \in \mathbb{Z}_M$ (see Section 2.6). This implies structure results for restrictions of A to such grids. Let $\Lambda := \Lambda(a, D(M))$ for some $a \in A$, so that $A \cap \Lambda$ is nonempty. It is easy to see that $\Phi_M|F_\nu$ for each $\nu \in \{i,j,k\}$. By the classic results on vanishing sums of roots of unity [2], [35], [36], [38], [32], [28], Φ_M divides $A \cap \Lambda$ if and only if

$$(A \cap \Lambda)(X) = \sum_{\nu \in \{i,j,k\}} Q_\nu(X) F_\nu(X),$$

where Q_i, Q_j, Q_k are polynomials with integer coefficients depending on both A and Λ.

A particularly simple case occurs when $A(X) = Q_\nu(X)F_\nu(X)$ for a single $\nu \in \{i,j,k\}$, so that A is fibered on all $D(M)$ grids in the same direction. However, much more complicated structures are also possible. For instance, $A \cap \Lambda$ may be M-fibered in different directions on different $D(M)$ grids Λ, or there may exist a $D(M)$ grid Λ such that $A \cap \Lambda$ contains nonintersecting M-fibers in two or three different directions. An additional issue is that the polynomials Q_i, Q_j, Q_k are not required to have nonnegative coefficients. In such cases, there may be points $a \in A$ such that $a * F_\nu \not\subset A$ for any ν, due to cancellations between M-fibers in different directions.

Our classification results, and our proof of (T2), split into cases according to the fibering properties of A. Theorems 3.1 and 3.2 summarize our main findings in the unfibered and fibered case, respectively.

Theorem 3.1. *Let $A \oplus B = \mathbb{Z}_M$, where $M = p_i^2 p_j^2 p_k^2$ is odd. Assume that $|A| = |B| = p_i p_j p_k$, $\Phi_M|A$, and there exists a $D(M)$ grid Λ such that $A \cap \Lambda$ is nonempty and is not M-fibered in any direction. Assume further, without loss of generality, that $0 \in \Lambda$. Then $A^\flat = \Lambda$, and the tiling $A \oplus B = \mathbb{Z}_M$ is T2-equivalent to $\Lambda \oplus B = \mathbb{Z}_M$ via fiber shifts. By Corollary 2.2, both A and B satisfy (T2).*

Theorem 3.2. *Let $A \oplus B = \mathbb{Z}_M$, where $M = p_i^2 p_j^2 p_k^2$ is odd. Assume that $|A| = |B| = p_i p_j p_k$, $\Phi_M|A$, and that for every $a \in A$, the set $A \cap \Lambda(a, D(M))$ is M-fibered in at least one direction (possibly depending on a).*

(I) *Suppose that there exists an element $a_0 \in A$ such that*

$$a_0 * F_\nu \subset A \quad \forall \nu \in \{i,j,k\}. \tag{3.1}$$

Then the tiling $A \oplus B = \mathbb{Z}_M$ is T2-equivalent to $\Lambda \oplus B = \mathbb{Z}_M$ via fiber shifts, where $\Lambda := \Lambda(a_0, D(M))$. By Corollary 2.2, both A and B satisfy (T2).

(II) *Assume that (3.1) does not hold for any $a_0 \in A$. Then at least one of the following holds.*

- *$A \subset \Pi(a, p_\nu)$ for some $a \in A$ and $\nu \in \{i, j, k\}$. By Theorem 2.5, both A and B satisfy (T2).*
- *There exists a $\nu \in \{i, j, k\}$ such that (possibly after interchanging A and B) the conditions of Theorem 2.6 are satisfied in the p_ν direction. By Corollary 2.7, both A and B satisfy (T2).*

A more detailed breakdown of the case (II) of Theorem 3.2 is provided in Theorem 9.1.

3.2. **Outline of the proof.** In the rest of this section, we provide an outline of the proof of Theorems 3.1 and 3.2. We assume that $A \oplus B = \mathbb{Z}_M$, where $M = p_i^2 p_j^2 p_k^2$ is odd, $|A| = |B| = p_i p_j p_k$, and $\Phi_M | A$. Some of our arguments apply to tilings with more general M. In order to be able to sketch the main ideas without interruptions, we postpone the discussion of such extensions until the end of this section.

We begin with general arguments that are needed in both fibered and unfibered cases. In Section 4, we develop technical tools we will use throughout the article. Lemma 4.1 is from [24]; several of the other results in that section are specific to the 3-prime setting.

Assume first that $\Phi_M | A$ and that there exists a $D(M)$ grid Λ such that $A \cap \Lambda$ is not M-fibered in any direction. In Sections 5 and 6, we prove that $A \cap \Lambda$ must then contain at least one of two special structures, either *diagonal boxes* (Proposition 5.2) or an *extended corner* (Proposition 5.5). Large parts of the argument are combinatorial and apply to all $A \subset \mathbb{Z}_M$ such that $\Phi_M | A$; however, to get the full strength of our results, we need to use saturating set techniques, hence the tiling assumption is necessary.

Some of our technical tools work only when all the "top differences" are divisors of A, i.e.,

$$\{m : \ D(M)|m|M\} \subset \mathrm{Div}(A). \tag{3.2}$$

We therefore must pay special attention to the cases where (3.2) fails. A classification of such structures is provided in Section 6. This analysis is also needed in the fibered case (Theorem 3.2) when fibering, or lack thereof, on lower scales must be considered.

We resolve diagonal boxes and extended corner structures in Sections 7 and 8, respectively. In Theorem 7.1, we prove that if $A \cap \Lambda$ contains diagonal boxes, then A is T2-equivalent to either Λ (in which case we are done) or to another tile A' containing an extended corner. We then prove in Theorem 8.1 that if $A \cap \Lambda$ contains an extended corner structure, then A is T2-equivalent to Λ. Theorem 3.1 follows by combining Theorem 7.1 and Theorem 8.1.

The main idea of that part of the proof is that all such tilings can be obtained via fiber shifts (Lemma 2.9) from the standard tiling

$$A^\flat \oplus B^\flat = \mathbb{Z}_M, \tag{3.3}$$

where $A^\flat = \Lambda(0, D(M))$, and $B^\flat$ is the standard tiling complement with $\Phi_{p_i} \Phi_{p_j} \Phi_{p_k} | B^\flat$. Similar tilings were constructed by Szabó ([43]; see also [25]). We prove that all tilings satisfying the assumptions of Theorems 3.1 must in fact have this form. Starting with an unfibered grid in the given tiling, and using saturating set methods, we are able to locate the shifted fibers and shift them back into place, returning to (3.3). This proves (T2) and provides full information about the structure of the tiling.

In Section 9, we consider the fibered case. In the simple case when the entire set A is M-fibered in the same direction, we can apply Corollary 2.7 and be done; however, it is possible for A to be fibered in different directions on different $D(M)$-grids. The proof breaks down into cases, according to how the fibers in different directions interact.

Suppose first that there exists an element $a_0 \in A$ such that (3.1) holds. This case turns out to be similar to that of unfibered grids and is resolved by similar methods, ending in T2-equivalence to (3.3).

Assume now that no such element exists. Our main intermediate result in this case is that, in fact, only two fibering directions are allowed (see Theorem 9.1 for more details). This breaks down further into cases according to fibering properties and cyclotomic divisibility, with each case terminating in either the subgroup reduction (Theorem 2.5) or slab reduction (Theorem 2.6 and Corollary 2.7).

While our final result is restricted to the case when $M = p_i^2 p_j^2 p_k^2$ is odd, many of our methods and intermediate results apply under weaker assumptions. Whenever a significant part of the argument can be run in a more general case with little or no additional effort, we do so, assuming that $M = p_i^{n_i} p_j^{n_j} p_k^{n_k}$ for more general $n_i, n_j, n_k \geq 2$ and $p_i, p_j, p_k \geq 2$. For example, the classification of unfibered grids in Sections 5 and 6 allows all n_ν to be arbitrary and M to be either odd or even. The resolution of the p_i extended corner case in Section 8 works for both odd and even M, with $n_i \geq 2$ arbitrary and with only a few additional lines needed to accommodate the even case. On the other hand, the arguments in Section 9 are limited to the odd $M = p_i^2 p_j^2 p_k^2$ case from the beginning.

Although we expect that the main conclusions should be the same in the even case, many of our technical tools work differently when one of the primes is equal to 2. We would like to draw the reader's attention to the basic fibering argument in Lemma 4.9. This argument does not work when $p_i = 2$, and indeed, in Section 6 we provide examples of unfibered grids in the even case where the fibering conclusions of the lemma fail. In particular, the unfibered structures in Lemma 6.5 do not have a counterpart in the odd case. Additionally, with fewer geometric restrictions coming from (2.19), saturating set arguments can be more difficult to run.

The constraint $n_i = n_j = n_k = 2$ is often needed in arguments based on divisor exclusion. For example, while (2.19) provides geometric restrictions on saturating sets, we often need additional constraints based on availability of divisors, and with $n_i = n_j = n_k = 2$ there are fewer divisors available to begin with. In the fibered case, several of our proofs terminate in an essentially 2-dimensional (therefore easier) problem after we have run out of scales in one direction. In order to allow arbitrary n_i, n_j, n_k throughout the argument, we expect that a systematic way to induct on scales may be necessary.

4. Toolbox

4.1. **Divisors.** The first lemma is Lemma 8.9 of [24], specialized to the 3-prime case.

Lemma 4.1 (**Enhanced divisor exclusion**). *Let $A \oplus B = \mathbb{Z}_M$, with $M = \prod_{\iota \in \{i,j,k\}} p_\iota^{n_\iota}$. Let $m = \prod_{\iota \in \{i,j,k\}} p_\iota^{\alpha_\iota}$ and $m' = \prod_{\iota \in \{i,j,k\}} p_\iota^{\alpha'_\iota}$, with $0 \leq \alpha_\iota, \alpha'_\iota \leq n_\iota$. Assume that at least one of m, m' is different from M, and that for every $\iota \in \{i,j,k\}$ we have*

$$\text{either } \alpha_\iota \neq \alpha'_\iota \text{ or } \alpha_\iota = \alpha'_\iota = n_\iota. \tag{4.1}$$

Then for all $x, y \in \mathbb{Z}_M$ we have

$$\mathbb{A}_m^M[x]\,\mathbb{A}_{m'}^M[x]\,\mathbb{B}_m^M[y]\,\mathbb{B}_{m'}^M[y] = 0.$$

In other words, there are no configurations $(a, a', b, b') \in A \times A \times B \times B$ such that

$$(a - x, M) = (b - y, M) = m, \quad (a' - x, M) = (b' - y, M) = m'. \tag{4.2}$$

Proof. If we did have a configuration as in (4.2), then, under the assumption (4.1) for all ι, we would have

$$(a - a', M) = (b - b', M) = \prod_{\iota \in \{i,j,k\}} p_\iota^{\min(\alpha_\iota, \alpha'_\iota)},$$

with the right side different from M. But that is prohibited by divisor exclusion. □

4.2. **Cyclotomic divisibility.**

Lemma 4.2. *Let $A \in \mathcal{M}(\mathbb{Z}_M)$, and let $m, s|M$ with $s \neq 1$. Suppose that for every $a \in A$, Φ_s divides $A \cap \Lambda(a, m)$. Then $\Phi_s|A$.*

Proof. Write $\mathbb{Z}_M = \bigcup_\nu \Lambda_\nu$, where Λ_ν are pairwise disjoint m-grids. Accordingly, $A(X) = \sum_\nu A_\nu(X)$, where $A_\nu = A \cap \Lambda_\nu$. If A is disjoint from Λ_ν, we have $A_\nu(X) \equiv 0$. If on the other hand $A \cap \Lambda_\nu \neq \emptyset$, then $\Phi_s|A_\nu$. Summing up in ν, we get $\Phi_s|A$. □

The next two lemmas are based on a combinatorial interpretation of divisibility by prime power cyclotomics. For $A \subset \mathbb{Z}_M$ and $1 \leq \alpha \leq n_i$, we have $\Phi_{p_i^\alpha}(X)|A(X)$ if and only if

$$|A \cap \Pi(x, p_i^\alpha)| = \frac{1}{p_i}|A \cap \Pi(x, p_i^{\alpha-1})| \quad \forall x \in \mathbb{Z}_M, \tag{4.3}$$

so that the elements of A are uniformly distributed mod p_i^α within each residue class mod $p_i^{\alpha-1}$. This in particular limits the number of elements that a tiling set may have in a plane on some scale.

Lemma 4.3 (Plane bound). *Let $A \oplus B = \mathbb{Z}_M$, where $M = p_i^{n_i} p_j^{n_j} p_k^{n_k}$ and $|A| = p_i^{\beta_i} p_j^{\beta_j} p_k^{\beta_k}$. Then for every $x \in \mathbb{Z}_M$ and $0 \leq \alpha_i \leq n_i$ we have*

$$|A \cap \Pi(x, p_i^{n_i - \alpha_i})| \leq p_i^{\alpha_i} p_j^{\beta_j} p_k^{\beta_k}.$$

Corollary 4.4. *Let $A \oplus B = \mathbb{Z}_M$, where $M = p_i^{n_i} p_j^{n_j} p_k^{n_k}$ and $|A| = p_i^{\beta_i} p_j^{\beta_j} p_k^{\beta_k}$ with $\beta_i > 0$. Suppose that for some $x \in \mathbb{Z}_M$ and $1 \leq \alpha_0 \leq n_i$*

$$|A \cap \Pi(x, p_i^{n_i - \alpha_0})| > p_i^{\beta_i - 1} p_j^{\beta_j} p_k^{\beta_k}, \tag{4.4}$$

then $\Phi_{p_i^{n_i - \alpha}}|A$ for at least one $\alpha \in \{0, \ldots, \alpha_0 - 1\}$.

Proof. Suppose that $\Phi_{p_i^{n_i-\alpha}} \nmid A$ for all $\alpha \in \{0, \ldots, \alpha_0 - 1\}$. It follows that there must exist a $\gamma \geq \alpha_0$ such that $\Phi_{p_i^{n_i-\gamma}}|A$. The latter implies, by (4.4), that $|A| > \prod_\nu p_\nu^{\beta_\nu}$, which is a contradiction. □

4.3. **Saturating sets.**

Lemma 4.5 (**No missing joints**). *Let $A \oplus B = \mathbb{Z}_M$, where $M = p_i^{n_i} p_j^{n_j} p_k^{n_k}$. Suppose that*

$$\{D(M)|m|M\} \cap \mathrm{Div}(B) = \emptyset, \tag{4.5}$$

and that for some $x \in \mathbb{Z}_M$ there exist $a_i, a_j, a_k \in A$ such that

$$(x - a_i, M) = M/p_i,\ (x - a_j, M) = M/p_j,\ (x - a_k, M) = M/p_k. \tag{4.6}$$

Then $x \in A$.

Proof. Suppose that $x \notin A$, and let Δ be the M-cuboid with vertices x, a_i, a_j, a_k. By (4.6), the saturating set A_x is contained in the vertex set of Δ. But that is impossible by (4.5). □

Lemma 4.6 (**Flat corner**). *Let $A \oplus B = \mathbb{Z}_M, M = p_i^2 p_j^{n_j} p_k^{n_k}, |A| = p_i p_j p_k$. Suppose that (4.5) holds, and that A contains the following 3-point configuration: for some $x \in \mathbb{Z}_M \setminus A$ there exist $a, a_j, a_k \in A$ such that*

$$(a - a_j, M) = (a_k - x, M) = M/p_j,\ (a - a_k, M) = (a_j - x, M) = M/p_k$$

Then $A_x \subset \ell_i(x)$, and the pair (A, B) has a $(1, 2)$-cofibered structure, with an M-cofiber in A at distance M/p_i^2 from x.

Proof. Fix $b \in B$. Then

$$A_{x,b} \subset \ell_i(x) \cup \ell_i(a) \cup \ell_i(a_j) \cup \ell_i(a_k).$$

By (4.5), we have the following.

- $A_{x,b} \cap \ell_i(x) \neq \emptyset$ implies that $\mathbb{A}_{M/p_i^2}[x]\mathbb{B}_{M/p_i^2}[b] > 0$, hence $M/p_i^2 p_j$, $M/p_i^2 p_k$, $M/p_i^2 p_j p_k \in \mathrm{Div}(A)$ and $M/p_i^2 \in \mathrm{Div}(B)$.
- $A_{x,b} \cap \ell_i(a) \neq \emptyset$ implies that $\mathbb{A}_{M/p_i^2 p_j p_k}[x|\ell_i(a)]\mathbb{B}_{M/p_i^2 p_j p_k}[b] > 0$, hence M/p_i^2, $M/p_i^2 p_j$, $M/p_i^2 p_k \in \mathrm{Div}(A)$ and $M/p_i^2 p_j p_k \in \mathrm{Div}(B)$.
- $A_{x,b} \cap \ell_i(a_j) \neq \emptyset$ implies that $\mathbb{A}_{M/p_i^2 p_k}[x|\ell_i(a_j)]\mathbb{B}_{M/p_i^2 p_k}[b] > 0$, hence M/p_i^2, $M/p_i^2 p_j$, $M/p_i^2 p_j p_k \in \mathrm{Div}(A)$ and $M/p_i^2 p_k \in \mathrm{Div}(B)$.
- $A_{x,b} \cap \ell_i(a_k) \neq \emptyset$ implies that $\mathbb{A}_{M/p_i^2 p_j}[x|\ell_i(a_k)]\mathbb{B}_{M/p_i^2 p_j}[b] > 0$, hence M/p_i^2, $M/p_i^2 p_k$, $M/p_i^2 p_j p_k \in \mathrm{Div}(A)$ and $M/p_i^2 p_j \in \mathrm{Div}(B)$.

It follows that $A_{x,b}$ cannot intersect more than one of the above lines. We now show it cannot intersect any line other than $\ell_i(x)$. Indeed, assume that $A_{x,b} \subset \ell_i(a)$, then

$$\begin{aligned}\phi(p_i^2 p_j p_k) &= \mathbb{A}_{M/p_i^2 p_j p_k}[x|\ell_i(a)]\mathbb{B}_{M/p_i^2 p_j p_k}[b] \\ &= \mathbb{A}_{M/p_i^2}[a] \sum_{y:(y-b,M)=M/p_j p_k} \mathbb{B}_{M/p_i^2}[y].\end{aligned}$$

Observe that $\mathbb{A}_{M/p_i^2}[a] \leq \phi(p_i^2)$, and since $M/p_i, M/p_i^2 \notin \mathrm{Div}(B)$, we have $\mathbb{B}_{M/p_i^2}[y] \leq 1$ for all $y \in \mathbb{Z}_M \setminus B$ with $(y - b, M) = M/p_j p_k$. It follows that both must hold with equality. Now, if $p_i > p_j$ then $\mathbb{A}_{M/p_i^2}[a] = \phi(p_i^2)$ implies that

$$\begin{aligned}|A \cap \Pi(a, p_k^{n_k})| &\geq \mathbb{A}[a] + \mathbb{A}_{M/p_i^2}[a] \\ &= 1 + \phi(p_i^2) > p_i p_j,\end{aligned}$$

which contradicts Lemma 4.3. If on the other hand $p_i > p_j$, then $\mathbb{B}_{M/p_i^2}[y] = 1$ for all y as above implies $\{D(M)|m|M\} \cap \mathrm{Div}(B) \neq \emptyset$, which contradicts (4.5). Hence $A_{x,b} \cap \ell_i(a) = \emptyset$.

Next, assume that $A_{x,b} \subset \ell_i(a_k)$, then

$$\begin{aligned}\phi(p_j p_i^2) &= \mathbb{A}_{M/p_i^2 p_j}[x|\ell_i(a_k)]\mathbb{B}_{M/p_i^2 p_j}[b] \\ &= \mathbb{A}_{M/p_i^2}[a_k] \sum_{y:(y-b,M)=M/p_j} \mathbb{B}_{M/p_i^2}[y]\end{aligned}$$

By the same argument as above, we deduce that $\mathbb{A}_{M/p_i^2}[a_k] = \phi(p_i^2)$ and $\mathbb{B}_{M/p_i^2}[y] = 1$ for all $y \in \mathbb{Z}_M \setminus B$ with $(y-b, M) = M/p_j$. When $p_i > p_j$, we therefore get $|A \cap \Pi(a_k, p_k^{n_k})| > p_i p_j$, which is a contradiction. When $p_i < p_j$, we must have $\{M/p_j, M/p_i p_j\} \cap \mathrm{Div}(B) \neq \emptyset$, contradicting (4.5). Thus $A_{x,b} \cap \ell_i(a_k) = \emptyset$. By symmetry, $A_{x,b} \cap \ell_i(a_j) = \emptyset$. □

4.4. **Fibering lemmas.** Lemma 4.7 below is a simple version of the de Bruijn-Rédei-Schoenberg theorem for cyclic groups $\mathbb{Z}_N$, where N has at most two distinct prime factors. This was essentially proved in [2]; see also [28, Theorem 3.3].

Lemma 4.7. (Cyclotomic divisibility for 2 prime factors) *Let $A \in \mathcal{M}(\mathbb{Z}_N)$ for some $N \mid M$ such that $p_i \nmid N$. Then:*

(i) $\Phi_N | A$ if and only if A is a linear combination of N-fibers in the p_j and p_k direction with non-negative integer coefficients.

(ii) Let Λ be a $D(N)$-grid. Assume that $\Phi_N | A$, and that there exists $c_0 \in \mathbb{N}$ such that $\mathbb{A}_N^N[x] \in \{0, c_0\}$ for all $x \in \Lambda$. Then $A \cap \Lambda$ is N-fibered in either the p_j or the p_k direction.

Lemma 4.8 is a localized version of the above.

Lemma 4.8. (Flat cuboids) *Let $A \subset \mathbb{Z}_M$. Assume that $\Phi_M | A$, and that there is a plane $\Pi := \Pi(z, p_i^{n_i})$ such that $A \cap \Pi$ is a disjoint union of M-fibers in the p_j and p_k directions. Then for every parallel plane $\Pi' := \Pi(z', p_i^{n_i})$, where $(z - z', M) = M/p_i$, the set $A \cap \Pi'$ is a disjoint union of M-fibers in the p_j and p_k directions.*

Proof. Consider a 2-dimensional cuboid Δ' with vertices $x', x'+d_j, x'+d_k, x'+d_j+d_k$, where $x' \in \Pi$, $(d_j, M) = M/p_j$, $(d_k, M) = M/p_k$. Let $x \in \Pi$ be the point such that $(x - x', M) = M/p_i$, and let Δ be the 2-dimensional cuboid with vertices $x, x+d_j, x+d_k, x+d_j+d_k$. By the fibering property of $A \cap \Pi$, we have $\mathbb{A}_M[\Delta] = 0$. Since $\Phi_M | A$, we also have $\mathbb{A}_M[\Delta - \Delta'] = 0$, hence $\mathbb{A}_M[\Delta'] = 0$. If we consider $A \cap \Pi'$ (after translation) as a subset of $\mathbb{Z}_{M/p_i^{n_i}}$, it follows that $\Phi_{M/p_i^{n_i}} \mid (A \cap \Pi')(X)$. By Lemma 4.7, $A \cap \Pi'$ is a union of fibers as claimed. □

Lemma 4.9. (Missing top difference implies fibering) *Let $N | M$ with $p_i p_j p_k | N$. Let $\Lambda := \Lambda(x_0, D(N))$ for some $x_0 \in \mathbb{Z}_N$. Assume that $A \subset \mathbb{Z}_N$ satisfies $\Phi_N | A$ and $\Lambda \cap A \neq \emptyset$.*

(i) Suppose that $p_i \neq 2$, and that

$$N/p_i \notin \mathrm{Div}_N(A \cap \Lambda). \tag{4.7}$$

Then $A \cap \Lambda$ is N-fibered in one of the p_j and p_k directions. In particular, if $N/p_i \notin \mathrm{Div}_N(A)$, then $A \cap \Lambda$ is N-fibered in one of the p_j and p_k directions for every $D(N)$-grid Λ.

(ii) Suppose that $N/p_i, N/p_j \notin \mathrm{Div}_N(A \cap \Lambda)$. Then $A \cap \Lambda$ is N-fibered in the p_k direction.

Proof. (i) We first prove that each $a \in A \cap \Lambda$ belongs to an N-fiber in the p_ν direction for at least one $\nu \in \{j, k\}$. Suppose, for contradiction, that there exists an $a \in A \cap \Lambda$ that does not have this property. Then there are $x_i, x_j \in \mathbb{Z}_N$ such that

$$\mathbb{A}^N[x_j] = \mathbb{A}^N[x_k] = 0 \text{ and } (x_j - a, N) = N/p_j,\ (x_k - a, N) = N/p_k.$$

Since $p_i > 2$ and $\mathbb{A}^N_{N/p_i}[a] = 0$, there exist at least two distinct elements $x_i, x_i' \in \mathbb{Z}_N$ satisfying

$$\mathbb{A}^N[x_i] = \mathbb{A}^N[x_i'] = 0,\ (a - x_i, N) = (a - x_i', N) = N/p_i.$$

Consider two N-cuboids, each with vertices at a, x_i, x_j, and with another vertex at x_k and x_k' respectively. By the cyclotomic divisibility assumption, each of those cuboids must be balanced. This can only happen if there are two elements $a_{ijk}, a_{ijk}' \in \mathbb{Z}_N$ at the opposite vertex of each cuboid from a (that is, with $(a - a_{ijk}, N) = (a - a_{ijk}', N) = D(N)$) satisfying $\mathbb{A}^N[a_{ijk}] = \mathbb{A}^N[a_{ijk}'] = c_0$. However, this leads to a contradiction, since $(a_{ijk} - a_{ijk}', N) = N/p_i$. We therefore conclude that each $a \in A \cap \Lambda$ belongs to an N-fiber in at least one direction as indicated.

Next, suppose that $a_j, a_k \in A \cap \Lambda$ belong to N-fibers in, respectively, the p_j and p_k direction. If $a_j * F_j$ and $a_k * F_k$ do not intersect, then $N/p_i \in \mathrm{Div}_N(a_j * F_j, a_k * F_k)$, contradicting the assumption (4.7). We may therefore assume that $a_j = a_k = a$ and that $a * F_j, a * F_k \subset A$.

Consider any N-cuboid with one vertex at a. Then the vertices at distance N/p_j and N/p_k from a belong to A, and, by (4.7), the vertices at distance N/p_ip_j and N/p_ip_k from a cannot be in A. The only way to balance the cuboid is for the vertex at distance N/p_jp_k from a to be in A. Allowing such cuboids to vary, we see that $A \cap \Pi \cap \Lambda$ must be N-fibered in both of the p_j and p_k directions. This proves part (i) of the lemma.

(ii) Assume that $N/p_i, N/p_j \notin \mathrm{Div}_N(A \cap \Lambda)$. At least one of p_i, p_j must be odd; without loss of generality, we may assume that $p_i \neq 2$. By part (i) of the lemma, $A \cap \Lambda$ must be N-fibered in one of the p_j and p_k directions on Λ. However, it cannot be N-fibered in the p_j direction, since $N/p_j \notin \mathrm{Div}_N(A \cap \Lambda)$. Part (ii) follows. □

5. Structure on unfibered grids

5.1. **Diagonal boxes.** Throughout this section, we will use the following notation. Let $M = p_i^{n_i} p_j^{n_j} p_k^{n_k}$, and let Λ be a fixed $D(M)$ grid such that $A \cap \Lambda \neq \emptyset$. We identify Λ with $\mathbb{Z}_{p_i} \oplus \mathbb{Z}_{p_j} \oplus \mathbb{Z}_{p_k}$, and represent each point $x \in \Lambda$ as $(\lambda_i x, \lambda_j x, \lambda_k x)$ in the implied coordinate system.

Definition 5.1. *Let $A \subset \mathbb{Z}_M$. We say that $A \cap \Lambda$* contains diagonal boxes *if there are nonempty sets $I \subset \mathbb{Z}_{p_i}$, $J \subset \mathbb{Z}_{p_j}$, $K \subset \mathbb{Z}_{p_k}$, such that*

$$I^c := \mathbb{Z}_{p_i} \setminus I,\ J^c := \mathbb{Z}_{p_j} \setminus J,\ K^c := \mathbb{Z}_{p_k} \setminus K$$

are also nonempty, and

$$(I \times J \times K) \cup (I^c \times J^c \times K^c) \subset A \cap \Lambda. \tag{5.1}$$

Proposition 5.2. *Let $A \oplus B = \mathbb{Z}_M$ be a tiling, $M = p_i^{n_i} p_j^{n_j} p_k^{n_k}$, and assume that $\Phi_M \mid A$. Suppose that there is a $D(M)$-grid Λ such that $A \cap \Lambda$ is not a union of disjoint M-fibers (possibly in different directions). Then $A \cap \Lambda$ contains diagonal boxes.*

Proof. Write $D = D(M)$ for short. We first construct a set $A_0 \subset A \cap \Lambda$ as follows. If $A \cap \Lambda$ contains no M-fibers, let $A_0 := A \cap \Lambda$. If $A \cap \Lambda$ does contain an M-fiber F, consider the set $(A \cap \Lambda) \setminus F$ (if there is more than one such fiber, just choose one arbitrarily and remove it). If this set contains no M-fibers, we let A_0 be that set; otherwise continue by induction. The procedure terminates when no more M-fibers can be found. The remaining set A_0 is

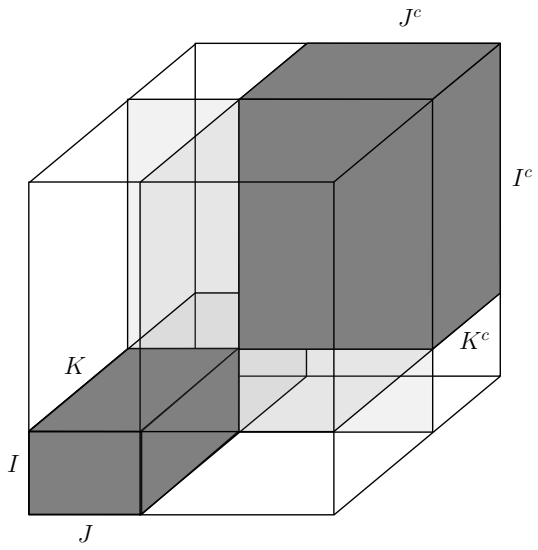

FIGURE 2. A pair of diagonal boxes.

nonempty by our assumption on A, contains no M-fibers, and $\Phi_M|A_0$; however, A_0 need not be a tiling complement.

We remark that A_0 is not necessarily uniquely determined by A, as the fiber removal procedure may lead to different outcomes depending on the order in which fibers are removed. In that event, we fix one such set A_0 and keep it fixed throughout the proof.

For future reference, we record a lemma.

Lemma 5.3. *Suppose that $x \in A \setminus A_0$. Then there exists a $\nu \in \{i,j,k\}$ such that $x * F_\nu \subset (A \setminus A_0)$.*

Proof. This follows directly from the construction, since any point $x \in A \setminus A_0$ would have been removed from A together with an M-fiber (in some direction) containing x. □

For each $x \in \Lambda$, we define

$$\begin{aligned} I(x) &= \{l \in \mathbb{Z}_{p_i} : \ (l, \lambda_j x, \lambda_k x) \in A_0\}, & I^c(x) &= \mathbb{Z}_{p_i} \setminus I(x), \\ J(x) &= \{l \in \mathbb{Z}_{p_j} : \ (\lambda_i x, l, \lambda_k x) \in A_0\}, & J^c(x) &= \mathbb{Z}_{p_j} \setminus J(x), \\ K(x) &= \{l \in \mathbb{Z}_{p_k} : \ (\lambda_i x, \lambda_j x, l) \in A_0\}, & K^c(x) &= \mathbb{Z}_{p_k} \setminus K(x). \end{aligned}$$

Let $a \in A_0$ be an element such that $|K(a)|$ is maximal, in the sense that

$$|K(a')| \le |K(a)| \ \ \forall a' \in A_0. \tag{5.2}$$

By translational invariance, we may assume that $a = (0,0,0)$. Observe that $I(a), J(a), K(a)$ are all nonempty since $a \in A_0$, and $I^c(a), J^c(a), K^c(a)$ are all nonempty since a does not belong to an M-fiber in A_0 in any direction.

Claim 1. *For all $a' = (0,0,l) \in A_0$, with $l \in K(a)$, we have $I(a') \times J(a') \times \{l\} \subset A_0$.*

Proof. Since $K(a') = K(a)$, it suffices to prove this with $a' = a$. Let $a_i = (l_i, 0, 0)$ and $a_j = (0, l_j, 0)$ for some $l_i \in I(a)$, $l_j \in J(a)$, and let $z = (l_i, l_j, 0) \in I(a) \times J(a) \times \{0\}$. We need to prove that $z \in A_0$. Suppose, for contradiction, that $z \notin A_0$, and consider the cuboid with vertices at a, a_i, a_j, and $(0,0,l_k)$. For $l_k \in K(a)$, the cuboid can only be balanced if both $(l_i, 0, l_k)$ and $(0, l_j, l_k)$ are elements of A_0. Furthermore, let $l_k \in K^c(a)$, then for the cuboid

to be balanced we still need at least one of $(l_i, 0, l_k)$ and $(0, l_j, l_k)$ to be in A_0. This implies that $\max(|K(a_i)|, |K(a_j)|) > |K(a)|$, contradicting the maximality assumption (5.2). Hence Claim 1 follows. □

Claim 2. *For all $a' = (0, 0, l) \in A_0$ with $l \in K(a)$, we have $I^c(a') \times J^c(a') \times K^c(a) \subset A_0$.*

Proof. Let $x \in I^c(a') \times J^c(a') \times K^c(a)$. Considering the M-cuboid with opposite vertices at a' and x, we see that the three vertices at distance $M/p_i, M/p_j, M/p_k$ from a' are not in A_0. Hence, in order for the cuboid to be balanced, we must have $x \in A_0$ as claimed. □

Claim 3. *Suppose that for all $l', l'' \in K(a)$, we have either $(I(a') \times J(a')) \subseteq (I(a'') \times J(a''))$ or $(I(a'') \times J(a'')) \subseteq (I(a') \times J(a'))$, where $a' = (0, 0, l')$, $a'' = (0, 0, l'')$. Then A_0 contains diagonal boxes.*

Proof. Under the assumptions of the claim, the sets $I(a') \times J(a')$ with $a' \in A_0$ and $M/p_k \mid a - a'$ have a minimal element. Without loss of generality, we may assume that

$$(I(a) \times J(a)) \subseteq (I(a') \times J(a')) \ \ \forall a' \in A_0 \text{ such that } M/p_k \mid a - a'.$$

Then Claims 1 and 2 imply that A_0 (and therefore A) contains the diagonal boxes

$$(I(a) \times J(a) \times K(a)) \cup (I^c(a) \times J^c(a) \times K^c(a)),$$

which proves the claim. □

It remains to consider the case when there exist $l', l'' \in K(a)$ such that for $a' = (0, 0, l')$, $a'' = (0, 0, l'')$, we have

$$I(a') \not\subset I(a'') \text{ and } J(a'') \not\subset J(a'). \tag{5.3}$$

For this to be possible, $K(a)$ must have at least two distinct elements, and each of $I(a')$ and $J(a'')$ must have at least one element different from 0. Since none of a, a', a'' belongs to an M-fiber in A_0, we must have

$$p_i, p_j, p_k > 2. \tag{5.4}$$

Furthermore, this configuration implies that

$$\{m : \ D|m|M\} \subset \mathrm{Div}(A_0) \subset \mathrm{Div}(A \cap \Lambda). \tag{5.5}$$

Indeed, we have

$$\{m : \ D|m|M\} \setminus \{M/p_i p_j\} \subset \mathrm{Div}((I(a') \times J(a') \times \{l'\}) \cup (I(a'') \times J(a'') \times \{l''\})),$$
$$M/p_i p_j \in \mathrm{Div}(I^c(a') \times J^c(a') \times \{l_k\}, I^c(a'') \times J^c(a'') \times \{l_k\}) \text{ for any } l_k \in K^c(a).$$

Claim 4. *Suppose that (5.3) holds for some $l', l'' \in K(a)$, with $a' = (0, 0, l')$, $a'' = (0, 0, l'')$. Then*

$$I^c(a'') \times J^c(a') \times K^c(a) \subset A_0.$$

Proof. We may assume that $a'' = a$ and $l'' = 0$. We will also write $l_k = l'$, so that $a' = (0, 0, l_k)$ Let $l_i \in I(a') \setminus I(a), l_j \in J(a) \setminus J(a'), l_i^c \in I^c(a')$ and $l_j^c \in J^c(a)$. We also fix $l_k^c \in K^c(a)$. For the purpose of this proof, we will need to consider points with coordinates $(\beta_i, \beta_j, \beta_k)$ such that

$$\beta_\nu \in \{0, l_\nu, l_\nu^c\} \text{ for } \nu \in \{i, j, k\}.$$

Let $x_i := (l_i, 0, 0)$, $x_j := (0, l_j, l_k)$, and $a_i := (l_i, 0, l_k), a_j := (0, l_j, 0)$. Then $x_i, x_j \notin A_0$, and $a_i, a_j \in A_0$. By Claim 2, we have

$$\begin{aligned} z_i &:= (l_i^c, l_j, l_k^c) \in A_0 \text{ since } l_j \in J^c(a'), \\ z_j &:= (l_i, l_j^c, l_k^c) \in A_0 \text{ since } l_i \in I^c(a). \end{aligned} \tag{5.6}$$

Let $z = (l_i, l_j, l_k^c) \in I^c(a) \times J^c(a') \times K^c(a)$ be the point such that

$$(z - z_i, M) = M/p_i, \ (z - z_j, M) = M/p_j. \tag{5.7}$$

We need to prove that $z \in A_0$.

We begin with the following reduction.

Claim 4'. *Let $x_{ij} = (l_i, l_j, 0)$ and $y_{ij} = (l_i, l_j, l_k)$. Suppose that at least one of the following holds: either*

$$\{x_{ij}, y_{ij}\} \cap A \neq \emptyset, \tag{5.8}$$

or else there exists a set $A' \subset \mathbb{Z}_M$, identical to A except possibly along the line $\ell_j(x_i)$, such that $A' \oplus B = \mathbb{Z}_M$ and

$$x_i * F_j \subset A', \text{ hence } x_{ij} \in A'. \tag{5.9}$$

Then $z \in A_0$.

Proof. For notational consistency, if (5.8) holds, we let $A' = A$. We have

$$(x_{ij} - z, M) = (y_{ij} - z, M) = M/p_k. \tag{5.10}$$

Suppose that at least one of x_{ij}, y_{ij} belongs to A'. By (5.7) and (5.5), the assumptions of Lemma 4.5 hold for z and A'. Hence $z \in A'$. Since A' and A may differ only along $\ell_j(x_i)$, we must in fact have $z \in A$.

Next, we claim that $z \in A_0$. Indeed, assume for contradiction that $z \in A \setminus A_0$. By Lemma 5.3, there is a $\nu \in \{i, j, k\}$ such that $z * F_\nu \subset (A \setminus A_0)$. But by (5.6) and (5.7), $z_i \in (z * F_i) \cap A_0$ and $z_j \in (z * F_j) \cap A_0$. It follows that $\nu = k$, and $z * F_k \subset (A \setminus A_0)$. By (5.10), we have $x_{ij}, y_{ij} \in z * F_k$, and in particular $x_{ij}, y_{ij} \in A$.

Consider now the point x_i. We have

$$(x_i - a, M) = M/p_i, \ \ (x_i - a_i, M) = M/p_k, \ \ (x_i - x_{ij}, M) = M/p_j, \tag{5.11}$$

with $a, a' \in A_0, x_{ij} \in A$. Taking also (5.5) into account, we see that x_i satisfies the assumptions of Lemma 4.5 applied to A. Therefore $x_i \in A$.

Since $x_i \notin A_0$, we must have $x_i \in A \setminus A_0$. By Lemma 5.3 and (5.11), it follows that $x_i * F_j \subset A \setminus A_0$. However, consider the point $x_{ij} \in x_i * F_j$. We have already seen that z must have been removed from A together with the fiber $z * F_k$, which also contains x_{ij}. This is a contradiction, since we are not allowed to remove the same point twice. □

To prove Claim 4, it remains to prove that at least one of the conditions (5.8), (5.9) in Claim 4' must hold. Assume first that $x_i \in A \setminus A_0$. By Lemma 5.3, $A \setminus A_0$ must contain an M-fiber $x_i * F_\nu$ for some $\nu \in \{i, j, k\}$. Since $a \in (x_i * F_i) \cap A$ and $a_i \in (x_i * F_k) \cap A$, we must have $\nu = j$. This clearly implies $x_{ij} \in A$. The same argument applies with i and j interchanged.

We are left with the case when $x_i, x_j, x_{ij}, y_{ij} \notin A$. Assume also for contradiction that $z \notin A_0$. Consider the M-cuboid with vertices at x_{ij}, z, z_i and z_j, and note that $z_i, z_j \in A_0$. In order to balance this cuboid in A_0, at least one of the points $y_i := (l_i^c, l_j, 0)$ and $y_j := (l_i, l_j^c, 0)$ must be in A_0. However, if $y_j \in A_0$, then we can apply Lemma 4.5 again, this time to A and

x_i, using (5.5), (5.11), and $(x_i - y_j, M) = M/p_j$. Hence $x_i \in A$. We assumed that $x_i \notin A_0$, hence by Lemma 5.3 there must be a $\nu \in \{i,j,k\}$ such that $x_i * F_\nu \subset A \setminus A_0$. However, the points $a, a_i, y_j \in A_0$ block all three directions, contradicting the assumption that $y_j \in A_0$.

We therefore have $y_i \in A_0$. Now, considering the saturating set A_{x_i}, we have

$$A_{x_i} \subset \bigcap_{\hat{a}\in\{a,a_i,z_j,y_i\}} \mathrm{Bispan}(x_i, \hat{a}) = \ell_j(x_i) \cup \{a, a_j\}.$$

By (5.5), we must in fact have $A_{x_i} \subset \ell_j(x_i)$. It follows that the pair (A,B) has a $(\mathcal{P}_A, \mathcal{P}_B)$-cofibered structure of depth γ in the p_j direction for some $\gamma \geq 2$, with $1 \in \mathcal{P}_A$ and an M-fiber in the p_j direction in A at distance M/p_j^γ from x_i. Applying Lemma 2.9 to shift that fiber to x_i, we obtain a set A' with the desired properties, thus concluding the proof of Claim 4. □

We can now complete the proof of the proposition. We claim that

$$(I \times J \times K) \cup (I^c \times J^c \times K^c) \subset A_0, \tag{5.12}$$

where

$$I = \bigcap_{l\in K(a)} I(a_l), \quad J = \bigcap_{l\in K(a)} J(a_l), \quad K = K(a),$$
$$I^c = \bigcup_{l\in K(a)} I^c(a_l), \quad J^c = \bigcup_{l\in K(a)} J^c(a_l), \quad K^c = K^c(a),$$

and $a_l = (0,0,l)$. It is clear that $I \times J \times K \subset A_0$. Suppose now that $(l_i, l_j, l_k) \in I^c \times J^c \times K^c$. Then there are $l, l' \in K(a)$ such that $l_i \in I^c(a_l)$ and $l_j \in J^c(a_{l'})$. If $l = l'$, or if $l_i \in I^c(a_{l'})$, then $(l_i, l_j, l_k) \in A_0$ by Claim 2 applied to $a_{l'}$. The case $l_j \in J^c(a_l)$ is similar. Assume therefore that $l_i \in I(a_{l'}) \setminus I(a_l)$ and $l_j \in J(a_l) \setminus J(a_{l'})$. But then (5.3) holds with a', a'' replaced by $a_l, a_{l'}$. Applying Claim 4, we see that $(l_i, l_j, l_k) \in A_0$ in that case as well. This ends the proof of the proposition. □

5.2. **Extended corners.** We continue to write $M = p_i^{n_i} p_j^{n_j} p_k^{n_k}$. Assume that $A \oplus B = \mathbb{Z}_M$ is a tiling, $\Phi_M \mid A$, and there exists a $D(M)$-grid Λ such that $A \cap \Lambda \neq \emptyset$ and $A \cap \Lambda$ is not M-fibered in any direction. By Proposition 5.2, if $A \cap \Lambda$ is not a union of disjoint M-fibers, then $A \cap \Lambda$ contains diagonal boxes. It remains to consider the case when $A \cap \Lambda$ is a union of disjoint M-fibers. In that case, we claim that $A \cap \Lambda$ contains the following structure.

Definition 5.4. *Suppose that $A \subset \mathbb{Z}_M$, and let Λ be a $D(M)$ grid.*

(i) We say that $A \cap \Lambda$ contains a p_i corner *if there exist $a, a_i \in A \cap \Lambda$ with $(a - a_i, M) = M/p_i$ satisfying*

$$A \cap (a * F_j * F_k) = a * F_j, \quad A \cap (a_i * F_j * F_k) = a_i * F_k. \tag{5.13}$$

(ii) We say that $A \cap \Lambda$ contains a p_i extended corner *if there exist $a, a_i \in A \cap \Lambda$ such that $(a - a_i, M) = M/p_i$ and*

- *$A \cap (a * F_j * F_k)$ is M-fibered in the p_j direction but not in the p_k direction,*
- *$A \cap (a_i * F_j * F_k)$ is M-fibered in the p_k direction but not in the p_j direction.*

We now prove that unfibered grids as described above must contain extended corners.

Proposition 5.5. *Let $D = D(M)$, and let Λ be a D-grid. Assume that $A \cap \Lambda$ is a union of disjoint M-fibers, but is not fibered in any direction. Then $A \cap \Lambda$ contains a p_ν extended corner for some $\nu \in \{i,j,k\}$.*

Proof. Fix A and Λ as in the statement of the proposition. We will say that $\kappa : A \cap \Lambda \to \{i, j, k\}$ is an *assignment function* if $A \cap \Lambda$ can be written as

$$A \cap \Lambda = \bigcup_{a \in A \cap \Lambda} (a * F_{\kappa(a)}),$$

where for any $a, a' \in A \cap \Lambda$, the fibers $a * F_{\kappa(a)}$ and $a' * F_{\kappa(a')}$ are either identical or disjoint. Thus, if $a' \in a * F_{\kappa(a)}$, then $\kappa(a') = \kappa(a)$.

We recall [24, Proposition 7.10]:

Proposition 5.6. *Let $M = p_i^{n_i} p_j^{n_j} p_k^{n_k}$, and let $D = D(M)$. Assume that $A \oplus B = \mathbb{Z}_M$ is a tiling and that there exists a D-grid Λ such that $A \cap \Lambda$ is a nonempty union of disjoint M-fibers. Then there is a subset $\{\nu_1, \nu_2\} \subset \{i, j, k\}$ of cardinality 2 such that $A \cap \Lambda$ is a union of disjoint M-fibers in the p_{ν_1} and p_{ν_2} directions.*

Hence there exists an assignment function κ that takes at most two distinct values. (In fact, the proof of [24, Proposition 7.10] shows that this is true for any assignment function.) Without loss of generality, we may assume that $\kappa(a) \in \{j, k\}$ for all $a \in A \cap \Lambda$. We claim that this implies that $A \cap \Lambda$ contains a p_i extended corner.

Split Λ into 2-dimensional grids $\Pi_\iota := x_\iota * F_j * F_k$, $\iota = 0, 1, \ldots, p_i - 1$. Then for each ι, the set $A \cap \Pi_\iota$ is a union of disjoint fibers in at least one of the p_j or p_k direction. Moreover, we are assuming that $A \cap \Lambda$ is not fibered in either the p_j or p_k direction. Therefore for each $\nu \in \{j, k\}$, there must be at least one $\iota(\nu)$ such that $A \cap \Pi_{\iota(\nu)}$ is fibered only in the p_ν direction. Choosing $a \in A \cap \Pi_{\iota(j)}$ and $a_i \in A \cap \Pi_{\iota(k)}$ with $(a - a_i) = M/p_i$, we see that the condition (ii) of Definition 5.4 is satisfied. □

A p_i corner is a special case of a p_i extended corner, with only one fiber in each of the planes through a and a_i in Λ. This is one of the special structures that occur when $\Phi_M \mid A$, $A \cap \Lambda$ is not M-fibered in any direction, and $\{D(M)|m|M\} \not\subset \mathrm{Div}(A)$ (see Section 6).

In addition to the present purpose of classification of unfibered grids in on scale M, we will also refer to Definition 5.4 (ii) in the fibered case, in the proofs of Proposition 9.14 (Claim 1) and Lemma 9.30.

6. Unfibered grids with missing top differences

Let $A \subset \mathbb{Z}_M$, and let Λ be a $D(M)$-grid such that $A \cap \Lambda \neq \emptyset$. The purpose of this section is to classify all possible unfibered grids $A \cap \Lambda$ under the assumption that $\Phi_M | A$ and that $\mathrm{Div}(A)$ does not contain all m such that $D(M)|m|M$. We do *not* assume in this section that A is a tiling complement.

6.1. **A structure result.**

Proposition 6.1. *Let $M = p_i^{n_i} p_j^{n_j} p_k^{n_k}$. Assume that $A \subset \mathbb{Z}_M$ satisfies $\Phi_M | A$. Let Λ be a $D(M)$-grid such that $A \cap \Lambda \neq \emptyset$. Suppose that $A \cap \Lambda$ is not M-fibered in any direction, and that (5.5) fails, i.e.*

$$\{m : \ D(M)|m|M\} \not\subset \mathrm{Div}(A \cap \Lambda). \tag{6.1}$$

Then $A \cap \Lambda$ is a union of at most one set of diagonal boxes

$$A_1 = (I_1 \times J_1 \times K_1) \cup (I_1^c \times J_1^c \times K_1^c), \tag{6.2}$$

where $I_1 \subset \mathbb{Z}_{p_i}$, $J_1 \subset \mathbb{Z}_{p_j}$, $K_1 \subset \mathbb{Z}_{p_k}$ are non-empty sets such that $I_1^c := \mathbb{Z}_{p_i} \setminus I_1$, $J_1^c := \mathbb{Z}_{p_j} \setminus J_1$, $K_1^c := \mathbb{Z}_{p_k} \setminus K_1$ are also non-empty, and possibly additional M-fibers in one or more

directions, disjoint from A_1 and from each other. Furthermore, if $A \cap \Lambda$ does contain a set of diagonal boxes (6.2), then at least one of the sets I_1, J_1, K_1 has cardinality 1, and at least one of the sets I_1^c, J_1^c, K_1^c has cardinality 1.

Remark 6.1. *For simplicity, we only state and prove Proposition 6.1 for sets $A \subset \mathbb{Z}_M$. However, if we assume instead that $A \in \mathcal{M}(\mathbb{Z}_N)$ for some $N|M$, and that (6.12) holds (i.e., $A \cap \Lambda$ is a multiset of constant multiplicity c_0), the same argument applies except that the diagonal boxes and fibers in the conclusion also have multiplicity c_0.*

Proof. We begin as in the proof of Proposition 5.2. Let $A_0 \subset A \cap \Lambda$ be a set constructed by removing M-fibers from $A \cap \Lambda$ until none are left, so that $A \cap \Lambda$ is the union of A_0 and some number of M-fibers in one or more directions, disjoint from A_0 and from each other. If $A_0 = \emptyset$, we are done. Otherwise, we proceed with Claims 1, 2, and 3 from the proof of Proposition 5.2, noting that this part does not require the use of saturating sets (hence A need not be a tile). At that point, the only remaining case in the proof of Proposition 5.2 is when there exist $l', l'' \in K(a)$ such that (5.3) holds. However, in that case we have (5.5), which contradicts (6.1). Therefore, under the assumptions of Proposition 6.1, A_0 contains diagonal boxes A_1 as in (6.2). Moreover, the cardinality statement must hold, since otherwise we would not have (6.1).

As in Proposition 5.2, A_0 need not be unique and may depend on the order in which the fibers are removed, and A_1 may then depend on the choice of A_0. We fix one such choice of A_0 and A_1, and keep it fixed for the remainder of the proof.

We claim that $A_0 = A_1$. To prove this, assume for contradiction that $A_0 \setminus A_1$ is nonempty. We clearly have $\Phi_M | A_0$ and $\Phi_M | A_1$, therefore $\Phi_M | A_0(X) - A_1(X)$. Since the set $A_0 \setminus A_1$ is nonempty and contains no fibers, it must contain another set of diagonal boxes

$$A_2 = (I_2 \times J_2 \times K_2) \cup (I_2^c \times J_2^c \times K_2^c), \tag{6.3}$$

with obvious notation. Furthermore, since $A_2 \subset A_0 \setminus A_1$, we must have

$$A_1 \cap A_2 = \emptyset. \tag{6.4}$$

We first claim that at least one of

$$I_1 = I_2,\ I_1 = I_2^c,\ J_1 = J_2,\ J_1 = J_2^c,\ K_1 = K_2,\ K_1 = K_2^c,$$

must hold. Indeed, by (6.4), we must have

$$(I_1 \times J_1 \times K_1) \cap (I_2 \times J_2 \times K_2) = \emptyset,$$

so that at least one of $I_1 \cap I_2$, $J_1 \cap J_2$, $K_1 \cap K_2$ is empty. Without loss of generality, we may assume that $K_1 \cap K_2 = \emptyset$, so that

$$K_1 \subset K_2^c, \quad K_2 \subset K_1^c. \tag{6.5}$$

We also have

$$(I_1 \times J_1 \times K_1) \cap (I_2^c \times J_2^c \times K_2^c) = \emptyset,$$

and since $K_1 \cap K_2^c \neq \emptyset$, one of $I_1 \cap I_2^c$ and $J_1 \cap J_2^c$ is empty. Without loss of generality, we may assume that $J_1 \cap J_2^c = \emptyset$, so that

$$J_1 \subset J_2, \quad J_2^c \subset J_1^c. \tag{6.6}$$

Next,

$$(I_1^c \times J_1^c \times K_1^c) \cap (I_2^c \times J_2^c \times K_2^c) = \emptyset.$$

By (6.6), we have $J_1^c \cap J_2^c \neq \emptyset$, therefore one of $I_1^c \cap I_2^c$ and $K_1^c \cap K_2^c$ is empty. If $K_1^c \cap K_2^c = \emptyset$, then $K_2^c \subset K_1$, which together with (6.5) shows that $K_1 = K_2^c$ and proves the claim. If $I_1^c \cap I_2^c = \emptyset$, we get that

$$I_1^c \subset I_2, \quad I_2^c \subset I_1. \tag{6.7}$$

Using that

$$(I_1^c \times J_1^c \times K_1^c) \cap (I_2 \times J_2 \times K_2) = \emptyset.$$

and taking (6.5) and (6.7) into account, we see that $J_1^c \cap J_2 = \emptyset$. But then $J_2 \subset J_1$, which together with (6.6) proves the claim.

We may assume without loss of generality that

$$K_1 = K_2^c. \tag{6.8}$$

This implies that

$$(I_1 \times J_1) \cap (I_2 \times J_2) = \emptyset \text{ and } (I_1^c \times J_1^c) \cap (I_2^c \times J_2^c) = \emptyset, \tag{6.9}$$

since otherwise we would have an M-fiber in the p_k direction in A_0. But also, considering the box layers with third coordinate $l \in K_1$ and $l' \in K_2$, we have

$$(I_1 \times J_1) \cap (I_2^c \times J_2^c) = \emptyset \text{ and } (I_1^c \times J_1^c) \cap (I_2 \times J_2) = \emptyset, \tag{6.10}$$

The first parts of (6.9) and (6.10) imply that either

$$I_1 \cap I_2 = J_1 \cap J_2^c = \emptyset, \tag{6.11}$$

or else the same holds with I, J interchanged. Assume that (6.11) holds. Then $I_2 \subset I_1^c$ and $J_2^c \subset J_1^c$, so that in order for the second parts of (6.9) and (6.10) to hold, we must have

$$I_1^c \cap I_2^c = J_1^c \cap J_2 = \emptyset.$$

But this implies that $I_1^c \subset I_2$ and $J_2^c \subset J_1^c$. Hence $I_1 = I_2^c$ and $J_1 = J_2$. But then A_0 contains M-fibers in the p_i direction, a contradiction. □

6.2. **Special unfibered structures: odd M.**

Lemma 6.2. *Let $M = p_i^{n_i} p_j^{n_j} p_k^{n_k}$ and $N = M/p_i^{\alpha_i} p_j^{\alpha_j} p_k^{\alpha_k}$ with $\alpha_\iota < n_\iota$ for all $\iota \in \{i, j, k\}$. Assume that $2 \nmid M$, and that $A \in \mathcal{M}(\mathbb{Z}_N)$ satisfies $\Phi_N|A$. Let Λ be a $D(N)$-grid such that $A \cap \Lambda \neq \emptyset$. Assume further that*

- *there exists a $c_0 \in \mathbb{N}$ such that*

$$\mathbb{A}_N^N[x] \in \{0, c_0\} \text{ for all } x \in \Lambda, \tag{6.12}$$

- *$A \cap \Lambda$ is not M-fibered in any direction.*

Then there is a permutation of $\{i, j, k\}$ such that

$$\{D(N)|m|N\} \setminus \mathrm{Div}(A \cap \Lambda) \subseteq \{N/p_i p_j, N/p_i p_k\}. \tag{6.13}$$

Moreover, (6.13) holds with equality if and only if there exists $x \in \mathbb{Z}_M \setminus A$ such that

$$\mathbb{A}_{N/p_i}^N[x] = c_0\phi(p_i),\ \mathbb{A}_{N/p_j p_k}^N[x] = c_0\phi(p_j p_k),$$

$$\mathbb{A}_m^N[x] = 0 \text{ for all } m \in \{D(N)|m|N\} \setminus \{N/p_i, N/p_j p_k\}.$$

*We will refer to this structure as a p_i-**full plane**.*

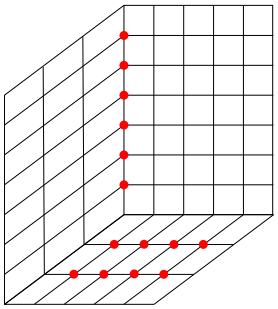

FIGURE 3. Full plane structure on a $D(M)$ grid.

Proof. We may assume that $M = N$ and $c_0 = 1$. We will also write $D = D(M)$ for short. By Proposition 6.1 and the assumption that $A \cap \Lambda$ is not fibered, $A \cap \Lambda$ must contain at least one of the following:

(a) two nonintersecting M-fibers in different directions, say p_i and p_j,

(b) diagonal boxes as in Definition 5.1, and possibly additional M-fibers in one or more directions, disjoint from the diagonal boxes and from each other.

In the first case, we have $\{D|m|M\} \setminus \mathrm{Div}(A \cap \Lambda) \subseteq \{M/p_i p_j\}$, just based on these two fibers. It remains to consider the second case. Suppose that $A \cap \Lambda$ contains diagonal boxes

$$(I \times J \times K) \cup (I^c \times J^c \times K^c)$$

with I, J, K, I^c, J^c, K^c all nonempty. In order for $\{D|m|M\} \setminus \mathrm{Div}(A \cap \Lambda)$ to be nonempty, we must have

$$\min(|I|, |J|, |K|) = \min(|I^c|, |J^c|, |K^c|) = 1. \tag{6.14}$$

We may assume without loss of generality that $|I^c| = |J| = 1$. Since $p_\iota \geq 3$ for all ι, it follows that $|I|, |J^c|$, and at least one of $|K|, |K^c|$ are greater than 1. Assume that $|K^c| > 1$. Then

$$\begin{aligned} M/p_i &\in \mathrm{Div}(I \times J \times K), \\ M/p_j, M/p_k, M/p_j p_k &\in \mathrm{Div}(I^c \times J^c \times K^c), \\ M/p_i p_j p_k &\in \mathrm{Div}(I \times J \times K, I^c \times J^c \times K^c). \end{aligned} \tag{6.15}$$

This implies (6.13). Furthermore, if (6.13) holds with equality, then $|K| = 1$, since otherwise we would also have $M/p_i p_k \in \mathrm{Div}(I \times J \times K)$. This proves the second conclusion of the lemma, with x equal to the unique element of $I^c \times J \times K$. Note that if we add an M-fiber in any direction to this structure, then equality in (6.13) can no longer hold. □

Lemma 6.3. *Let $M = p_i^{n_i} p_j^{n_j} p_k^{n_k}$ and $N = M/p_i^{\alpha_i} p_j^{\alpha_j} p_k^{\alpha_k}$ with $\alpha_\iota < n_\iota$ for all $\iota \in \{i, j, k\}$. Assume that $2 \nmid M$, and that $A \in \mathcal{M}(\mathbb{Z}_N)$ satisfies $\Phi_N|A$. Let Λ be a $D(N)$-grid such that $A \cap \Lambda \neq \emptyset$. Assume further that*

- *(6.12) holds for all $x \in \Lambda$,*
- *$A \cap \Lambda$ is not fibered in any direction,*

and that

$$\{D|m|N\} \setminus Div(A \cap \Lambda) = \{N/p_i p_j\}. \tag{6.16}$$

Then $A \cap \Lambda$ has one of the following, mutually exclusive, possible structures:

(*i*) (*p_k-**corner**, cf. Definition 5.4 (i).) For each $x \in \mathbb{Z}_N$, the set $A \cap \Lambda \cap \Pi(x, p_k^{n_k-\alpha_k})$ is either empty or consists of a single N-fiber in one of the p_i or p_j directions. Since $A \cap \Lambda$ is not fibered, there has to be at least one of each.*

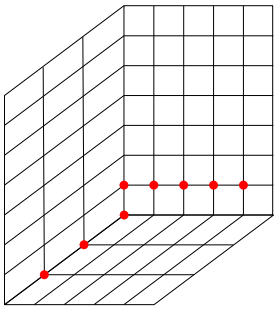

FIGURE 4. Corner structure on a $D(M)$ grid.

(*ii*) **(p_k-almost corner)** *There exist $x_0, x_1, \dots, x_{\phi(p_k)} \in \mathbb{Z}_N$ with $(x_l - x_{l'}, N) = N/p_k$ for $l \neq l'$, and two disjoint sets $\mathcal{L}_i, \mathcal{L}_j \subset \mathbb{Z}_{p_k}$ satisfying $|\mathcal{L}_i|, |\mathcal{L}_j| > 1$ and $\mathcal{L}_i \cup \mathcal{L}_j = \{0, 1, \dots, \phi(p_k)\}$, such that for all $z \in \Lambda$ we have*

$$\mathbb{A}_N^N[z] = \begin{cases} c_0 & \text{if } (z - x_l, N) = N/p_i \text{ for some } l \in \mathcal{L}_i \\ & \text{or } (z - x_l, N) = N/p_j \text{ for some } l \in \mathcal{L}_j \\ 0 & \text{otherwise.} \end{cases}$$

In particular, $\mathbb{A}_N^N[x_l] = 0$ and $\mathbb{A}_{N/p_i}^N[x_l] = c_0\phi(p_i)$ for all $l \in \mathcal{L}_i$, and similarly with i and j interchanged.

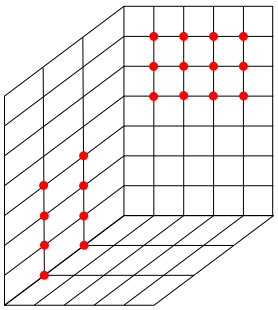

FIGURE 5. An almost corner structure on a $D(M)$ grid.

Proof. We may assume that $M = N$ and $c_0 = 1$, and proceed as in the proof of Lemma 6.2. Suppose first that (a) holds (i.e., $A \cap \Lambda$ contains two non-overlapping M-fibers in two different directions), but A does not contain diagonal boxes. Then the M-fibers must be in the p_i and p_j directions, or else (6.16) would be violated. Moreover, having any two such fibers in the same plane $\Pi(x, p_k^{n_k-\alpha_k})$ would also violate (6.16), hence (i) holds in this case.

In case (b), $A \cap \Lambda$ contains diagonal boxes. We proceed as in the proof of Lemma 6.2 to get (6.14). Since $M/p_ip_j \notin \mathrm{Div}(A)$, we must have

$$\min(|I|, |J|) = \min(|I^c|, |J^c|) = 1.$$

Therefore we may again assume without loss of generality that $|I^c| = |J| = 1$ and that $|K^c| > 1$, and get (6.15). We now consider two cases.

- If $|K| = 1$, the diagonal boxes are as in the conclusion of Lemma 6.2, and instead of (6.16) we have (6.13) with equality.
- If $|K| > 1$, the diagonal boxes present the structure described in (ii), with $\mathcal{L}_i = K$ and $\mathcal{L}_j = K^c$.

The only way we can add an M-fiber to either of these structures without adding M/p_ip_j to $\mathrm{Div}(A)$ is to add an M-fiber in the p_k direction rooted at x (in the first case, with x defined as in Lemma 6.2), or at x_0 (in the second case, with x_0 specified in (ii)) That, however, puts us in the case (i) of the lemma. □

6.3. **Special unfibered structures: even M.**

Lemma 6.4. *Let $M = p_i^{n_i}p_j^{n_j}p_k^{n_k}$ with $2|M$, and $N = M/p_i^{\alpha_i}p_j^{\alpha_j}p_k^{\alpha_k}$ with $\alpha_\iota < n_\iota$ for all $\iota \in \{i,j,k\}$. Assume that $A \in \mathcal{M}(\mathbb{Z}_N)$ satisfies $\Phi_N|A$. Let Λ be a $D(N)$-grid such that $A \cap \Lambda \neq \emptyset$. Assume further that*

- *(6.12) holds for all $x \in \Lambda$,*
- *$A \cap \Lambda$ is not fibered in any direction.*

Assume that $N/p_\iota \in \mathrm{Div}(A \cap \Lambda)$ for all $\iota \in \{i,j,k\}$, but $\{D(N)|m|N\} \not\subset \mathrm{Div}(A \cap \Lambda)$. Then for some permutation of $\{i,j,k\}$ we have

$$\{D(N)|m|N\} \setminus \mathrm{Div}(A \cap \Lambda) = \{N/p_ip_j\},$$

and $A \cap \Lambda$ has the p_k corner structure in the sense of Definition 5.4 (i): for each $x \in \Lambda$, the set $A \cap \Lambda \cap \Pi(x, p_k^{n_k-\alpha_k})$ is either empty or consists of a single N-fiber in one of the p_i or p_j directions. Since $A \cap \Lambda$ is not fibered, there has to be at least one of each.

Proof. We may assume that $M = N$ and $c_0 = 1$. Assume without loss of generality that $p_k = 2$. In this case, in order for $N/p_k \in \mathrm{Div}(A \cap \Lambda)$, we must have at least one N-fiber F in the p_k direction in $A \cap \Lambda$. Let A_1 be the set obtained from $A \cap \Lambda$ by removing all N-fibers in the p_k direction. Since $A \cap \Lambda$ is not fibered, A_1 is nonempty and satisfies (6.1). By Proposition 6.1, it must contain either at least one fiber in another direction or a set of diagonal boxes, each disjoint from F.

- Suppose that A_1 contains diagonal boxes as in Definition 5.1. Without loss of generality, we may assume that F is rooted at a point $a \in I^c \times J \times K$. Then

$$\begin{aligned} M/p_i, M/p_ip_k &\in \mathrm{Div}(F, I \times J \times K),\\ M/p_j, M/p_jp_k &\in \mathrm{Div}(F, I^c \times J^c \times K^c),\\ M/p_ip_jp_k &\in \mathrm{Div}(I \times J \times K, I^c \times J^c \times K^c). \end{aligned}$$

  It follows that the only missing divisor can be M/p_ip_j. In order to avoid that divisor within each box, we must have

$$\min(|I|, |J|) = \min(|I^c|, |J^c|) = 1.$$

  Taking into account the differences $(a' - a'', M)$, where $a' \in F$ and a'' belongs to one of the boxes, we see that the only possible case is $|I^c| = |J| = 1$. Then A contains the M-fiber in the p_i direction rooted at the unique point $x \in I^c \times J \times K$, and the M-fiber in the p_j direction rooted at the unique point $x' \in I^c \times J \times K^c$. Any other points in $A \cap \Lambda$ would add M/p_ip_j to $\mathrm{Div}(A \cap \Lambda)$. Thus the conclusion of the lemma holds.

- If A_1 contains no diagonal boxes, then it must contain a fiber in at least one other direction. This case is identical to the corresponding case of Lemma 6.3 (i), for some permutation of $\{i,j,k\}$.

□

Lemma 6.5. *Let $M = p_i^{n_i}p_j^{n_j}p_k^{n_k}$ with $2|M$, and $N = M/p_i^{\alpha_i}p_j^{\alpha_j}p_k^{\alpha_k}$ with $\alpha_\iota < n_\iota$ for all $\iota \in \{i,j,k\}$. Assume that $A \in \mathcal{M}(\mathbb{Z}_N)$ satisfies $\Phi_N|A$. Let Λ be a $D(M)$-grid such that $A \cap \Lambda \neq \emptyset$. Assume further that $p_k = 2$, and that*

- *(6.12) holds for all $x \in \Lambda$,*
- *$A \cap \Lambda$ is not fibered in any direction,*
- *$\{N/p_i, N/p_j, N/p_k\} \not\subset \mathrm{Div}(A \cap \Lambda)$.*

Then

$$N/p_k \notin \mathrm{Div}(A \cap \Lambda), \tag{6.17}$$

and there is a pair of diagonal boxes

$$A_0 = (I \times J \times K) \cup (I^c \times J^c \times K^c) \subset \Lambda,$$

as in Definition 5.1, such that for all $z \in A \cap \Lambda$ we have

$$\mathbb{A}_N^N[z] = \begin{cases} c_0 & \text{if } z \in A_0, \\ 0 & \text{otherwise.} \end{cases}$$

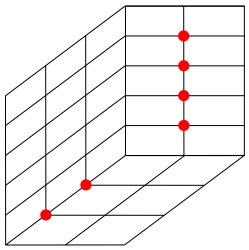

FIGURE 6. An even almost corner structure on a $D(M)$ grid with $M/p_k \notin \mathrm{Div}(A)$, $p_k = 2$.

Proof. We may assume that $M = N$ and $c_0 = 1$. If M/p_i or M/p_j is not in $\mathrm{Div}(A \cap \Lambda)$, then $A \cap \Lambda$ is fibered by Lemma 4.9, contradicting the assumptions of the lemma. Therefore (6.17) holds. Invoking Proposition 6.1 again, we see that $A \cap \Lambda$ must contain either diagonal boxes or at least two non-overlapping M-fibers in different directions. The second case cannot be reconciled with (6.17). Therefore $A \cap \Lambda$ must contain a set A_0 of diagonal boxes. Notice that adding an M-fiber in any direction to A_0 would introduce M/p_k as a divisor of $A \cap \Lambda$. Therefore $A \cap \Lambda = A_0$. □

7. Resolving diagonal boxes

Theorem 7.1. *Let $A \oplus B = \mathbb{Z}_M$ be a tiling, $M = p_i^2p_j^2p_k^2$ with $p_i, p_j, p_k \geq 3$, $|A| = |B| = p_ip_jp_k$, and assume that $\Phi_M \mid A$. Let $D = D(M)$, and let Λ be a $D(M)$-grid such that $A \cap \Lambda \neq \emptyset$. Assume further that $A \cap \Lambda$ is not fibered in any direction, and that one of the following holds: either*

$$\{m : \ D(M)|m|M\} \subset \mathrm{Div}(A \cap \Lambda) \tag{7.1}$$

and $A\cap\Lambda$ contains diagonal boxes as in Definition 5.1, or else $A\cap\Lambda$ has one of the structures described in Lemma 6.2 (full plane) or Lemma 6.3 (ii) (almost corner). Then at least one of the following is true:

- *The tiling $A\oplus B=\mathbb{Z}_M$ is T2-equivalent to $\Lambda\oplus B=\mathbb{Z}_M$ via fiber shifts. Thus Λ is a translate of $A^\flat$, and by Corollary 2.2, both A and B satisfy (T2).*
- *The tiling $A\oplus B=\mathbb{Z}_M$ is T2-equivalent to a tiling $A'\oplus B$, where $A'\cap\Lambda$ contains a p_ν corner structure as in Definition 5.4 (i) for some $\nu\in\{i,j,k\}$.*

We remark that, in the case when (7.1) holds, $A\cap\Lambda$ might be larger than just a pair of diagonal boxes. For example, it could contain diagonal boxes and some number of M-fibers in various directions disjoint from the boxes. However, any such additional structures can only make our task easier.

We split the proof into cases. Since $p_i,p_j,p_k\geq 3$, at least one of I and I^c must have cardinality greater than 1, and similarly for each of the pairs J,J^c and K,K^c. We claim that it suffices to consider the following two cases.

Case (DB1): The tiling $A\oplus B$ satisfies the assumptions of Theorem 7.1, and additionally $\min(|I|,|J^c|,|K^c|)\geq 2$.

Case (DB2): The tiling $A\oplus B$ satisfies the assumptions of Theorem 7.1, and additionally $|I^c|=|J^c|=|K^c|=1$.

Indeed, if either $|I|=|J|=|K|=1$ or $|I^c|=|J^c|=|K^c|=1$, then we are in the case (DB2), possibly after relabelling I,J,K as I^c,J^c,K^c and vice versa. Suppose now that neither of these holds, say $|I|\geq 2$ and $|J^c|\geq 2$. Since $p_k\geq 3$, at least one of K and K^c must have cardinality at least 2. If $|K^c|\geq 2$, we are in the case (DB1). If $|K|\geq 2$, we are in the case (DB1) again, with p_i and p_j interchanged, and with the sets I,J,K relabelled as I^c,J^c,K^c and vice versa. All other cases are identical up to a permutation of the indices i,j,k.

We will proceed to resolve the cases (DB1) and (DB2) in Sections 7.2 and 7.3, respectively. Throughout this section, we continue to use the notation of Section 5.1.

7.1. **Preliminary results.**

Lemma 7.2. *Let $A\oplus B=\mathbb{Z}_M$ be a tiling, $M=p_i^2p_j^2p_k^2$, $|A|=|B|=p_ip_jp_k$, and assume that $\Phi_M\mid A$. Let $D=D(M)$, and let Λ be a D-grid such that $A\cap\Lambda\neq\emptyset$. Assume that (7.1) holds. Suppose that $A\cap\Lambda$ is not fibered, and that it contains diagonal boxes*

$$A_0:=(I\times J\times K)\cup(I^c\times J^c\times K^c)\subset A\cap\Lambda, \tag{7.2}$$

with I,J,K,I^c,J^c,K^c as in Definition 5.1. If one of the "complementary boxes" is contained in A, say

$$(I^c\times J\times K)\subset A, \tag{7.3}$$

then the tiling $A\oplus B=\mathbb{Z}_M$ is T2-equivalent to $\Lambda\oplus B=\mathbb{Z}_M$. Consequently, A and B both satisfy T2.

Proof. Throughout the proof, we assume that $A\oplus B=\mathbb{Z}_M$ is a tiling satisfying the assumptions of the lemma.

Claim 1. *Assume that (7.3) holds. Then either $A \oplus B = \mathbb{Z}_M$ is T2-equivalent to $\Lambda \oplus B = \mathbb{Z}_M$, or*

$$(I^c \times J \times K^c) \cap A = \emptyset. \tag{7.4}$$

Proof. Suppose that there is an $a \in (I^c \times J \times K^c) \cap A$, and let

$$Z = \Pi(a, p_i^2) \cap I^c \times J^c \times K.$$

For any $z \in Z$, let a_j, a_k be points such that $(z - a_j, M) = (a_k - a, M) = M/p_j$ and $(z - a_k, M) = (a_j - a, M) = M/p_k$. Then $a_j, a_k \in A$, since $a_j \in I^c \times J \times K$ and $a_k \in I^c \times J^c \times K^c$. By Lemma 4.6, either $z \in A$, or else $A_z \subset \ell_i(x)$, with a $(1, 2)$ cofibered structure for (A, B) in the p_i direction and an M-fiber in A at distance M/p_i^2 from z as a cofiber. We can use Lemma 2.9 to shift that fiber to z. Repeating this procedure for all $z \in Z \setminus A$, we get a new set $A_1 \subset \mathbb{Z}_M$ such that $A_1 \oplus B = \mathbb{Z}_M$ and A_1 is T2-equivalent to A. Moreover, $Z \subset A_1$, and, since $A \cap \Lambda \subset A_1 \cap \Lambda$, (7.1) holds with A replaced by A_1.

Now, any $x \in (I^c \times J^c \times K) \setminus Z$ satisfies the assumptions of Lemma 4.5 applied to A_1, with $(x - z, M) = M/p_i$ for some $z \in Z$, $(x - a_j', M) = M/p_j$ for some $a_j' \in I^c \times J \times K$, and $(x - a_k', M) = M/p_k$ for some $a_k' \in I^c \times J^c \times K^c$. It follows that

$$I^c \times J^c \times K \subset A_1. \tag{7.5}$$

We can then apply Lemma 4.6 and the fiber shifting argument again, first to all points in $I^c \times J \times K^c$ with the flat corner configurations in planes perpendicular to the p_i direction, then to all points in $I \times J^c \times K$ with the flat corner configurations in planes perpendicular to the p_k direction. Thus A_1 is T2-equivalent to a set A_2 that satisfies $A_2 \oplus B = \mathbb{Z}_M$, continues to obey (7.3)–(7.5), and moreover has the property that

$$(I^c \times J \times K^c) \cup (I \times J^c \times K) \subset A_2.$$

Finally, we cannot have any points in $I \times J^c \times K^c$ that are not in A_2, since that would contradict Lemma 4.5. Thus $A_2 = \Lambda$. □

Claim 1'. *Assume that (7.3) holds. Then either $A \oplus B = \mathbb{Z}_M$ is T2-equivalent to $\Lambda \oplus B = \mathbb{Z}_M$, or*

$$(I^c \times J^c \times K) \cap A = \emptyset. \tag{7.6}$$

Proof. This is identical to the proof of Claim 1, with the p_j and p_k directions interchanged, and with J and K^c replaced by J^c and K. □

Claim 2. *Assume that (7.3), (7.4), and (7.6) hold. Then*

$$I \times J^c \times K^c \subset A. \tag{7.7}$$

Proof. Let $x \in I \times J^c \times K^c$. Considering any M-cuboid with one vertex at x and another in $I^c \times J \times K$, we see that it can only be balanced if $x \in A$. □

We can now finish the proof of the lemma. It suffices to consider the case when (7.3) and (7.7) hold, so that

$$Y := (I \times J \times \mathbb{Z}_{p_i}) \cup (I^c \times J^c \times \mathbb{Z}_{p_i}) \subset A. \tag{7.8}$$

If there are any elements $a \in (A \cap \Lambda) \setminus Y$, we can repeat the argument in the proof of Claim 1 to prove that the tiling $A \oplus B = \mathbb{Z}_M$ is T2-equivalent to $\Lambda \oplus B = \mathbb{Z}_M$. If on the other

hand $A \cap \Lambda = Y$, then this set is fibered in the p_i direction, contradicting the assumptions of the lemma. □

Corollary 7.3. *Let $A \oplus B = \mathbb{Z}_M$ be a tiling, $M = p_i^2 p_j^2 p_k^2$, $|A| = |B| = p_i p_j p_k$, and assume that $\Phi_M \mid A$. Let $D = D(M)$, and let Λ be a D-grid such that $A \cap \Lambda \neq \emptyset$. Assume that (5.5) holds. Suppose that $A \cap \Lambda$ is not fibered, that it contains diagonal boxes (7.2) with I, J, K, I^c, J^c, K^c as in Definition 5.1, and that it also contains at least one M-fiber disjoint from these boxes. Then the tiling $A \oplus B = \mathbb{Z}_M$ is T2-equivalent to $\Lambda \oplus B = \mathbb{Z}_M$. Consequently, A and B both satisfy T2.*

Proof. We may assume without loss of generality that the M-fiber $F \subset A \cap \Lambda$ is in the p_j direction, with $F \subset I^c \times \mathbb{Z}_{p_j} \times K$. By translational invariance, we may further assume that

$$F = \{0\} \times \mathbb{Z}_{p_j} \times \{0\},$$

with $0 \in I^c$ and $0 \in K$.

We first claim that there is a set $A_1 \subset \mathbb{Z}_M$, either equal to A or T2-equivalent to it, such that $A_1 \oplus B = \mathbb{Z}_M$ and

$$\{0\} \times J \times K \subset A_1. \tag{7.9}$$

Indeed, if $K = \{0\}$, then $\{0\} \times J \times K \subset F \subset A$ and there is nothing to prove. Otherwise, let $l_j \in J$ and $l_k \in K \setminus \{0\}$. Then the point $x = (0, l_j, l_k)$ either belongs to A, or else it satisfies the assumptions of Lemma 4.6, with $(0, l_j, 0)$, $(l_i, l_j, 0)$, and (l_i, l_j, l_k) all in A for any $l_i \in I$, so that the flat corner configuration is perpendicular to the p_j direction. By Lemma 4.6, in the latter case we have $A_x \subset \ell_j(x)$, implying a $(1, 2)$ cofibered structure. We can then use Lemma 2.9 to shift the M-cofiber in A to x. After all such shifts have been performed, we arrive at A_1.

By a similar argument but with p_i and p_k interchanged, we may further replace A_1 by a T2-equivalent set A_2 such that $A_2 \oplus B = \mathbb{Z}_M$, (7.9) continues to hold for A_2, and

$$I^c \times J^c \times \{0\} \subset A_2. \tag{7.10}$$

Next, we replace A_2 by a T2-equivalent set A_3 such that $A_3 \oplus B = \mathbb{Z}_M$, (7.9) and (7.10) both continue to hold for A_3, and

$$I^c \times J \times \{0\} \subset A_3. \tag{7.11}$$

Indeed, consider a point $z = (l_i', l_j, 0\} \in I^c \times J \times \{0\}$, with $l_i' \neq 0$. (If no such l_i' exists, (7.11) holds with $A_3 = A_2$.) If $z \notin A_2$, then it satisfies the assumptions of Lemma 4.6, with $(0, l_j, 0)$, $(0, l_j', 0)$, and $(l_i, l_j', 0)$ all in A_2 for any $l_j' \in J^c$, so that the flat corner configuration is perpendicular to the p_k direction. Then $A_z \subset \ell_k(z)$, and again our conclusion follows by Lemma 2.9.

Finally, we may pass to another T2-equivalent set A_4 such that $A_4 \oplus B = \mathbb{Z}_M$ and

$$I^c \times J \times K \subset A_4. \tag{7.12}$$

If $I^c = \{0\}$ or $K = \{0\}$, this follows from (7.9) or (7.11), with $A_4 = A_3$. Otherwise, we let $w = (l_i', l_j, l_k\} \in I^c \times J \times K$ with $l_i' \neq 0$ and $l_k \neq 0$, and repeat the argument from the proof of (7.9) with the first coordinate 0 replaced by l_i'.

With (7.12) in place, the corollary now follows from Lemma 7.2. □

7.2. **Case (DB1).** Assume that $A \oplus B = \mathbb{Z}_M$ is a tiling satisfying the assumptions of Theorem 7.1. Let $D = D(M)$, and let Λ be the $D(M)$ grid provided by the assumption of the theorem. Additionally, we assume that

$$\min(|I|, |J^c|, |K^c|) \geq 2. \tag{7.13}$$

If $A \cap \Lambda$ has one of the structures described in Lemma 6.2 (full plane) or Lemma 6.3 (ii) (almost corner), then in both cases we have $A \cap \Lambda = (I \times J \times K) \cup (I^c \times J^c \times K^c)$ with no other points permitted, so that

$$(I^c \times J \times K) \cap A = \emptyset. \tag{7.14}$$

(Note that (7.13) covers the cases of a p_i full plane and a p_j almost corner structure. See the end of this section for more details.)

If on the other hand (7.1) holds and $I^c \times J \times K \subset A$, the conclusion of Theorem 7.1 follows by Lemma 7.2. We may therefore assume that there exists a point

$$x \in (I^c \times J \times K) \setminus A. \tag{7.15}$$

Lemma 7.4. *Assume (DB1), and let x satisfy (7.15). Then for every $b \in B$ we have exactly one of the following:*

$$A_{x,b} \subset \ell_j(x), \text{ with } \mathbb{A}_{M/p_j^2}[x]\mathbb{B}_{M/p_j^2}[b] = \phi(p_j^2), \tag{7.16}$$

$$A_{x,b} \subset \ell_k(x), \text{ with } \mathbb{A}_{M/p_k^2}[x]\mathbb{B}_{M/p_k^2}[b] = \phi(p_k^2). \tag{7.17}$$

Furthermore:

- $\mathbb{A}_{M/p_j}[x] \cdot \mathbb{A}_{M/p_k}[x] = 0$,
- *if $\mathbb{A}_{M/p_j}[x] > 0$, then (7.16) cannot hold for any $b \in B$, and if $\mathbb{A}_{M/p_k}[x] > 0$, then (7.17) cannot hold for any $b \in B$,*
- *if (7.16) holds for some $b \in B$, then the product $\langle \mathbb{A}[x], \mathbb{B}[b] \rangle$ is saturated by a $(1,2)$-cofiber pair in the p_j direction, with the A-cofiber at distance M/p_j^2 from x and the B-fiber rooted at b. The same is true for (7.17), with j and k interchanged.*

Proof. The assumptions (DB1) and (7.15) imply in particular that

$$\mathbb{A}_{M/p_i}[x] \geq 2, \tag{7.18}$$

so that $A_x \subset \Pi(x, p_i^2)$. Moreover,

$$A_x \subset \bigcap_{a:(x-a,M)=M/p_jp_k} \mathrm{Bispan}(x,a) = \Pi(x,p_j^2) \cup \Pi(x,p_k^2),$$

which together with (7.18) proves that $A_x \subset \ell_j(x) \cup \ell_k(x)$. By Lemma 4.1, for each $b \in B$ we must in fact have either $A_{x,b} \subset \ell_j(x)$ or $A_{x,b} \subset \ell_k(x)$. If $A_{x,b} \subset \ell_j(x)$, then the second part of (7.16) follows since $M/p_j \in \mathrm{Div}(A)$, and Lemma 2.10 implies the existence of a cofiber pair as described in the last part of the lemma. The same applies with j and k interchanged.

Next, suppose that $\mathbb{A}_{M/p_j}[x] > 0$. Then $A_x \subset \mathrm{Bispan}(x,a)$ for any $a \in A$ with $(x-a, M) = M/p_j$. But this is clearly not compatible with (7.16) for any $b \in B$. The same applies with j and k interchanged.

On the other hand, one of (7.16) and (7.17) must hold for each b, therefore we cannot have both $\mathbb{A}_{M/p_j}[x] > 0$ and $\mathbb{A}_{M/p_j}[x] > 0$. This ends the proof of the lemma. □

Lemma 7.5. *Assume (DB1). Suppose that there is a point $x \in \Lambda$ such that (7.15) holds and*

$$\max(\mathbb{A}_{M/p_j}[x], \mathbb{A}_{M/p_k}[x]) > 0. \tag{7.19}$$

Then the conclusion of Theorem 7.1 holds.

Proof. Assume without loss of generality that $\mathbb{A}_{M/p_j}[x] > 0$. Then $A_x \subset \ell_j(x)$ by Lemma 7.4. By Lemma 2.10, the pair (A, B) has a (1,2)-cofibered structure in the p_j direction with a cofiber in A at distance M/p_j^2 from x. We apply Lemma 2.9 to shift the M-cofiber in A to x. Let A_1 be the set thus obtained.

We note that (7.19) does not hold for either the full plane structure or the almost corner structure. Therefore (7.1) must hold, and since $A \cap \Lambda \subset A_1 \cap \Lambda$, the same holds for A_1. Furthermore, A_1 contains the diagonal boxes inherited from A as well as the added M-fiber through x in the p_j direction, disjoint from the boxes. By Corollary 7.3, the tiling $A_1 \oplus B = \mathbb{Z}_M$ (therefore also $A \oplus B = \mathbb{Z}_M$) is T2-equivalent to $\Lambda \oplus B = \mathbb{Z}_M$. □

It remains to consider the complementary case when

$$\mathbb{A}_{M/p_j}[x] = \mathbb{A}_{M/p_k}[x] = 0 \ \ \forall x \in (I^c \times J \times K) \setminus A. \tag{7.20}$$

Lemma 7.6. *Assume that (DB1) holds, that $(I^c \times J \times K) \setminus A \neq \emptyset$, and that (7.20) holds. Fix an index $l_i \in I^c$ such that $(\{l_i\} \times J \times K) \setminus A$ is nonempty, and define*

$$\mathcal{X} = \mathcal{X}(l_i) := \{l_i\} \times J \times K.$$

Then either A is T2-equivalent to a set $A' \subset \mathbb{Z}_M$ such that $A' \oplus B = \mathbb{Z}_M$ and every point $x' \in \mathcal{X}$ belongs to a fiber in A' in the p_j direction, or else the same holds with j and k interchanged.

Proof. We fix l_i as in the statement of the lemma, and keep it fixed for the duration of the proof. Let also $\Pi := \Pi(x, p_i^2)$ for any $x \in \mathcal{X}$, and note that $\mathcal{X} \subset \Pi$.

Let $x \in \mathcal{X} \setminus A$. By (7.20), we must have $x' \notin A$ for all $x' \in \mathcal{X}$ with $(x' - x, M) \in \{M/p_j, M/p_k\}$. Applying (7.20) to all such x', we see that $\mathcal{X} \cap A = \emptyset$. Let

$$\mathcal{X}_j = \{x \in \mathcal{X} : \ \exists b \in B \text{ such that } A_{x,b} \subset \ell_j(x)\},$$

and similarly for $\mathcal{X}_k$. By Lemma 7.4, we have $\mathcal{X} = \mathcal{X}_j \cup \mathcal{X}_k$.

Suppose that $\mathcal{X}_k \neq \emptyset$, with $x_0 \in \mathcal{X}_k$, and let $x_1, x_2, \dots, x_{|J|-1}$, be the distinct points in $\mathcal{X}$ such that $(x_0 - x_\nu, M) = M/p_j$ for $\nu \neq 0$. By the definition of $\mathcal{X}_k$, there exists $b \in B$ such that

$$A_{x_\nu, b} \subset \ell_k(x_\nu) \tag{7.21}$$

for $\nu = 0$. We claim that (7.21) also holds for $\nu = 1, \dots, |J|-1$, with the same $b \in B$. Indeed, suppose that $A_{x_\nu,b} \subset \ell_j(x_\nu)$ for some $\nu \geq 1$. By Lemma 2.10, the product $\langle \mathbb{A}[x_\nu], \mathbb{B}[b] \rangle$ would be saturated by a $(1, 2)$ cofiber pair in the p_j direction. But then we would also have $A_{x_0,b} \subset \ell_j(x_0)$, with the product saturated by the same cofiber pair, contradicting the assumption that (7.21) holds for $\nu = 0$.

By Lemma 2.10 again, it follows that if $\mathcal{X}_k \neq \emptyset$, then A contains an M-fiber in the p_k direction at distance M/p_k^2 from each of the points $x_0, x_1, x_2, \dots, x_{|J|-1}$. Since $\ell_k(x_\nu) \subset \Pi$, all these fibers are contained in Π.

If we had both $\mathcal{X}_j \neq \emptyset$ and $\mathcal{X}_k \neq \emptyset$, we would get such sets of fibers in both directions, all contained in Π. But then Π contains $|J^c|\,|K^c|$ points of $I^c \times J^c \times K^c$, at least $|J|$ fibers in A

in the p_k direction, and at least $|K|$ fibers in A in the p_j direction, disjoint from $\Lambda \cap \Pi$ and from each other. Thus

$$|A \cap \Pi| \geq (p_j - |J|)(p_k - |K|) + p_k|J| + p_j|K| = p_j p_k + |J||K|.$$

This contradicts Lemma 4.3.

It follows that at least one of $\mathcal{X}_j$ and $\mathcal{X}_k$ must be empty. Assume without loss of generality that $\mathcal{X}_j = \mathcal{X}$ and $\mathcal{X}_k = \emptyset$. It follows that $A_x \subset \ell_j(x)$ for all $x \in \mathcal{X}$. By Lemma 2.10, the pair (A, B) has a $(1, 2)$-cofibered structure in the p_k direction, and for every $x \in \mathcal{X}$ there is a cofiber in A at distance M/p_j^2 from x. We can now use Lemma 2.9 to shift all of the aforementioned cofibers in the p_j direction, obtaining the new set A' as indicated in the conclusion of the lemma. □

We can now conclude the proof of Theorem 7.1 under the assumption that (7.13), (7.15), and (7.20) hold.

- If (7.1) holds, then the set $A' \subset \mathbb{Z}_M$ from Lemma 7.6 still satisfies (7.1), and $A' \cap \Lambda$ contains the diagonal boxes inherited from A as well as the additional fibers added in Lemma 7.6. By Corollary 7.3, A' is T2-equivalent to Λ, and the theorem follows.
- Suppose that $A \cap \Lambda$ has the p_i full plane structure (Lemma 6.2), with $|I^c| = |J| = |K| = 1$. Then there is only one index $l_i \in I^c$, and for that l_i, $\mathcal{X} = \{x\}$ is a single point. In this case, Lemma 7.6 identifies a direction p_ν for some $\nu \in \{j, k\}$, such that the pair (A, B) has a $(1, 2)$-cofibered structure in the p_ν direction with a cofiber in A at distance M/p_ν^2 from x. Assume without loss of generality that $\nu = j$. After shifting the cofiber to x as permitted by Lemma 2.9, we arrive at a p_j corner structure, where $A' \cap \Lambda$ is the union of one M-fiber in the p_i direction and $(p_j - 1)$ M-fibers in the p_k directions, each in a different plane perpendicular to the p_j direction.
- Assume now that $A \cap \Lambda$ has the p_j almost corner structure (Lemma 6.3 (ii)), with $|K| = |I^c| = 1$ and $|J|, |J^c| > 1$. Then there is again only one index $l_i \in I^c$, and the set $\mathcal{X} = \{l_i\} \times J \times K$ for that value of l_i has dimensions $1 \times |J| \times 1$. We again apply Lemma 7.6. If the fiber identified and shifted in the lemma was in the p_j direction, then we are in the p_j corner situation again, with $A' \cap \Lambda$ consisting of non-overlapping fibers in the p_i and p_k directions. If on the other hand the shifted fibers were in the p_k direction, then we have $\{D|m|M\} \subset \mathrm{Div}(A')$, and $A' \cap \Lambda$ contains the diagonal boxes inherited from A as well as the added fibers disjoint from the boxes. By Corollary 7.3, the tiling $A' \oplus B = \mathbb{Z}_M$ (therefore also $A \oplus B = \mathbb{Z}_M$) is T2-equivalent to $\Lambda \oplus B = \mathbb{Z}_M$.

7.3. **Case (DB2).** We now assume that $A \oplus B = \mathbb{Z}_M$ is a tiling satisfying the assumptions of Theorem 7.1 on a D-grid Λ. Additionally, we assume that

$$|I^c| = |J^c| = |K^c| = 1. \tag{7.22}$$

Since p_i, p_j, p_k are all odd, the assumption (7.22) implies that $\min(|I|, |J|, |K|) \geq 2$, and in particular (7.1) holds. Let also

$$I^c \times J^c \times K^c = \{a_0\}.$$

Lemma 7.7. *Assume (DB2). Suppose that there exists a point $x \in \Lambda \setminus A$ such that A_x is contained in one of the lines $\ell_i(x)$, $\ell_j(x)$, $\ell_k(x)$. Then the tiling $A \oplus B = \mathbb{Z}_M$ is T2-equivalent to $\Lambda \oplus B = \mathbb{Z}_M$. Hence the conclusion of Theorem 7.1 holds.*

Proof. Suppose that $x \in \Lambda \setminus A$ and $A_x \subset \ell_k(x)$. By Lemma 2.10, the pair (A, B) has a (1,2)-cofibered structure in the p_k direction. By Lemma 2.9, the cofiber in $A \setminus \Lambda$ can be shifted to a fiber in Λ rooted at x. This yields a new tiling $A' \oplus B = \mathbb{Z}_M$, T2-equivalent to $A \oplus B = \mathbb{Z}_M$, such that $A' \cap \Lambda$ contains both the diagonal boxes $\{a_0\} \cup (I \times J \times K)$ and the added M-fiber. In light of (7.1), we can apply Corollary 7.3 to conclude the proof. □

Lemma 7.8. *Assume (DB2). If $(A \cap \Lambda) \setminus (\{a_0\} \cup (I \times J \times K)) \neq \emptyset$, then the tiling $A \oplus B = \mathbb{Z}_M$ is T2-equivalent to $\Lambda \oplus B = \mathbb{Z}_M$.*

Proof. Observe first that if $A \cap \Lambda$ contains a flat corner configuration in the sense of Lemma 4.6, then the conclusion holds. Indeed, Lemma 4.6 implies that the point $x \notin A$ in the flat corner configuration satisfies the assumptions of Lemma 7.7, which we then apply.

Suppose that there is an element $a \in A \cap (I^c \times J \times K)$. In order to avoid a flat corner configuration perpendicular to the p_j direction at points $x \in I^c \times \{\lambda_j a\} \times K$, all such points must belong to A. Similarly, we must have $I^c \times J \times \{\lambda_k a\} \subset A$, in order to avoid a flat corner perpendicular to the p_k direction at those points. By Lemma 4.5 and (7.1), we must in fact have $I^c \times J \times K \subset A$. But then the conclusion follows by Lemma 7.2. Since the assumption (DB2) is symmetric with respect to all permutations of the indices $\{i, j, k\}$, the same argument applies when $A \cap (I \times J^c \times K)$ or $A \cap (I \times J \times K^c)$ is nonempty.

Assume now that $a \in A \cap (I^c \times J^c \times K)$. Consider an M-cuboid with one vertex at a_0, another at a, and a third one at $a' \in I \times J \times K$ with $(a - a', M) = M/p_i p_j$ and $(a_0 - a, M) = D$. In order to balance this cuboid, at least one of the vertices in $I^c \times J \times K$, $I \times J^c \times K$, or $I \times J \times K^c$, must be in A. But then we are in the situation from the last paragraph. Since this case is also symmetric with respect to all permutations of $\{i, j, k\}$, we are done. □

Lemma 7.9. *Assume (DB2), and let $N_k = M/p_k$. If $M/p_k^2 \in \mathrm{Div}(A)$, then $\Phi_{N_k} | A$.*

Proof. Assume for contradiction that $\Phi_{N_k} | B$, and apply Lemma 4.9 to B, on scale $N = N_k$. Since $N_k, N_k/p_k \notin \mathrm{Div}(B)$, it follows that B is N_k-fibered in one of the p_i and p_j directions. However, that is impossible, given that $\mathrm{Div}(B) \cap \{D|m|M\} = \{M\}$ due to (7.1). □

We now begin the proof of Theorem 7.1 under the assumption (DB2). By Lemma 7.8, it suffices to consider the case when

$$(A \cap \Lambda) \setminus (\{a_0\} \cup (I \times J \times K)) = \emptyset. \tag{7.23}$$

Let $x \in I \times J^c \times K$, so that in particular $x \notin A$. Fix $b \in B$. Then

$$A_{x,b} \subset \ell_i(x) \cup \ell_i(a_0) \cup \ell_k(x) \cup \ell_k(a_0).$$

We claim that $A_{x,b}$ cannot intersect more than one of the above lines. Indeed, by (7.1) we have

- $A_{x,b} \cap \ell_i(x) \neq \emptyset$ implies $\mathbb{A}_{M/p_i^2}[x]\mathbb{B}_{M/p_i^2}[b] > 0$, hence $M/p_i^2 p_k \in \mathrm{Div}(A)$ and $M/p_i^2 \in \mathrm{Div}(B)$.
- $A_{x,b} \cap \ell_k(x) \neq \emptyset$ implies $\mathbb{A}_{M/p_k^2}[x]\mathbb{B}_{M/p_k^2}[b] > 0$, hence $M/p_i p_k^2 \in \mathrm{Div}(A)$ and $M/p_k^2 \in \mathrm{Div}(B)$.
- $A_{x,b} \cap \ell_i(a_0) \neq \emptyset$ implies $\mathbb{A}_{M/p_i^2 p_k}[x|\ell_i(a_0)]\mathbb{B}_{M/p_i^2 p_k}[b] > 0$, hence $M/p_i^2 \in \mathrm{Div}(A)$ and $M/p_i^2 p_k \in \mathrm{Div}(B)$.
- $A_{x,b} \cap \ell_k(a_0) \neq \emptyset$ implies $\mathbb{A}_{M/p_i p_k^2}[x|\ell_k(a_0)]\mathbb{B}_{M/p_i p_k^2}[b] > 0$, hence $M/p_k^2 \in \mathrm{Div}(A)$ and $M/p_i p_k^2 \in \mathrm{Div}(B)$.

It is then easy to check that $A_{x,b}$ cannot have nonempty intersection with any two of the above lines, either by Lemma 4.1 or due to direct divisor conflict.

Lemma 7.10. *Assume (DB2) and (7.23), and let x be as above. Then $A_{x,b} \cap \ell_k(a_0) = \emptyset$.*

Proof. Assume for contradiction that

$$A_{x,b} \subset \ell_k(a_0). \tag{7.24}$$

Then $M/p_k^2 \in \mathrm{Div}(A)$, hence by Lemma 7.9 we have $\Phi_{M/p_k}|A$. We are also assuming as part of (DB2) that $\Phi_M|A$. It follows that A is null with respect to all cuboids of type $(M,(1,1,2),1)$.

Consider any such cuboid with one vertex at a_0, a second one at a point $x_j \in I^c \times J \times K^c$ so that $(x_j - a_0, M) = M/p_j$, and a third one at a point $z \in \ell_k(x)$ with $(x-z,M) = M/p_k^2$. By (7.22), none of the cuboid vertices in the plane $\Pi(a_0, p_k^2)$ except for a_0 are in A. In order to balance this cuboid, one of the vertices z_i and z_j must be in A, where $(z_i - z, M) = M/p_i$ and $(z_j - z, M) = M/p_j$. However, if $z_j \in A$, then $M/p_i p_k^2 \in \mathrm{Div}(\{z_j\}, I \times J \times K) \subset \mathrm{Div}(A)$, which contradicts (7.24) due to divisor conflict. It follows that $z_i \in A$.

Repeating this argument for all $z \in \ell_k(x)$ with $(x-z,M) = M/p_k^2$, we get that

$$\mathbb{A}_{M/p_k^2}[a_0] = \phi(p_k^2). \tag{7.25}$$

If $p_i < p_k$, then we must in fact have $p_i \le p_k - 2$ (since both are odd primes), so that $\phi(p_k^2) = p_k(p_k-1) > p_i p_k$. In this case, (7.25) contradicts Lemma 4.3.

Assume now that $p_i > p_k$. In this case, (7.24) and (7.25) imply that

$$\begin{aligned} 1 &= \frac{1}{\phi(p_i p_k^2)} \mathbb{A}_{M/p_i p_k^2}[x|\ell_k(a_0)] \mathbb{B}_{M/p_i p_k^2}[b] \\ &= \frac{1}{\phi(p_i p_k^2)} \mathbb{A}_{M/p_k^2}[a_0] \mathbb{B}_{M/p_i p_k^2}[b] \\ &= \frac{1}{\phi(p_i)} \mathbb{B}_{M/p_i p_k^2}[b]. \end{aligned}$$

But when $p_i > p_k$, this is not possible without introducing M/p_i or $M/p_i p_k$ as divisors of B, which would contradict (7.1). □

Repeating Lemmas 7.9 and 7.10 with k replaced by i, we get that for each $b \in B$ we must have one of

$$A_{x,b} \subset \ell_i(x) \text{ or } A_{x,b} \subset \ell_k(x).$$

Now let $x' \in (I \times J \times K^c) \cap \Pi(x, p_i^2)$. The same analysis, with the p_j and p_k directions interchanged, shows that for each $b \in B$ we must have one of

$$A_{x',b} \subset \ell_i(x') \text{ or } A_{x',b} \subset \ell_j(x').$$

If $A_{x,b} \subset \ell_k(x)$ and $A_{x',b'} \subset \ell_j(x')$ for some $b, b' \in B$, we use the same fiber crossing argument as in the case (DB1). By Lemma 2.10, A must contain an M-fiber in the p_k direction at distance M/p_k^2 from x, as well as an M-fiber in the p_j direction at distance M/p_j^2 from x', both of them in the plane $\Pi := \Pi(x, p_i^2)$ and disjoint from each other. The same plane also contains $(p_j - 1)(p_k - 1)$ elements of $I \times J \times K$. Therefore

$$|\Pi \cap A| \ge (p_j - 1)(p_k - 1) + p_j + p_k = p_j p_k + 1 > p_j p_k,$$

which contradicts Lemma 4.3.

It follows that either $A_{x,b} \subset \ell_i(x)$ for all $b \in B$, or else $A_{x',b} \subset \ell_i(x')$ for all $b \in B$. An application of Lemma 7.7 concludes the proof of the theorem.

8. Corners

In this section we address the extended corner structure, defined in Definition 5.4. We will assume the following.

Assumption (C): Assume that $A \oplus B = \mathbb{Z}_M$, where $M = p_i^{n_i} p_j^{n_j} p_k^{n_k}$, $|A| = p_i p_j p_k$, and $\Phi_M | A$. Moreover, assume that A contains a p_i extended corner in the sense of Definition 5.4 on a $D(M)$ grid Λ, that is, there exist $a, a_i \in A \cap \Lambda$ with $(a - a_i, M) = M/p_i$ such that

- $A \cap (a * F_j * F_k)$ is M-fibered in the p_j direction but not in the p_k direction,
- $A \cap (a_i * F_j * F_k)$ is M-fibered in the p_k direction but not in the p_j direction.

In particular, we may choose a, a_i so that

$$a * F_j \subset A, \quad a_i * F_k \subset A, \tag{8.1}$$

$$a_i * F_j \not\subset A, \quad a * F_k \not\subset A. \tag{8.2}$$

We fix such a, a_i for the rest of this section.

The following theorem is our main result for the extended corner structure.

Theorem 8.1. *Assume that (C) holds with $n_j = n_k = 2$. Then the tiling $A \oplus B = \mathbb{Z}_M$ is T2-equivalent to $\Lambda \oplus B = \mathbb{Z}_M$ via fiber shifts. By Corollary 2.2, both A and B satisfy (T2).*

We first note that when A contains a p_i corner,

$$\{D(M) | m | M\} \setminus \mathrm{Div}(A) \subseteq \{M/p_j p_k\} \tag{8.3}$$

Lemma 8.2 (Size-Divisor Lemma). *Assume that $A \subset \mathbb{Z}_M$, and that there exist $a, a_i \in A$ with $(a - a_i, M) = M/p_i$ such that*

$$a * F_j \subset A, \quad a_i * F_k \subset A, \quad a * F_k \not\subset A. \tag{8.4}$$

(In particular, this is true if (C) holds.) Suppose that $\Phi_M \Phi_{M/p_j} | A$. Then we have the following.

(i) For all $a_k \in A$ such that $(a_k - a_i, M) = M/p_k$ and $\mathbb{A}_{M/p_i}[a_k | \ell_k(a)] = 0$, we have

$$\mathbb{A}_{M/p_j^2}[a] + \mathbb{A}_{M/p_j^2}[a_k] \geq \phi(p_j^2),$$

(ii) M/p_j^2, $M/p_i p_j^2 p_k \in \mathrm{Div}(A)$. Moreover, if $\mathbb{A}_{M/p_j^2}[a] > 0$, then $M/p_i p_j^2 \in \mathrm{Div}(A)$, and if $\mathbb{A}_{M/p_j^2}[a] < \phi(p_j^2)$, then $M/p_j^2 p_k \in \mathrm{Div}(A)$.

Proof. Under the assumptions of the lemma, A is $\mathcal{T}$-null with respect to the cuboid type $\mathcal{T} = (M, \vec{\delta}, 1)$, where $\vec{\delta} = (1, 2, 1)$. Now the lemma follows by considering all possible $\mathcal{T}$ cuboids with vertices at a, a_i, a_k and x_k, where $x_k \notin A$ satisfies $(x_k - a, M) = M/p_k$ and $(x_k - a_k, M) = M/p_i$. □

Corollary 8.3. *Assume (C). Then $\Phi_{N_j} \Phi_{N_k} \nmid A$.*

Proof. Assume that $\Phi_{N_j} \Phi_{N_k} | A$. By Lemma 8.2 (i), we have

$$|A \cap (\Pi(a, p_i^{n_i}) \cup \Pi(a_i, p_i^{n_i}))| \geq p_j + \phi(p_j^2) + p_k + \phi(p_k^2) = p_j^2 + p_k^2 > 2 p_j p_k$$

hence $\max\{|A \cap \Pi(a, p_i^{n_i})|, |A \cap \Pi(a_i, p_i^{n_i})|\} > p_j p_k$, which contradicts Lemma 4.3. □

By Corollary 8.3, it suffices to consider the following sets of assumptions.

Assumption (C1): Assume that (C) holds, $p_k < p_j$, and $\Phi_{N_k}|B$.

Assumption (C2): Assume that (C) holds, $p_k < p_j$, $\Phi_{N_j}|B$, and

$$M/p_jp_k \notin \mathrm{Div}(B). \tag{8.5}$$

(In particular, (8.5) holds if either $\mathbb{A}_{M/p_jp_k}[a] > 0$ or $\mathbb{A}_{M/p_jp_k}[a_i] > 0$.)

Assumption (C3): Assume that (C) holds, $p_k < p_j$, $\Phi_{N_j}|B$, and

$$\mathbb{A}_{M/p_jp_k}[a] = \mathbb{A}_{M/p_jp_k}[a_i] = 0.$$

We first prove in Section 8.1 that under either of the assumptions (C1) or (C2), the conclusion of Theorem 8.1 holds. We then prove in Section 8.2 that if (C3) holds, then we must also have $\Phi_{N_k}|B$, so that we may return to (C1) to complete the proof.

8.1. **Cases (C1) and (C2).** The first two cases are similar and will be considered together.

Lemma 8.4. *Assume that (C1) holds with $n_k = 2$. Then:*

(i) Every element of B belongs to an N_k-fiber in B in either the p_j or p_k direction (not necessarily the same for all b).

(ii) Furthermore, suppose that there is a $b \in B$ that does not belong to an N_k fiber in the p_j direction in B. Then for all $x_j \in \mathbb{Z}_M \setminus A$ such that $(a_i - x_j, M) = M/p_j$ we have $A_{x_j,b} \subset \ell_k(x_j)$, with the product saturated by an N_k-fiber in B in the p_k direction, rooted at b, and an M-cofiber in A at distance M/p_k^2 from x_j.

Moreover, this lemma does not require the assumption $p_k < p_j$, therefore the same conclusions with j and k interchanged hold under the assumptions (C2) and (C3).

Proof. Observe that, by (8.3),

$$\mathbb{B}^{N_k}_{N_k/p_i}[b] = 0 \text{ for all } b \in B. \tag{8.6}$$

Assume first that $p_i = 2$, and suppose that there is a $b_0 \in B$ that does not belong to an N_k-fiber in either the p_k or the p_j direction. In this case, by (8.6) and Lemma 6.5 with $N = N_k$ we have

$$N_k/p_k = M/p_k^2 \in \mathrm{Div}(B). \tag{8.7}$$

Let $x_j \in \mathbb{Z}_M \setminus A$ with $(a_i - x_j, M) = M/p_j$, and consider the saturating set A_{x_j}. Let a_j be the element of A satisfying $(a - a_j, M) = M/p_j, (x_j - a_j, M) = M/p_i$. We have

$$A_{x_j,b_0} \subset \ell_k(x_j) \cup \ell_k(a_i) \cup \ell_k(a) \cup \ell_k(a_j).$$

By (8.3), no top level divisors except possibly M/p_jp_k are available to contribute to the product $\langle \mathbb{A}[x_j], \mathbb{B}[b_0]\rangle$. Therefore, if $A_{x_j,b_0} \cap \ell_k(a_j) \neq \emptyset$, we must have

$$\mathbb{A}_{M/p_ip_k^2}[x_j|\ell_k(a_j)] = \mathbb{A}_{M/p_k^2}[a_j] > 0$$

which contradicts (8.7). Similarly $A_{x_j,b_0} \cap \ell_k(a) \neq \emptyset$ implies

$$\mathbb{A}_{M/p_ip_jp_k^2}[x_j|\ell_k(a)] = \mathbb{A}_{M/p_k^2}[a] > 0$$

which, again, contradicts (8.7). Next, if $A_{x_j,b_0} \cap \ell_k(a_i) \neq \emptyset$, then by (8.7)

$$\mathbb{A}^{N_k}_{N_k/p_j}[x_j|\ell_k(a_i)]\mathbb{B}^{N_k}_{N_k/p_j}[b_0] = \mathbb{A}_{M/p_jp_k}[x_j|\ell_k(a_i)]\mathbb{B}_{M/p_jp_k}[b_0] > 0, \tag{8.8}$$

and if $A_{x_j,b} \cap \ell_k(x_j) \neq \emptyset$, then

$$\mathbb{A}^{N_k}_{N_k/p_k}[x_j]\mathbb{B}^{N_k}_{N_k/p_k}[b_0] = \mathbb{A}_{M/p_k^2}[x_j]\mathbb{B}_{M/p_k^2}[b_0] > 0. \tag{8.9}$$

By Lemma 4.1, for a fixed b_0 we cannot have both (8.8) and (8.9). Hence either $A_{x_j,b_0} \subset \ell_k(a_i)$ or $A_{x_j,b_0} \subset \ell_k(x_j)$. In the first case, we have

$$\mathbb{A}_{M/p_jp_k}[x_j|\ell_k(a_i)]\mathbb{B}_{M/p_jp_k}[b_0] = \mathbb{A}_{M/p_k}[a_i]\mathbb{B}_{M/p_jp_k}[b_0] = \phi(p_jp_k);$$

since $\mathbb{A}_{M/p_k}[a_i] = \phi(p_k)$, it follows that $\mathbb{B}_{M/p_jp_k}[b_0] = \mathbb{B}^{N_k}_{N_k/p_j}[b_0] = \phi(p_j)$, hence b_0 belongs to an N_k-fiber in the p_j direction, contradicting the choice of b_0. In the second case, we have

$$\mathbb{A}_{M/p_k^2}[x_j]\mathbb{B}_{M/p_k^2}[b_0] = \phi(p_k^2);$$

since $M/p_k \in \mathrm{Div}(A)$, this is only possible if A and B contain a configuration as in part (ii) of the lemma. In this case, it follows that b_0 belongs to an N_k-fiber in the p_k direction, contradicting our initial choice of b_0. This concludes the proof of (i). Furthermore, if the conclusion of (ii) fails for any x_j chosen as above, then we must always be in the first case for that x_j, and therefore every $b \in B$ must in fact belong to an N_k fiber in the p_j direction, a contradiction.

Assume now that $p_i > 2$. Then part (i) of the lemma follows directly from Lemma 4.9, (8.6) and the fact that B satisfies (6.12) with $N = N_k$ and $c = 1$. For (ii), suppose that there is a $b \in B$ that does not belong to an N_k fiber in B in the p_j direction. By (i), there is an N_k fiber in B in the p_k direction containing b. In particular, this implies that (8.7) holds. The rest of the proof of (ii) is the same as for the $p_i = 2$ case. □

Corollary 8.5. *(i) Assume (C1) with $n_k = 2$. Then the pair (A, B) has a $(1, 2)$-cofibered structure in the p_k direction. Moreover, for each $x_j \in \mathbb{Z}_M \setminus A$ such that $(a_i - x_j, M) = M/p_j$, A has a cofiber at distance M/p_k^2 from x_j.*

(ii) Assume (C2) with $n_j = 2$, then the same conclusion holds with k and j interchanged.

Proof. (i) Let $b \in B$. By Lemma 8.4, it suffices to prove that we cannot have $\mathbb{B}_{M/p_jp_k}[b] = \phi(p_j)$. In fact, we will show

$$\mathbb{B}_{M/p_jp_k}[b] \leq \phi(p_k); \tag{8.10}$$

moreover, this holds independently of the assumption $\Phi_{N_k}|B$. Indeed, by the corner assumption we have $\mathbb{A}_{M/p_jp_k}[x_k] \geq \phi(p_j)$ for all $x_k \in \mathbb{Z}_M \setminus A$ with $(x_k - a, M) = M/p_k$. Hence

$$1 \geq \frac{1}{\phi(p_jp_k)}\mathbb{A}_{M/p_jp_k}[x_k]\mathbb{B}_{M/p_jp_k}[b] \geq \frac{1}{\phi(p_k)}\mathbb{B}_{M/p_jp_k}[b]$$

as required.

(ii) By (8.5), we have $\mathbb{B}_{M/p_jp_k}[b] = 0$. The conclusion follows again from Lemma 8.4. □

The rest of the proof is the same in cases (C1) and (C2), except that p_j and p_k are interchanged. Without loss of generality, we assume that (C1) holds.

By Lemma 2.9, we may shift each of the cofibers provided by Corollary 8.5 to its respective point x_j. Let $A' \subset \mathbb{Z}_M$ be the set thus obtained. Then A is T2-equivalent to A', we have $A' \oplus B = \mathbb{Z}_M$, and for every $x_j \in \mathbb{Z}_M \setminus A$ with $(x_j - a_i, M) = M/p_j$, we have $x_j * F_k \subset A'$. Furthermore, A' differs from A only along the lines $\ell_k(x_j)$ with x_j as above, hence it follows

from (C) that for every $a_j \in A$ with $(a_j - a_i, M) = M/p_j$, we also have $a_j * F_k \subset A'$. Therefore $a_i * F_j * F_k \subset A'$. By Lemma 4.3, we must in fact have

$$A' \cap \Pi(a_i, p_i^{n_i}) = a_i * F_j * F_k. \tag{8.11}$$

Corollary 8.6. *Assume that (C1) holds with $n_k = 2$. Then $A' \subset \Pi(a, p_i^{n_i-1})$, where a is as in (C). Moreover, $\Phi_{p_i^{n_i}} | A'$.*

Proof. By (8.11) and (C), we have $|A' \cap \Pi(a_i, p_i^{n_i-1})| > p_j p_k$. The conclusion now follows from Corollary 4.4. □

If $n_i = n_k = 2$, we have $p_i \parallel |B|$, hence Corollary 8.6 and Theorem 2.5 imply that A' and B satisfy (T2). Since A and A' are T2-equivalent, the same is true for A. The additional arguments below are needed to prove the full conclusion of Theorem 8.1, namely, T2-equivalence between A' (hence A) and Λ for $n_i \geq 2$. This is needed for the classification result in Theorem 3.1, as well as for the applications in Section 9.

Corollary 8.7. *Assume that (C1) holds with $n_j = n_k = 2$, and define A' as above. Then the pair (A', B) has a $(1, 2)$-cofibered structure in the p_j direction, with cofibers in A' at distance M/p_j^2 from each $x_k \in \mathbb{Z}_M \setminus A$ such that $(x_k - a, M) = M/p_k$.*

Proof. By (8.11) and (C), we have $\{D(M)|m|M\} \subset \mathrm{Div}(A')$. The corollary follows by applying Lemma 4.6 to each x_k. □

We can now complete the proof under the assumption (C1). By Corollary 8.6, we have $\Phi_{p_i^{n_i}} \mid A'$. This together with (8.11) implies that

$$A' \subset \bigcup_{z \in a_i * F_i} \Pi(z, p_i^{n_i}), \tag{8.12}$$

and that for each plane $\Pi_z := \Pi(z, p_i^{n_i})$ with $z \in a_i * F_i$, we have

$$|A' \cap \Pi_z| = p_j p_k. \tag{8.13}$$

Applying Lemma 4.8 to A', and using (8.11), we get that each set $A' \cap \Pi_z$ is a disjoint union of M-fibers in the p_j and p_k directions. However, we also know that B is N_j-fibered in the p_j direction and N_k-fibered in the p_k direction, hence

$$\{M/p_j^2, M/p_k^2, M/p_j^2 p_k^2\} \subset \mathrm{Div}(B). \tag{8.14}$$

It follows that each set $A' \cap \Pi_z$ must in fact be M-fibered in one of the p_j and p_k directions. Assuming without loss of generality that $A' \cap \Pi_z$ is M-fibered in the p_j direction for some $z \in a_i * F_i$, and taking (8.13) into account, we get

$$A' \cap \Pi_z = \{u_1, \dots, u_{p_k}\} * F_j,$$

where for each $\nu \neq \nu'$ we have $(u_\nu - u_{\nu'}, M) \in \{M/p_k, M/p_j^2 p_k\}$. Using the cofibered structure in the p_j direction, and considering each $u_\nu * F_j$ as cofiber, we apply Lemma 2.9 if necessary to reduce to the case where $p_j \mid a_i - u_\nu$ for all ν. This aligns the fibers in Π_z to a grid $u_1 * F_j * F_k$, fibered in both directions. If we do not have $p_k \mid a_i - u_1$ at this point, we apply Lemma 2.9 again, this time in the p_k direction. Repeating this procedure in each plane, we get that A' is T2-equivalent to $A^\flat = \Lambda$. This ends the proof in this case.

8.2. **Case (C3).** In this case, we are assuming that $\Phi_{N_j}|B$ and $\mathbb{A}_{M/p_jp_k}[a] = \mathbb{A}_{M/p_jp_k}[a_i] = 0$. By (C), this implies that $\mathbb{A}_{M/p_k}[a] = \mathbb{A}_{M/p_j}[a_i] = 0$.

Lemma 8.8. *Assume (C3) with $n_k = 2$. Let $a_j \in A$ and $x_j \in \mathbb{Z}_M \setminus A$ satisfy $(a_j - x_j, M) = M/p_i$ and $(a - a_j, M) = (a_i - x_j, M) = M/p_j$. If $\Phi_{N_k} \nmid B$, then $A_{x_j} \subset \ell_k(a_i)$.*

Proof. The assumptions of the lemma imply that $\Phi_{N_k}|A$. Applying Lemma 8.2 to A with j and k interchanged, we have

$$M/p_k^2,\ M/p_ip_jp_k^2 \in \mathrm{Div}(A), \tag{8.15}$$

$$\mathbb{A}_{M/p_k^2}[a_i] + \mathbb{A}_{M/p_k^2}[a_j] \geq \phi(p_k^2). \tag{8.16}$$

Let $b \in B$, and consider the saturating set $A_{x_j,b}$. We have

$$A_{x_j,b} \subset \ell_k(x_j) \cup \ell_k(a_i) \cup \ell_k(a) \cup \ell_k(a_j).$$

Using (8.3) and (8.15), we conclude that $A_{x_j,b} \cap \ell_k(x_j) = \emptyset$ since $M/p_k, M/p_k^2 \notin \mathrm{Div}(B)$, and similarly $A_{x_j,b} \cap \ell_k(a) = \emptyset$ since $M/p_ip_j, M/p_ip_jp_k, M/p_ip_jp_k^2 \notin \mathrm{Div}(B)$.

Suppose that $A_{x_j,b} \cap \ell_k(a_j) \neq \emptyset$. By (8.3) again, this implies that

$$\mathbb{A}_{M/p_ip_k^2}[x_j|\ell_k(a_j)]\mathbb{B}_{M/p_ip_k^2}[b] > 0,$$

so that $\mathbb{A}_{M/p_ip_k^2}[a|\ell_k(a_i)] = \mathbb{A}_{M/p_k^2}[a_i] = 0$. By (8.16) we have $\mathbb{A}_{M/p_k^2}[a_j] = \phi(p_k^2)$ for all $(a_j - a, M) = M/p_j$. We will prove this contradicts Lemma 4.3. Indeed, we have

$$|A \cap \Pi(a, p_i^{n_i})| \geq \mathbb{A}_M[a] + \mathbb{A}_{M/p_j}[a] + \sum_{(a-a_j,M)=M/p_j} \mathbb{A}_{M/p_k^2}[a_j] = p_j + \phi(p_jp_k^2)$$

A simple calculation shows that the last expression exceeds p_jp_k if and only if $p_j > 1 + \frac{1}{\phi(p_k)}$, which holds true since $p_j > p_k \geq 2$. We therefore conclude that $A_{x_j,b} \subset \ell_k(a_i)$. □

Proposition 8.9. *(i) Assume (C3) with $n_j = n_k = 2$. Then $\Phi_{N_k}|B$.*

(ii) Assume that $p_i = 2$, and that (C3) holds with $n_k = 2$. (In this case, we do not need to assume that $n_j = 2$.) Then $\Phi_{N_k}|B$.

Proof. We prove (ii) first, since the proof in this case is immediate and straightforward. Assume, by contradiction, that $p_i = 2$ and $\Phi_{N_k} \nmid B$, so that $\Phi_{N_k}|A$. By Lemma 8.2, (8.16) holds for all $a_j \in A$ with $(a - a_j, M) = M/p_j$. By Lemma 4.3 we have

$$\begin{aligned} p_ip_k &\geq |A \cap \Pi(a, p_j^2)| \\ &\geq \mathbb{A}_M[a] + \mathbb{A}_M[a_i] + \mathbb{A}_{M/p_k}[a_i] + \mathbb{A}_{M/p_k^2}[a_i] \\ &= p_k + 1 + \mathbb{A}_{M/p_k^2}[a_i]. \end{aligned}$$

Since $p_i = 2$, we get $\mathbb{A}_{M/p_k^2}[a_i] \leq \phi(p_k)$. From (8.16) we deduce $\mathbb{A}_{M/p_k^2}[a_j] \geq \phi(p_k^2) - \phi(p_k)$, hence

$$\begin{aligned} |A \cap \Pi(a, p_i^{n_i})| &\geq \mathbb{A}_M[a] + \mathbb{A}_{M/p_j}[a] + \sum_{a_j:(a_j-a,M)=M/p_j} \mathbb{A}_{M/p_k^2}[a_j] \\ &\geq p_j + \phi(p_j)(\phi(p_k^2) - \phi(p_k)) \\ &= p_j + \phi(p_j)\phi(p_k)^2 \end{aligned}$$

We show the latter exceeds p_jp_k, which contradicts Lemma 4.3. Indeed, $p_j+\phi(p_j)\phi(p_k)^2 > p_jp_k$ if and only if $\phi(p_j)\phi(p_k)^2 > p_j\phi(p_k)$. This, in turn, is equivalent to $p_k > 2+\frac{1}{\phi(p_j)}$, which clearly holds true since $p_k \geq 3$.

We now prove (i). Assume that $p_i > 2$, and let $b \in B$. Applying Lemma 8.4 to B with p_j and p_k interchanged, we get that at least one of the following holds:

$$\mathbb{B}^{N_j}_{N_j/p_j}[b] = \phi(p_j), \tag{8.17}$$

$$\mathbb{B}^{N_j}_{N_j/p_k}[b] = \phi(p_k). \tag{8.18}$$

Assume, by contradiction, that $\Phi_{N_k} \nmid B$. Then $\Phi_{N_k}|A$. Let a, a_i, a_j, and x_j be as in Lemma 8.8. By Lemma 8.8, we have $A_{x_j,b} \subset \ell_k(a_i)$, hence

$$\frac{1}{\phi(p_jp_k)}\mathbb{A}_{M/p_jp_k}[x_j|\ell_k(a_i)]\mathbb{B}_{M/p_jp_k}[b] + \frac{1}{\phi(p_jp_k^2)}\mathbb{A}_{M/p_jp_k^2}[x_j|\ell_k(a_i)]\mathbb{B}_{M/p_jp_k^2}[b] = 1. \tag{8.19}$$

Notice that

$$\mathbb{A}_{M/p_jp_k}[x_j|\ell_k(a_i)] = \phi(p_k),\ \mathbb{B}_{M/p_jp_k}[b] \leq \phi(p_k) \tag{8.20}$$

where the former follows from the corner structure and the latter from (8.10). Applying this to (8.19), we get

$$\mathbb{A}_{M/p_jp_k^2}[x_j|\ell_k(a_i)]\mathbb{B}_{M/p_jp_k^2}[b] > 0, \tag{8.21}$$

and in particular $M/p_jp_k^2 \in \mathrm{Div}(B)$. We claim that this implies that

$$\mathbb{A}_{M/p_jp_k^2}[x_j|\ell_k(a_i)] = \mathbb{A}_{M/p_k^2}[a_i] = \phi(p_k^2) \tag{8.22}$$

Indeed, suppose that (8.22) fails, then $\mathbb{A}_{M/p_k^2}[a_i] < \phi(p_k^2)$. By (8.16), we have $\mathbb{A}_{M/p_k^2}[a_j] > 0$, hence $\mathbb{A}_{M/p_jp_k^2}[a] > 0$. It follows that $M/p_jp_k^2 \in \mathrm{Div}(A) \cap \mathrm{Div}(B)$, which is a contradiction.

Lemma 8.10. *Assume that (C3) holds with $n_j = n_k = 2$, but $\Phi_{N_k} \nmid B$. Then there is at least one $b_0 \in B$ for which (8.17) holds and (8.18) fails.*

Proof. Suppose that the lemma is false, so that (8.18) holds for all $b \in B$. We first prove that then

$$\forall b \in B,\quad p_k \,|\, \mathbb{B}_{M/p_jp_k^2}[b]. \tag{8.23}$$

Indeed, by the corner structure and (8.15) we have

$$M/p_j, M/p_k, M/p_k^2 \notin \mathrm{Div}(B), \tag{8.24}$$

Let $b \in B$. By (8.18), there is a p_k-tuple of elements $\{b_0 = b, b_1, \ldots, b_{p_k-1}\} \subset B$ such that $b_\nu \in \Lambda(b, M/p_jp_k)$. By (8.24), we must have $(b_\nu - b_{\nu'}, M) = M/p_jp_k$ for all $\nu \neq \nu'$.

Suppose now that $b' \in B$ satisfies $(b-b', M) = p_jp_k^2$, then b' belongs to a similar p_k-tuple $\{b'_0 = b', b'_1, \ldots, b'_{p_k-1}\} \subset B$ with $(b'_\nu - b'_{\nu'}, M) = M/p_jp_k$ for all $\nu \neq \nu'$. By (8.24) again, we must have $(b_\nu - b'_{\nu'}, M) = p_jp_k^2$ for all ν, ν'. Hence the p_k-tuples associated with different elements of B are either identical or disjoint. This proves (8.23).

Applying (8.20), (8.18), (8.22), and (8.23) to (8.19), we get that

$$1 = \frac{\phi(p_k)^2}{\phi(p_jp_k)} + \frac{\phi(p_k^2)\cdot Cp_k}{\phi(p_jp_k^2)} = \frac{\phi(p_k)+Cp_k}{\phi(p_j)}$$

for some integer C. But this implies $p_j = p_k(C+1)$, which is impossible since p_j is prime. This proves the lemma. □

By Lemma 8.10, there exists a $b_0 \in B$ such that $\mathbb{B}^{N_j}_{N_j/p_k}[b_0] < \phi(p_k)$ and $\mathbb{B}^{N_j}_{N_j/p_j}[b_0] = \phi(p_j)$. In particular, $M/p_j^2 \in \mathrm{Div}(B, b_0)$. Applying (8.21) to b_0, we get that $M/p_jp_k^2 \in \mathrm{Div}(B, b_0)$. Finally, if b', b'' are elements of B with $(b_0 - b', M) = M/p_j^2$ and $(b_0 - b', M) = M/p_jp_k^2$, then $(b'' - b', M) = M/p_j^2p_k^2$. Therefore

$$M/p_j^2, M/p_jp_k^2, M/p_j^2p_k^2 \in \mathrm{Div}(B). \tag{8.25}$$

Lemma 8.11. *Assume that (C3) holds with $n_j = n_k = 2$, but $\Phi_{N_k} \nmid B$. Then*

$$|A \cap \Pi(a_i, p_i^{n_i})| = p_k^2. \tag{8.26}$$

Proof. Using (8.22) and the fact that $\mathbb{A}_{M/p_k}[a_i] = \phi(p_k)$, we have

$$|A \cap \ell_k(a_i)| = p_k^2. \tag{8.27}$$

On the other hand, we claim that

$$\mathbb{A}_m[a_i] = 0 \ \forall m | M \text{ such that } p_i^{n_i} | m | M, \ m \neq M, M/p_k, M/p_k^2. \tag{8.28}$$

Indeed, we can exclude the divisors in (8.25). Furthermore, $M/p_j, M/p_jp_k \notin \mathrm{Div}(A, a_i)$ by the p_i corner assumption. It remains to check that $M/p_j^2p_k \notin \mathrm{Div}(A, a_i)$. If we had an element $a' \in A$ with $(a_i - a', M) = M/p_j^2p_k$, then there would be an element $a_k \in a_i * F_k \subset A$ with $(a_k - a', M) = M/p_j^2$, which is again prohibited by (8.25).

By (8.28), all elements of A in the plane $\Pi(a_i, p_i^{n_i})$ must in fact lie on the line $\ell_k(a_i)$, so that

$$|A \cap \Pi(a_i, p_i^{n_i})| = |A \cap \ell_k(a_i)| = p_k^2$$

as claimed. □

Lemma 8.12. *Assume that (C3) holds with $n_j = n_k = 2$, but $\Phi_{N_k} \nmid B$. Let $a_k \in A, x_k \in \mathbb{Z}_M \setminus A$ with $(a_k - x_k, M) = M/p_i, (a_i - a_k, M) = (a - x_k, M) = M/p_k$. Then for b_0 as above,*

$$A_{x_k, b_0} \subset \ell_j(x_k) \tag{8.29}$$

Proof. We have

$$A_{x_k,b_0} \subset \ell_j(x_k) \cup \ell_j(a_i) \cup \ell_j(a) \cup \ell_j(a_k).$$

Observe first that $A_{x_k,b_0} \cap (\ell_j(a_k) \cup \ell_j(a_i)) = \emptyset$, for otherwise, by (8.3) we would have either $\mathbb{A}_{M/p_j^2}[a_i] > 0$ or $\mathbb{A}_{M/p_j^2}[a_k] > 0$; both are not allowed due to (8.25).

Next, we claim that we cannot have both $A_{x_k,b_0} \cap \ell_j(a) \neq \emptyset$ and $A_{x_k,b_0} \cap \ell_j(x_k) \neq \emptyset$. Indeed, the former implies $\mathbb{A}_{M/p_jp_k}[x_k|\ell_j(a)]\mathbb{B}_{M/p_jp_k}[b_0] > 0$ and the latter implies $\mathbb{A}_{M/p_j^2}[x_k]\mathbb{B}_{M/p_j^2}[b_0] > 0$; having both would contradict Lemma 4.1.

Suppose that $A_{x_k,b_0} \subset \ell_j(a)$. Then

$$1 = \frac{1}{\phi(p_jp_k)}\mathbb{A}_{M/p_jp_k}[x_k|\ell_j(a)]\mathbb{B}_{M/p_jp_k}[b_0] + \frac{1}{\phi(p_j^2p_k)}\mathbb{A}_{M/p_j^2p_k}[x_k|\ell_j(a)]\mathbb{B}_{M/p_j^2p_k}[b_0] \tag{8.30}$$

We chose b_0 so that $\mathbb{B}_{M/p_jp_k}[b_0] < \phi(p_k)$. By the corner assumption, $\mathbb{A}_{M/p_jp_k}[x_k|\ell_j(a)] = \phi(p_j)$. We also have $\mathbb{A}_{M/p_j^2p_k}[x_k|\ell_j(a)] = \mathbb{A}_{M/p_j^2}[a] = 0$, by (8.25). Therefore the right side of (8.30) is strictly less than $\frac{1}{\phi(p_jp_k)}\phi(p_j)\phi(p_k) = 1$, a contradiction. The lemma follows. □

We now complete the proof of the proposition. By Lemma 8.12, we have $A_{x_k,b_0} \subset \ell_j(x_k)$, hence

$$\mathbb{A}_{M/p_j^2}[x_k]\mathbb{B}_{M/p_j^2}[b_0] = \phi(p_j^2)$$

for all $x_k \in \mathbb{Z}_M \setminus A$ with $(x_k - a, M) = M/p_k$. This can only happen when

$$\mathbb{A}_{M/p_j^2}[x_k] = p_j \tag{8.31}$$

and $\mathbb{B}_{M/p_j^2}[b_0] = \phi(p_j)$. Applying this to all x_k as above (while keeping in mind that $\mathbb{A}_{M/p_k}[a] = 0$), we have $|A \cap \ell_j(x_k)| \geq p_j$. It follows that

$$|A \cap (\Pi(a, p_i^{n_i}))| \geq p_j p_k > p_k^2.$$

On the other hand, by (8.26) we have $|A \cap \Pi(a_i, p_i^{n_i})| = p_k^2$. It follows that $\Phi_{p_i^{n_i}} \nmid A$, since otherwise the number of elements of A in both planes would be the same.

It remains to prove that we must also have $\Phi_{p_i^{n_i}} \nmid B$, which provides the final contradiction. Indeed, we use the following

$$\frac{1}{\phi(p_j)}\mathbb{A}_{M/p_j}[a] = \frac{1}{p_j}\mathbb{A}_{M/p_j^2}[x_k] = \frac{1}{\phi(p_k^2)}\mathbb{A}_{M/p_k^2}[a_i] = \frac{1}{\phi(p_k)}\mathbb{A}_{M/p_k}[a_i] = 1,$$

where the first and last part follow from the corner structure, the second part from (8.31), and the third part from (8.22). It is easy to see that this configuration implies that $\{p_i^{n_i-1} \parallel m | M\} \subset \mathrm{Div}(A)$. This means that for every $b \in B$

$$|B \cap (\Pi(b, p_i^{n_i-1}))| = |B \cap (\Pi(b, p_i^{n_i}))|$$

hence $\Phi_{p_i^{n_i}} \nmid B$. This gives the desired contradiction and ends the proof of the proposition. □

Proposition 8.9 implies that (C1) holds, and we can now follow the rest of the proof for that case.

9. Fibered grids

Throughout most of this section we will work under the following assumption.

Assumption (F): We have $A \oplus B = \mathbb{Z}_M$, where $M = p_i^2 p_j^2 p_k^2$ is odd. Furthermore, $|A| = |B| = p_i p_j p_k$, $\Phi_M | A$, and A is fibered on $D(M)$-grids.

Let $\mathcal{I}$ be the set of elements of A that belong to an M-fiber in the p_i direction, that is,

$$\mathcal{I} = \{a \in A \,|\, \mathbb{A}_{M/p_i}[a] = \phi(p_i)\}.$$

The sets $\mathcal{J}$ and $\mathcal{K}$ are defined similarly. The assumption (F) implies that every element of A belongs to an M-fiber in some direction, hence $A = \mathcal{I} \cup \mathcal{J} \cup \mathcal{K}$. We emphasize that this does *not* have to be a disjoint union and that it is possible for an element of A to belong to two or three of these sets.

Our main result on fibered grids is the following theorem.

Theorem 9.1. *Assume that (F) holds.*

(I) *If $\mathcal{I} \cap \mathcal{J} \cap \mathcal{K} \neq \emptyset$, then the tiling $A \oplus B = \mathbb{Z}_M$ is T2-equivalent to $\Lambda \oplus B = \mathbb{Z}_M$, where $\Lambda := \Lambda(0, D(M))$. By Corollary 2.2, both A and B satisfy (T2).*

(II) *Assume that $\mathcal{I} \cap \mathcal{J} \cap \mathcal{K} = \emptyset$. Then, after a permutation of the i, j, k indices if necessary, the following holds.*

(II a) *At least one of the sets $\mathcal{I}, \mathcal{J}, \mathcal{K}$ is empty. Without loss of the generality, we may assume that $\mathcal{I} = \emptyset$, so that $A \subset \mathcal{J} \cup \mathcal{K}$.*

(II b) *If $A \subset \mathcal{J}$ or $A \subset \mathcal{K}$, then A is M-fibered in the p_j or p_k direction, respectively. Consequently, the conditions of Theorem 2.6 are satisfied in that direction. By Corollary 2.7, both A and B satisfy (T2).*

(II c) *Suppose that $\mathcal{I} = \emptyset$, and that $\mathcal{J} \setminus \mathcal{K}$ and $\mathcal{K} \setminus \mathcal{J}$ are both nonempty.*

- *If $\Phi_{p_i}|A$, then, after interchanging A and B, the conditions of Theorem 2.6 are satisfied in the p_i direction. By Corollary 2.7, both A and B satisfy (T2).*
- *If $\Phi_{p_i^2}|A$, then $A \subset \Pi(a, p_i)$ for any $a \in A$. By Theorem 2.5, both A and B satisfy (T2).*

The proof of Theorem 9.1 is organized as follows. We will consider the following sets of assumptions.

Assumption (F'): We have $A \oplus B = \mathbb{Z}_M$, where $M = p_i^2 p_j^2 p_k^2$. (Note that M is not required to be odd). Furthermore, $|A| = |B| = p_i p_j p_k$, $\Phi_M|A$, and A is fibered on $D(M)$-grids.

Assumption (F1): We have $A \oplus B = \mathbb{Z}_M$, where $M = p_i^2 p_j^2 p_k^2$ is odd. Furthermore, $|A| = |B| = p_i p_j p_k$, $\Phi_M|A$, A is fibered on $D(M)$-grids, and $\mathcal{I}, \mathcal{J}, \mathcal{K}$ are pairwise disjoint.

Assumption (F2): We have $A \oplus B = \mathbb{Z}_M$, where $M = p_i^2 p_j^2 p_k^2$ is odd. Furthermore, $|A| = |B| = p_i p_j p_k$, $\Phi_M|A$, A is fibered on $D(M)$-grids, $\mathcal{I} \cap \mathcal{J} \cap \mathcal{K} = \emptyset$, and $\mathcal{J} \cap \mathcal{K} \neq \emptyset$.

Assumption (F3): We have $A \oplus B = \mathbb{Z}_M$, where $M = p_i^2 p_j^2 p_k^2$ is odd. Furthermore, $|A| = |B| = p_i p_j p_k$, $\Phi_M|A$, A is fibered on $D(M)$-grids, $\mathcal{I} = \emptyset$, $\mathcal{J} \setminus \mathcal{K} \neq \emptyset$, and $\mathcal{K} \setminus \mathcal{J} \neq \emptyset$.

We prove part (I) of Theorem 9.1 in Corollary 9.6; in fact, this part holds under the weaker assumption (F'). Assume now that $\mathcal{I} \cap \mathcal{J} \cap \mathcal{K} = \emptyset$. In that case, we first prove part (II a) of Theorem 9.1. While the conclusion is the same, the methods of proof will be very different, so that it is preferable to split this part into two results.

Proposition 9.2. *Assume that (F1) holds. Then one of the sets $\mathcal{I}, \mathcal{J}, \mathcal{K}$ is empty.*

Proposition 9.3. *Assume that (F2) holds. Then $\mathcal{I} = \emptyset$.*

Relabeling the primes if necessary, we may assume that $\mathcal{I} = \emptyset$ in the case (F1) as well. If A is M-fibered in one of the p_j or p_k directions, then (II b) holds, and we are done. It remains to consider the case covered in (F3). The following result completes the proof of the theorem.

Proposition 9.4. *Assume (F3). Then the conclusion (II c) of Theorem 9.1 holds.*

We briefly discuss the notation used in this section. For $N|M$, we will use $\mathbb{I}^N$, $\mathbb{J}^N$, $\mathbb{K}^N$ to denote the N-boxes associated with $\mathcal{I}$, $\mathcal{J}$, $\mathcal{K}$. We continue to write

$$F_i = \{0, M/p_i, \dots, (p_i - 1)M/p_i\}, \tag{9.1}$$

with F_j, F_k defined similarly. Recall also that

$$M_i = M/p_i^2, \quad M_j = M/p_j^2, \quad M_k = M/p_k^2. \tag{9.2}$$

Note that $M_i = p_j^2 p_k^2$ has only two distinct prime factors, and similarly for M_j, M_k. In particular, all M_ν-cuboids are 2-dimensional for $\nu \in \{i, j, k\}$, and all conclusions of Lemma 4.7 apply on that scale. Thus, if $\Phi_{M_i}|A$, then A mod M_i is a linear combination of M_i-fibers in the p_j and p_k directions, with non-negative integer coefficients. In particular, if $\Phi_{M_i}|A$ and

$$\mathbb{A}_{M_i}^{M_i}[x] \in \{0, c_0\} \quad \forall x \in \mathbb{Z}_M,$$

then A is M_i-fibered in one of the p_j and p_k directions on every $D(M_i)$-grid. Similar statements hold with A replaced by B, as well as for other permutations of the indices i, j, k.

9.1. **Intersections of $\mathcal{I}, \mathcal{J}, \mathcal{K}$.** It is possible for any of the sets $\mathcal{I}, \mathcal{J}$ and $\mathcal{K}$ to intersect the others. Furthermore, given a $D(M)$-grid Λ, $A \cap \Lambda$ can contain fibers in some direction without necessarily being fibered in that direction. For example, consider the set

$$A_0 = F_i * \big[(F_j * a) \cup (F_k * a)\big]. \tag{9.3}$$

Then $a \in \mathcal{I} \cap \mathcal{J} \cap \mathcal{K}$, but A_0 is not fibered in either the p_j or the p_k direction.

Nonetheless, the condition (F) places significant limits on the ways in which $\mathcal{I}, \mathcal{J}, \mathcal{K}$ may intersect, as provided by the following structure lemma. In fact, the weaker condition (F') is sufficient for this purpose.

Lemma 9.5. *Assume (F'), and suppose that $a \in \mathcal{J} \cap \mathcal{K}$ for some $a \in A$. Let $D = D(M)$.*

(i) If $A \cap \Lambda(a, D)$ is M-fibered in at least one of the p_j and p_k directions, then

$$A \cap \Pi(a, p_i^2) = a * F_j * F_k.$$

In particular, this holds if $a \notin \mathcal{I}$.

(ii) If $A \cap \Lambda(a, D)$ is M-fibered in the p_i direction, then $A_0 \subset A \cap \Lambda(a, D)$, where A_0 is the set in (9.3).

Proof. (i) Assume, without loss of generality, that $A \cap \Lambda(a, D)$ is M-fibered in the p_k direction. Since $a \in \mathcal{J}$, we have $a * F_j \subset A$, and the fibering assumption implies that $a * F_j * F_k \subset A$. The set $a * F_j * F_k$ is contained in the plane $\Pi(a, p_i^{n_i})$ and has cardinality $p_j p_k$. By Lemma 4.3, there are no other elements of A in that plane.

Part (ii) is obvious. □

Corollary 9.6. *Assume (F'), and suppose that $\mathcal{I} \cap \mathcal{J} \cap \mathcal{K} \neq \emptyset$. Then the tiling $A \oplus B = \mathbb{Z}_M$ is T2-equivalent to $\Lambda(a, D(M)) \oplus B = \mathbb{Z}_M$. Consequently, A and B satisfy (T2).*

Proof. Let $a \in \mathcal{I} \cap \mathcal{J} \cap \mathcal{K}$. Without loss of generality, we may assume that $A \cap \Lambda(a, D(M))$ is M-fibered in the p_i direction. It follows that $\{D(M)|m|M\} \subset \mathrm{Div}(A)$, and, by Lemma 9.5 (ii), we have $A_0 \subset A$.

For every $a_{jk} \in A$ with $(a - a_{jk}, M) = M/p_j p_k$, we have $a_{jk} * F_i \subset A$. Suppose now that there is a $z \in \mathbb{Z}_M \setminus A$ with $(a - z, M) = M/p_j p_k$. By Lemma 4.6, we have $A_z \subset \ell_i(z)$, so that the pair (A, B) has a (1,2)-cofibered structure in the p_i direction with the cofiber in A at distance M/p_i^2 from z. For each such z, we apply Lemma 2.9 to shift the cofiber, obtaining a new set A' with $A' \oplus B = \mathbb{Z}_M$ such that $z * F_i \subset A'$ and A' is T2-equivalent to A. After all such shifts have been performed, we see that A is T2-equivalent to $\Lambda(a, D(M))$. □

9.2. **Toolbox for fibered grids.** We start by pointing out a special case when Theorem 9.1 is very easy to prove.

Lemma 9.7. *Let $A \oplus B = \mathbb{Z}_M$, where $M = p_i^2 p_j^2 p_k^2$. If Φ_M divides both A and B, then at least one of the sets A and B is M-fibered in some direction. By Corollary 2.7, both A and B satisfy (T2).*

Proof. Each of the differences M/p_i, M/p_j, M/p_k can belong to at most one of $\mathrm{Div}(A)$ and $\mathrm{Div}(B)$. By pigeonholing, at least one of $\mathrm{Div}(A)$ and $\mathrm{Div}(B)$ must avoid at least two of these differences. Assume without loss of generality that $M/p_i, M/p_j \notin \mathrm{Div}(A)$. By the

assumptions of the lemma, we also have $\Phi_M|A$. Applying Lemma 4.9 (ii) to A, we get that A is M-fibered in the p_k direction, as claimed. □

Next, we discuss cyclotomic divisibility. Since $F_i(X) = (X^M - 1)/(X^{M/p_i} - 1)$, we have

$$\Phi_s|F_i \;\Leftrightarrow\; p_i^2|s|M, \tag{9.4}$$

and similarly for F_j and F_k. In particular,

$$\prod_{\alpha_i=1}^{2} \Phi_{M/p_i^{\alpha_i}}(X) \prod_{\alpha_j=1}^{2} \Phi_{M/p_j^{\alpha_j}}(X) \Big| \mathcal{K}(X), \tag{9.5}$$

and similarly for other permutations of the indices i, j, k.

Lemma 9.8. *Assume (F'). For each $\alpha_k \in \{1, 2\}$, we have*

$$\Phi_{M/p_k^{\alpha_k}}|A \;\Leftrightarrow\; \Phi_{M/p_k^{\alpha_k}}|\mathcal{K}.$$

In particular, if $\Phi_{M/p_k^{\alpha_k}} \nmid A$ for some $\alpha_k \in \{1, 2\}$, then $\mathcal{K} \neq \emptyset$. Similar conclusions hold for other permutations of the indices i, j, k.

Proof. The lemma follows immediately from (9.5) if $\mathcal{I}, \mathcal{J}, \mathcal{K}$ are mutually disjoint, since then we have $A(X) = \mathcal{I}(X) + \mathcal{J}(X) + \mathcal{K}(X)$. In the general case, we need a mild workaround as follows.

Write $A(X) = \mathcal{I}'(X) + \mathcal{J}'(X) + \mathcal{K}'(X)$, where the sets $\mathcal{I}', \mathcal{J}', \mathcal{K}'$ are pairwise disjoint and $\mathcal{I}', \mathcal{J}', \mathcal{K}'$ are M-fibered in the p_i, p_j, and p_k direction, respectively. This can be done by splitting up $\mathbb{Z}_M$ into pairwise disjoint $D(M)$ grids Λ_τ and adding $A \cap \Lambda_\tau$ to one of $\mathcal{I}', \mathcal{J}', \mathcal{K}'$, according to the direction in which $A \cap \Lambda_\tau$ is M-fibered. (If $A \cap \Lambda_\tau$ is fibered in more than one direction, choose one arbitrarily and add $A \cap \Lambda_\tau$ to the corresponding set.)

It follows from (9.5) that $\Phi_{M/p_k^{\alpha_k}}|A$ if and only if $\Phi_{M/p_k^{\alpha_k}}|\mathcal{K}'$. To pass from $\mathcal{K}'$ to $\mathcal{K}$, we write $\mathcal{K}(X) = \mathcal{K}'(X) + \mathcal{K}_i(X) + \mathcal{K}_j(X)$, where $\mathcal{K}_i \subset \mathcal{I}'$ and $\mathcal{K}_j \subset \mathcal{J}'$. By Lemma 9.5, $\mathcal{K}_i$ is a union of pairwise disjoint sets of the form $a * F_i * F_k$, where $a \in A$. In particular, by (9.5) we have $\Phi_{M/p_k^{\alpha_k}}|\mathcal{K}_i(X)$. Applying the same argument to $\mathcal{K}_j$, we see that $\Phi_{M/p_k^{\alpha_k}}|\mathcal{K}'$ if and only if $\Phi_{M/p_k^{\alpha_k}}|\mathcal{K}$, and the lemma follows. □

We finish with two counting lemmas.

Lemma 9.9. *Let $A \oplus B = \mathbb{Z}_M, M = p_i^2 p_j^2 p_k^2, |A| = |B| = p_i p_j p_k$. Then*

$$\{m/p_k^\alpha : \; \alpha \in \{0, 1, 2\}, m \in \{M, M/p_i, M/p_j, M/p_i p_j\}\} \cap \mathrm{Div}(B) \neq \emptyset. \tag{9.6}$$

Proof. Suppose that (9.6) fails. It follows that any $M/p_i p_j p_k^2$-grid may contain at most one element of B. Since $\mathbb{Z}_M$ is a disjoint union of $p_i p_j$ such grids, it follows that $|B| \leq M/p_i p_j p_k^2$, contradicting our assumption that $|B| = M/p_i p_j p_k$. □

Lemma 9.10. *Assume that (F') holds.*

(i) Let $a_k \in \mathcal{K}$ and $\Pi_k := \Pi(a_k, p_i^{\alpha_i})$ for some $\alpha_i \in \{1, 2\}$. Suppose that $|A \cap \Pi_k| = p_j p_k$ and $\mathcal{I} \cap \Pi_k = \emptyset$. Then $A \cap \Pi_k \subset \mathcal{K}$.

(ii) Let $a_i \in \mathcal{I}$ and $\Pi_i := \Pi(a_i, p_i)$. Suppose that $|A \cap \Pi_i| = p_j p_k$, $A \cap \Pi_i \subset \mathcal{I} \cup \mathcal{K}$, and $\mathcal{I} \cap \mathcal{K} \cap \Pi_i = \emptyset$. Then $p_i < p_j$.

The same conclusions hold with j and k interchanged.

Proof. (i) Suppose first that $\mathcal{J}\cap\mathcal{K}\cap\Pi_k\neq\emptyset$, and let $a\in\mathcal{J}\cap\mathcal{K}\cap\Pi_k$. Then $a\notin\mathcal{I}$. By Lemma 9.5 (i), we have $A\cap\Pi_k=a*F_j*F_k$, so that $A\cap\Pi_k\subset\mathcal{J}\cap\mathcal{K}$.

Assume now that $\mathcal{J}\cap\mathcal{K}\cap\Pi_k=\emptyset$. Since $a_k\in\mathcal{K}$, there exists a nonnegative integer c_j and a positive integer c_k such that

$$p_jp_k=|A\cap\Pi_k|=c_jp_j+c_kp_k$$

This clearly implies $c_j=0$ and $c_k=p_j$, thus proving $\mathcal{J}\cap\Pi_k=\emptyset$.

(ii) In this case, $A\cap\Pi_i$ is a disjoint union of M-fibers in the p_i and p_k directions. Since $a_i\in\mathcal{I}$, there exists a positive integer c_i and a nonnegative integer c_k such that

$$p_jp_k=|A\cap\Pi_k|=c_ip_i+c_kp_k.$$

It follows that $c_i=c_i'p_k$ for some positive integer c_i'. But then $p_j=c_i'p_i+c_k$, so that $p_j>p_i$ as claimed. □

9.3. **Fibering on lower scales.**

Lemma 9.11. *Assume that (F) holds.*

(i) Let $\Lambda:=\Lambda(a_0,D(N_i))$ for some $a_0\in\mathcal{I}$. If $\mathcal{I}\cap\mathcal{J}\cap\Lambda=\emptyset$ and A is N_i-fibered on Λ, it cannot be fibered in the p_j direction.

(ii) If $\Phi_{N_i}|B$ and

$$\{M/p_j,M/p_k,M/p_ip_j,M/p_ip_k\}\cap\mathrm{Div}(B)=\emptyset, \tag{9.7}$$

then B must be N_i-fibered in the p_i direction. Note in particular that if $\{D(M)|m|M\}\subset\mathrm{Div}(A)$, then (9.7) holds.

Proof. (i) Assume, by contradiction, that A is N_i-fibered in the p_j direction on Λ. Then $\mathbb{A}^{N_i}_{N_i}[a]=p_i$ and $\mathbb{A}^{N_i}_{N_i/p_j}[a]=p_i\phi(p_j)$ for all $a\in\mathcal{I}\cap\Lambda$, meaning that $\mathcal{I}\cap\Lambda$ is also M-fibered in the p_j direction. This contradicts the assumption that $\mathcal{I}\cap\mathcal{J}\cap\Lambda$ is empty.

(ii) By (9.7), we have $\mathbb{B}^{N_i}_{N_i/p_j}[b]=\mathbb{B}^{N_i}_{N_i/p_k}[b]=0$ for all $b\in B$. It follows from Lemma 4.9 (ii) that B is N_i-fibered in the p_i direction. □

By Lemma 9.8, if $\Phi_{N_k}|A$, then $\Phi_{N_k}|\mathcal{K}$. We consider the question of whether, in these circumstances, $\mathcal{K}$ is permitted to have an unfibered grid on a lower scale.

Lemma 9.12. *Assume that (F) holds. Suppose that $\Phi_{N_k}|A$ and that there exists an $D(N_k)$ grid on which $\mathcal{K}$ is not fibered. Then:*

- $\{D(M)|m|M\}\cup\{M/p_k^2,M/p_ip_jp_k^2\}\subset\mathrm{Div}(A)$,
- *there exists an $x\in\mathbb{Z}_M$ such that $\mathbb{K}^{N_k}_{N_k/p_k}[x]=\phi(p_k^2)$.*

Proof. Assume that $\Phi_{N_k}|A$. By Lemma 9.8, we have $\Phi_{N_k}|\mathcal{K}$. Consider $\mathcal{K}$ as a multiset in $\mathbb{Z}_{N_k}$ with constant multiplicity p_k. We claim that if $\mathcal{K}$ is not fibered on a $D(N_k)$-grid Λ, it must satisfy the conclusion of either Lemma 6.2 or Lemma 6.3 with $N=N_k$ on that grid. Indeed, if this is not the case, then we must have

$$\{m:\ D(N_k)|m|N_k\}\subset\mathrm{Div}_{N_k}(\mathcal{K}).$$

But then, for every $D(N_k)|m|N_k$ there exist $a,a'\in\mathcal{K}$ (depending on m) such that $(a-a',N_k)=m$. Since $\mathbb{K}^{N_k}_{N_k}[a]=\mathbb{K}^{N_k}_{N_k}[a']=\phi(p_k)$, we have

$$\left\{m/p_k^\alpha:\ \alpha\in\{0,1,2\},m\in\{M,M/p_i,M/p_j,M/p_ip_j\}\right\}\subset\mathrm{Div}(\mathcal{K}).$$

The latter contradicts Lemma 9.9.

On the other hand, suppose that $\mathcal{K} \in \mathcal{M}(\mathbb{Z}_{N_k})$ has at least $p_j + 1$ distinct points in some plane $\Pi(x, p_i^2)$ in $\mathbb{Z}_{N_k}$, each of multiplicity p_k. Then $|A \cap \Pi(x, p_i^2)| \geq (p_j + 1)p_k$ in $\mathbb{Z}_M$, contradicting Lemma 4.3. The same argument applies with i and j interchanged.

Among the structures described in Lemmas 6.2 and 6.3 with $N = N_k$, the only ones that avoid configurations as in the last paragraph are as follows.

- $\mathcal{K} \cap \Lambda$ has the p_k full plane structure as in Lemma 6.2, so that for some $x \in \mathbb{Z}_M$ we have $\mathbb{K}^{N_k}_{N_k/p_k}[x] = \phi(p_k^2)$ and $\mathbb{K}^{N_k}_{N_k/p_ip_j}[x] = p_k\phi(p_ip_j)$.
- $\mathcal{K} \cap \Lambda$ has a p_i or p_j corner structure as in Lemma 6.3 (i).
- $\mathcal{K} \cap \Lambda$ has a p_i or p_j almost corner structure as in Lemma 6.3 (ii), so that (possibly after a permutation of i and j) there exist $x_1, x_2, x_3, x_4 \in \mathbb{Z}_M$ such that $(x_\nu - x'_\nu, N_k) = N_k/p_i$ for $\nu \neq \nu'$, $\mathbb{K}^{N_k}_{N_k/p_k}[x_1] = \mathbb{K}^{N_k}_{N_k/p_k}[x_2] = \phi(p_k^2)$ and $\mathbb{K}^{N_k}_{N_k/p_j}[x_3] = \mathbb{K}^{N_k}_{N_k/p_j}[x_4] = p_k\phi(p_j)$.

We address the second case, the first and third case being similar. Indeed, a p_i corner structure in $\mathcal{K}$ on the N_k scale means that there exist $a, a' \in \mathcal{K}$ with $(a - a', N_k) = N_k/p_i$ so that

$$\mathbb{K}^{N_k}_{N_k/p_k}[a] = \mathbb{K}_{M/p_k^2}[a] = \phi(p_k^2) \tag{9.8}$$

and

$$\mathbb{K}^{N_k}_{N_k/p_j}[a'] = \mathbb{K}_{M/p_j}[a'] + \mathbb{K}_{M/p_jp_k}[a'] = p_k\phi(p_j). \tag{9.9}$$

Now consider $\mathcal{K}$ on scale M. We have $(a - a', M) \in \{M/p_i, M/p_ip_k\}$, with the fiber chain in (9.8) attached to a. By (9.9) and the fact that $a' \in \mathcal{K}$, we also have $a' * F_j * F_k \subset \mathcal{K}$. Hence the conclusions of the lemma hold with $x = a$. □

Corollary 9.13. *Assume (F). If $\Phi_{N_k}|A$ and $p_k > \min_\nu p_\nu$, then $\mathcal{K}$ is N_k-fibered on each $D(N_k)$-grid in one of the p_i and p_j directions. In particular, $\mathcal{K} \subset \mathcal{I} \cup \mathcal{J}$.*

Proof. Assume without loss of generality that $p_i = \min_\nu p_\nu$. By Lemma 9.8, we have $\Phi_{N_k}|A$ if and only if $\Phi_{N_k}|\mathcal{K}$. By Lemma 9.12, if there exists a $D(N_k)$-grid on which $\mathcal{K}$ is unfibered, then there must exist $x \in \mathbb{Z}_M$ such that $\mathbb{K}^{N_k}_{N_k/p_k}[x] = \phi(p_k^2)$, thus

$$\begin{aligned} |A \cap \Pi(x, p_j^2)| &\geq \mathbb{K}^{N_k}_{N_k/p_k}[x] = \phi(p_k^2) \\ &> p_ip_k \end{aligned}$$

which contradicts Lemma 4.3. Thus $\mathcal{K}$ must be fibered on all $D(N_k)$-grids. On the other hand, by the same argument as above, $\mathcal{K}$ cannot be N_k-fibered in the p_k direction on any $D(N_k)$-grid. □

9.4. **Proof of Proposition 9.2.** In this section, we are assuming (F1), which we state here again for the reader's convenience.

Assumption (F1). We have $A \oplus B = \mathbb{Z}_M$, where $M = p_i^2p_j^2p_k^2$ is odd. Furthermore, $|A| = |B| = p_ip_jp_k$, $\Phi_M|A$, A is fibered on $D(M)$-grids, and

$$\mathcal{I}, \mathcal{J}, \mathcal{K} \text{ are pairwise disjoint.} \tag{9.10}$$

We must prove that at least one of the sets $\mathcal{I}, \mathcal{J}$ or $\mathcal{K}$ has to be empty. To this end we assume the contrary, i.e.,

$$\mathcal{I}, \mathcal{J}, \mathcal{K} \neq \emptyset, \tag{9.11}$$

and prove by contradiction that (9.11) cannot hold. We may assume, without loss of generality, that

$$p_i < p_j < p_k. \tag{9.12}$$

By Corollary 9.13, this implies that Φ_{N_j} and Φ_{N_k} cannot divide A, so that

$$\Phi_{N_j}\Phi_{N_k}|B. \tag{9.13}$$

We start with a cyclotomic divisibility result.

Proposition 9.14. *Assume that (F1), (9.11), and (9.13) hold. Then $\Phi_{N_i} \nmid A$.*

Proof. The proof is divided into several steps. In each of the following claims, the assumptions of the proposition are assumed to hold.

Claim 1. *If $\Phi_{N_i}|A$, then $\mathcal{I}$ is N_i-fibered in the p_i direction, so that for every $a_i \in \mathcal{I}$,*

$$\mathbb{I}_M[a_i] + \mathbb{I}_{M/p_i}[a_i] + \mathbb{I}_{M/p_i^2}[a_i] = p_i^2. \tag{9.14}$$

Proof. It suffices to prove that $\mathcal{I}$ is N_i-fibered on each $D(N_i)$-grid. Once we know that, Lemma 9.11 (i) together with (9.10) implies that the N_i-fibering must be in the p_i direction, and the claim follows.

Assume, by contradiction, that there exists a $D(N_i)$ grid over which $\mathcal{I}$ is not fibered. By Lemma 9.12,

$$\{D(M)|m|M\} \cup \{M/p_i^2, M/p_i^2p_jp_k\} \subset \mathrm{Div}(A) \tag{9.15}$$

and there exists $x_0 \in \mathbb{Z}_M$ with

$$\mathbb{I}^{N_i}_{N_i/p_i}[x_0] = \phi(p_i^2). \tag{9.16}$$

The proof of Lemma 9.12 implies further that $\mathcal{I} \cap \Lambda(x_0, D(N_i))$ must contain one of the structures described in Lemmas 6.2 and 6.3 with $N = N_i$. Additionally, (9.13) and Lemma 9.11 (ii) imply that B is N_j-fibered in the p_j direction and N_k-fibered in the p_k direction, hence

$$M/p_j^2, M/p_k^2, M/p_j^2p_k^2 \in \mathrm{Div}(B). \tag{9.17}$$

We claim that

$$\mathbb{I}^{N_i}_{N_i}[x_0] = 0. \tag{9.18}$$

Suppose this is not true, then

$$\mathbb{A}_M[x_0] + \mathbb{A}_{M/p_i}[x_0] + \mathbb{A}_{M/p_i^2}[x_0] = p_i^2. \tag{9.19}$$

Let $a_j \in \mathcal{J}$ and $a_k \in \mathcal{K}$. Recall from Lemma 2.9 that the fibering in B allows one to shift the fibers rooted at a_j and a_k by distance M/p_j^2 and M/p_k^2, respectively. Let $x_j, x_k \in \mathbb{Z}_M$ with

$$(a_\nu - x_\nu, M) = M/p_\nu^2 \ \text{ and } \ p_\nu^2|x_0 - x_\nu \ \text{ for } \ \nu \in \{j,k\}.$$

Let A' be the set obtained by shifting the fibers rooted at a_j and a_k to x_j and x_k respectively, so that $x_j * F_j, x_k * F_k \in A'$. By Lemma 2.9, $A' \oplus B = \mathbb{Z}_M$ and A' is T2-equivalent to A.

Let $a_1, a_2 \in \ell_i(x_0)$ be the points with $p_i^2 | a_1 - x_j$, $p_i^2 | a_2 - x_k$. By (9.19), $\ell_i(x_0) \subset A \cap A'$. It follows from (9.17) that we cannot have $M/p_k^2 \in \mathrm{Div}(A')$, hence $(a_1 - x_j, M) = M/p_k$. Similarly, $(a_2 - x_k, M) = M/p_j$. But now $M/p_i^2 p_j, M/p_i^2 p_k \in \mathrm{Div}(A')$, hence $M/p_i^2 p_j, M/p_i^2 p_k \notin \mathrm{Div}(B)$. Together with (9.15), this contradicts Lemma 9.9. This proves (9.18).

We therefore conclude that $\mathcal{I}$ has either a full plane structure (Lemma 6.2) or an almost corner structure (Lemma 6.3 (ii)). In either one of these cases, there exists a point

$$a_i^* \in \mathcal{I} \cap \Lambda(x_0, D(M)) \tag{9.20}$$

such that for each $\nu \in \{j, k\}$, $A \cap (a_i^* * F_i * F_\nu)$ is M-fibered in the p_i direction, but is not M-fibered in the p_ν direction.

Let $a_j \in \mathcal{J}$, and let $x_j \in \ell_j(a_j)$ be the point such that $p_j^2 | x_0 - x_j$. We consider two cases.

(a) If it is possible to choose a_j so that $x_j \in a_j * F_j$, we fix that choice, and let $A' := A$.
(b) Otherwise, let A' be the set obtained from A by shifting the fiber $a_j * F_j$ to x_j if necessary, so that $x_j * F_j \subset A'$. By Lemma 2.9, $A' \oplus B = \mathbb{Z}_M$ and A' is T2-equivalent to A. Since we are not in case (a), $A' \cap (x_j * F_i * F_j)$ contains no other M-fibers in the p_j direction.

We show that either $x_j \in \Lambda(x_0, D(M))$ or

$$M/p_i^2 p_k \notin \mathrm{Div}(B). \tag{9.21}$$

- Suppose that $p_i | x_0 - x_j$. If $(x_0 - x_j, M) \in \{M, M/p_i, M/p_k, M/p_i p_k\}$, then clearly $x_j \in \Lambda(x_0, D(M))$. If on the other hand $(x_0 - x_j, M) \in \{M/p_k^2, M/p_i p_k^2\}$, then by (9.20), there exists a fiber $a * F_i \subset I$ with $a \in \Lambda(x_0, D(M))$ and $p_i^2 | a - x_j$. Then $M/p_k^2 \in \mathrm{Div}(a, x_j * F_j) \subset \mathrm{Div}(A')$, contradicting (9.17).
- Assume now that $p_i \nmid x_0 - x_j$. If $(x_0 - x_j, M) = M/p_i^2 p_k$, then by (9.16) together with the fact that $p_i \geq 3$, there must be an $a \in A$ with $(a - x_0, M) = M/p_i^2$ and $(a - x_j, M) = M/p_i^2 p_k$, proving (9.21) since $x_j \in A'$. Otherwise, we have $(x_0 - x_j, M) = M/p_i^2 p_k^2$, but then by (9.16), there must be $a \in A$ with $(a - x_0, M) = M/p_i^2$ and $(a - x_j, M) = M/p_k^2$, contradicting (9.17).

Suppose now that $x_j \in \Lambda(x_0, D(M))$. We claim that, in this case, A' contains a p_k extended corner structure consisting of the fiber in $x_j * F_i * F_j$ in the p_j direction and at least one fiber in $a_i^* * F_i * F_j$ in the p_i direction. Indeed, by (9.20) and (9.10), we have

$$\Lambda(x_j, D(M)) \cap (\mathcal{J} \cup \mathcal{K}) = \emptyset. \tag{9.22}$$

In particular, we must be in case (b) above, and $A \cap (x_j * F_i * F_j) = x_j * F_j$. This together with the choice of a_i^* proves that the conditions of Definition 5.4 (ii) hold.

In that case, however, we proved in Theorem 8.1 that A' (therefore A) is T2-equivalent to a $D(M)$ grid. It follows that A satisfies (T2), therefore $A \oplus B^\flat = \mathbb{Z}_M$, where $\Phi_{p_\nu} | B^\flat$ for all $\nu \in \{i, j, k\}$. This contradicts (9.15), since clearly $M/p_i^2 \in \mathrm{Div}(B^\flat)$.

We are therefore left with (9.21). By the same argument with p_j and p_k interchanged, we must also have $M/p_i^2 p_j \notin \mathrm{Div}(B)$. Together with (9.15), this again contradicts Lemma 9.9. □

Claim 2. *If $\Phi_{N_i} | A$, then $\Phi_{M_i} \nmid A$, and therefore $\Phi_{M_i} | B$.*

Proof. By Lemma 9.8, if $\Phi_{M_i} | A$, then $\Phi_{M_i} | \mathcal{I}$. Since M_i has only 2 distinct prime divisors, we may apply Lemma 4.7 on the scale M_i. We conclude that every element of $\mathcal{I}$ belongs to either an M_i-fiber in the p_j direction, in which case we have $|A \cap \Pi(a_i, p_k^2)| \geq p_i^2 p_j$, or to an

M_i-fiber in the p_k direction, in which case we have $|A\cap\Pi(a_i,p_j^2)|\geq p_i^2p_k$. Since both bounds contradict Lemma 4.3, we deduce $\Phi_{M_i}|B$, and the lemma follows. □

The next claim is a direct consequence of Claim 2, (9.14), and Lemma 4.7.

Claim 3. *If $\Phi_{N_i}|A$, then $\mathbb{B}_{M_i}^{M_i}[y]\in\{0,1\}$ for all $y\in\mathbb{Z}_M$. Moreover, for every $b\in B$, either*

$$\mathbb{B}_{M_i/p_j}^{M_i}[b']=\phi(p_j) \text{ for all } b'\in B\cap\Lambda(b,D(M_i)), \tag{9.23}$$

and, since $p_i=\min_\nu p_\nu$,

$$M/p_ip_j, M/p_i^2p_j\in \mathrm{Div}(B), \tag{9.24}$$

or

$$\mathbb{B}_{M_i/p_k}^{M_i}[b']=\phi(p_k) \text{ for all } b'\in B\cap\Lambda(b,D(M_i)) \tag{9.25}$$

and

$$M/p_ip_k, M/p_i^2p_k\in \mathrm{Div}(B). \tag{9.26}$$

Claim 4. *If $\Phi_{N_i}|A$, then for all $a_i\in\mathcal{I}$ we have $\mathbb{I}_{M_i/p_j}^{M_i}[a_i]=\mathbb{I}_{M_i/p_k}^{M_i}[a_i]=0$.*

Proof. If there exist $b_1,b_2\in B$ such that (9.23) holds with $b=b_1$ and (9.25) holds with $b=b_2$, then the claim follows from (9.24) and (9.26). Assume therefore that (9.23) holds for all $b\in B$, and, consequently, $\mathbb{I}_{M_i/p_j}^{M_i}[a_i]=0$ for all $a_i\in\mathcal{I}$.

Assume, by contradiction, that $\mathbb{I}_{M_i/p_k}^{M_i}[a_i]>0$ for some $a_i\in\mathcal{I}$. It follows from (9.14) that

$$M/p_i, M/p_i^2, M/p_ip_k, M/p_i^2p_k\in \mathrm{Div}(A). \tag{9.27}$$

We claim that

$$M/p_jp_k\notin \mathrm{Div}(B). \tag{9.28}$$

Assuming this, we prove Claim 4 as follows. By (9.11), (9.27), and (9.28), we have $N_k/p_i, N_k/p_j\notin \mathrm{Div}_{N_k}(B)$. By (9.13) and Lemma 4.9 (ii), B must be N_k-fibered in the p_k direction, so that

$$\mathbb{B}_{N_k/p_k}^{N_k}[b]=\phi(p_k)\ \ \forall b\in B.$$

Let $a_k\in\mathcal{K}$. Since a_i satisfies (9.14), we may assume (moving a_i to a different point in the same fiber chain if necessary) that

$$p_i^2|a_k-a_i. \tag{9.29}$$

Then the pair (A,B) has a (1,2)-cofibered structure, with a_k*F_k as a cofiber. Let $x_k'\in\ell_k(a_k)$ be the point such that $p_k^2|a_i-x_k'$. If $p_k|a_k-x_k'$, we note that $x_k'\in A$, and let $A':=A$. If on the other hand $(a_k-x_k',M)=M/p_k^2$, we use Lemma 2.9 to shift a_k*F_k to x_k', obtaining a new set A' such that $x_k'\in A'$, $A'\oplus B=\mathbb{Z}_M$, and A' is T2-equivalent to A.

By (9.24) and (9.29), we must have $(a_i-x_k',M)=M/p_j^2$, so that $M/p_j^2\in\mathrm{Div}(A')$. In particular, $M/p_j^2\notin\mathrm{Div}(B)$. On the other hand, we also have $\Phi_{N_j}|B$, and

$$\mathbb{B}_{N_j/p_k}^{N_j}[b]=\mathbb{B}_{N_j/p_j}^{N_j}[b]=0 \text{ for all } b\in B.$$

By Lemma 4.9, B must be N_j-fibered in the p_i direction, so that $\mathbb{B}_{N_j/p_i}^{N_j}[b]=\phi(p_i)$ for every $b\in B$. Together with (9.11), this means that every grid $\Lambda(b,M/p_ip_j)$ with $b\in B$ contains exactly p_i points of B. On the other hand, the assumption that (9.23) holds for all $b\in B$

implies that every such grid contains exactly p_j points of B. This contradiction proves the claim, assuming (9.28).

We now prove (9.28). Assume, by contradiction, that $b, b' \in B$ with $(b-b', M) = M/p_jp_k$. Let $y, y' \in \mathbb{Z}_M \setminus B$ with $(b-y, M) = (b'-y', M) = M/p_k$, $(b-y', M) = (b'-y, M) = M/p_j$, and consider the saturating set B_{y,a_i}. Then

$$B_{y,a_i} \subset \ell_i(b) \cup \ell_i(y) \cup \ell_i(b') \cup \ell_i(y').$$

If $B_{y,a_i} \cap (\ell_i(y) \cup \ell_i(y'))$ is nonempty, then $\{M/p_ip_k, M/p_i^2p_k\} \cap \mathrm{Div}(B)$ must be nonempty, while if $B_{y,a_i} \cap (\ell_i(b) \cup \ell_i(b'))$ is nonempty, then $\{M/p_i, M/p_i^2\} \cap \mathrm{Div}(B)$ must be nonempty. Both of these contradict (9.27). □

Claim 5. *If $\Phi_{N_i}|A$ and $M/p_jp_k \notin \mathrm{Div}(B)$, then there must exist $b_j, b_k \in B$ such that (9.23) holds with $b = b_j$ and (9.25) holds with $b = b_k$.*

Proof. Assume, by contradiction, that the conclusion is false. Without loss of generality, we may assume that (9.23) holds for all $b \in B$.

Since $M/p_jp_k \notin \mathrm{Div}(B)$, we have $\mathbb{B}^{N_j}_{N_j/p_k}[b] = 0$ for all $b \in B$. By (9.13) and Lemma 4.9, B must be fibered on $D(N_j)$ grids, so that for every $b \in B$ either

$$\mathbb{B}^{N_j}_{N_j/p_j}[b] = \phi(p_j), \tag{9.30}$$

or

$$\mathbb{B}^{N_j}_{N_j/p_i}[b] = \mathbb{B}^{M}_{M/p_ip_j}[b] = \phi(p_i). \tag{9.31}$$

Suppose that there exists $b_0 \in B$ satisfying (9.30). Applying (9.23) to all $b' \in B$ with $(b_0 - b', M) = M/p_j^2$, we get

$$|B \cap \Pi(b_0, p_k^2)| \geq p_j^2 > p_ip_j,$$

which contradicts Lemma 4.3.

Hence (9.31) must hold for all $b \in B$, so that B is a union of disjoint N_j-fibers in the p_i direction, each of cardinality p_i. On the other hand, by Claim 3 and (9.23), for any $b \in B$ we have

$$\begin{aligned}\mathbb{B}^{N_j/p_i^2}_{N_j/p_j^2}[b] &= \mathbb{B}^{M_i/p_j}_{M_i/p_j}[b] \\ &= 1 + \mathbb{B}^{M_i}_{M_i/p_j}[b] \\ &= 1 + \phi(p_j) = p_j.\end{aligned}$$

This implies that p_j is divisible by p_i, which is obviously false. □

Claim 6. *If $M/p_jp_k \notin \mathrm{Div}(B)$, then $\Phi_{N_i} \nmid A$.*

Proof. Assume, by contradiction, that $M/p_jp_k \notin \mathrm{Div}(B)$ and $\Phi_{N_i}|A$. Then the conclusions of Claims 1-5 apply. By Claim 5, we may find b_j, b_k such that (9.23) holds with $b = b_j$ and (9.25) holds with $b = b_k$. We fix these elements for the duration of the proof.

We claim that

$$M/p_j^2, M/p_k^2 \in \mathrm{Div}(B). \tag{9.32}$$

We prove the first part of (9.32), the second part being identical with p_j and p_k interchanged.

As in the proof of Claim 5, we use (9.13) and the assumption that $M/p_jp_k \notin \mathrm{Div}(B)$ to conclude that either (9.30) or (9.31) holds for every $b \in B$. Suppose that (9.31) holds for

$b = b_j$. Then $B \cap \Lambda(b_j, D(M_i))$ is a union of disjoint N_j-fibers in the p_i direction, and the same argument as in the proof of Claim 5 implies that p_j is divisible by p_i, which is obviously false. Thus (9.30) holds with $b = b_j$, and in particular $M/p_j^2 \in \mathrm{Div}(B)$.

With (9.32) in place, we complete the proof as follows. Fix let $a_j \in \mathcal{J}$ and $a_i \in \mathcal{I}$ such that $p_i^{n_i} | a_j - a_i$. (This is possible by (9.14).) Taking (9.24), (9.26) and (9.32) into account, we see that $(a_i - a_j, M) = M/p_j^2 p_k^2 = p_i^2$. By (9.14) again, we must in fact have $1, p_i, p_i^2 \in \mathrm{Div}(A)$.

We deduce that $M_i/p_j^2 p_k^2 \notin \mathrm{Div}_{M_i}(B)$, and, therefore, one of $B \subset b * p_j\mathbb{Z}$ or $B \subset b * p_k\mathbb{Z}$ must hold for any fixed $b \in B$. That, however, contradicts (9.32). □

Claim 6 proves Proposition 9.14 in the case when $M/p_jp_k \notin \mathrm{Div}(B)$. From here on, we will therefore assume that

$$M/p_jp_k \in \mathrm{Div}(B). \tag{9.33}$$

Claim 7. *If $\Phi_{N_i}|A$ and (9.33) holds, then $\Phi_{M_i/p_j}\Phi_{M_i/p_k}|B$.*

Proof. Assume, by contradiction, that $\Phi_{M_i/p_j}|A$. Let $a_i \in \mathcal{I}$, and let $x_k \in \mathbb{Z}_M$ with $(a_i - x_k, M) = M/p_k$. By Claim 4, we have $\mathbb{A}_{M_i}^{M_i}[x_k] = 0$. Furthermore, (9.14) and the assumption that $M/p_jp_k \notin \mathrm{Div}(A)$ imply that

$$\mathbb{A}_{M_i/p_j}^{M_i/p_j}[x_k] = \mathbb{A}_{M_i}^{M_i}[x_k] + \sum_{x_k' : (x_k - x_k', M) = M/p_j} \mathbb{A}_{M_i}^{M_i}[x_k'] = 0,$$

Considering all M_i/p_j cuboids with vertices at a_i and x_k, we see that for every $x_j \in \mathbb{Z}_M$ with $(a_i - x_j, M) = M/p_j^2$ we have

$$\mathbb{A}_{M_i/p_j}^{M_i/p_j}[x_j] = p_i^2,$$

thus

$$|A \cap \Pi(a_i, p_k^2)| \geq p_jp_i^2 > p_ip_j$$

contradicting Lemma 4.3. Since this argument is symmetric with respect to j and k, the claim follows. □

Claim 8. *If $\Phi_{N_i}|A$ and (9.33) holds, then B is M_i-fibered in both of the p_j and p_k directions, so that for all $b \in B$ we have*

$$\frac{1}{\phi(p_j)}\mathbb{B}_{M_i/p_j}^{M_i}[b] = \frac{1}{\phi(p_k)}\mathbb{B}_{M_i/p_k}^{M_i}[b] = 1. \tag{9.34}$$

Proof. By Claims 2 and 7, we have $\Phi_{M_i}\Phi_{M_i/p_k}|B$. Therefore B is $\mathcal{T}$-null with respect to the cuboid type $\mathcal{T} = (M_i, \vec{\delta}, 1)$, where $\vec{\delta} = (0, 1, 2)$. Suppose that for some $b_0 \in B$ we have $\mathbb{B}_{M_i/p_j}^{M_i}[b_0] < \phi(p_j)$. Fix $y_j \in \mathbb{Z}_M$ such that $(b_0 - y_j, M) = M/p_j$ and $\mathbb{B}_{M_i}^{M_i}[y_j] = 0$, and consider all cuboids of type $\mathcal{T}$ with vertices at b_0 and y_j. In order to balance these cuboids, we must have $\mathbb{B}_{M_i}^{M_i}[y] = 1$ for all $y \in \mathbb{Z}_M$ with $(b_0 - y, M) = M/p_k^2$. But now we get $|B \cap \Pi(b_0, p_j^2)| \geq p_k^2 > p_ip_k$, which contradicts Lemma 4.3. Since this argument is symmetric in j and k, the claim follows. □

Claim 9. *If $\Phi_{N_i}|A$ and (9.33) holds, then for all $b \in B$ we have $\mathbb{B}_{N_j/p_i}^{N_j}[b] = \phi(p_i)$.*

Proof. Assume, by contradiction, that there exists $b_0 \in B$ with $\mathbb{B}^{N_j}_{N_j/p_i}[b_0] < \phi(p_i)$. Since $p_k > p_j$ and $M/p_k \in \mathrm{Div}(A)$, we must also have $\mathbb{B}^{N_j}_{N_j/p_k}[b_0] < \phi(p_k)$. Let $y_i, y_k \in \mathbb{Z}_M$ with $(b_0 - y_i, N_j) = N_j/p_i, (b_0 - y_k, N_j) = N_j/p_k$ and such that $\mathbb{B}^{N_j}_{N_j}[y_i] = \mathbb{B}^{N_j}_{N_j}[y_k] = 0$. Recall that $\Phi_{N_j}|B$, and consider all N_j cuboids with vertices at b, y_i and y_k. We get that for every z with $(z - b_0, N_j) = N_j/p_j$, we have

$$\mathbb{B}^{N_j}_{N_j}[z] + \mathbb{B}^{N_j}_{N_j/p_ip_k}[z] \geq 1. \tag{9.35}$$

But by Claim 8, we also have

$$|B \cap \Pi(b, p_j)| \geq p_jp_k \quad \forall b \in B.$$

Applying this to all $b \in B$ contributing to (9.35), and summing over z with $(z - b_0, N_j) = N_j/p_j$, we get $|B| \geq p_j \cdot (p_jp_k) > p_ip_jp_k$, a contradiction. □

Claim 10. *If (9.33) holds, then $\Phi_{N_i} \nmid A$.*

Proof. Assume, for contradiction, that (9.33) holds and $\Phi_{N_i}|A$. By Claim 8, B must satisfy (9.34), and in particular

$$\mathbb{B}^{M_i}_{M_i/p_j}[b] = \phi(p_j) \text{ for all } b \in B.$$

By Claim 9,

$$\mathbb{B}^{N_j}_{N_j/p_i}[b] = \mathbb{B}_{M/p_ip_j}[b] = \phi(p_i) \text{ for all } b \in B.$$

As in the proof of Claim 6, these two properties imply that p_j is divisible by p_i, which is false. □

Claim 10 concludes the proof of Proposition 9.14. □

By Proposition 9.14, it remains to prove Proposition 9.2 under the assumption that

$$\prod_{\nu \in \{i,j,k\}} \Phi_{M/p_\nu}|B. \tag{9.36}$$

Lemma 9.15. *Assume (F1) and (9.36). Then at least one of the sets $\mathcal{I}, \mathcal{J}$, or $\mathcal{K}$ must be empty.*

Proof. Assume, by contradiction, that (9.11) holds. By (9.36), $\Phi_{N_i}|B$. Since $p_i = \min_\nu p_\nu$ and $M/p_j, M/p_k \notin \mathrm{Div}(B)$, we must have

$$\frac{1}{\phi(p_j)}\mathbb{B}^{N_i}_{N_i/p_j}[b], \frac{1}{\phi(p_k)}\mathbb{B}^{N_i}_{N_i/p_k}[b] < 1 \text{ for all } b \in B,$$

and, in particular, B cannot be N_i-fibered in either the p_j or the p_k direction. By Lemma 4.9, it follows that $M/p_i^2 \in \mathrm{Div}(B)$.

Suppose first that $\Phi_{M_i}|A$, so that $\Phi_{M_i}|\mathcal{I}$. We have $\mathbb{I}^{M_i}_{M_i}[x] \in \{0, p_i\}$ for all $x \in \mathbb{Z}_M$, and $N_i/p_i \notin \mathrm{Div}_{N_i}(A)$. By Lemma 4.9, $\mathcal{I}$ must be N_i-fibered in one of the p_j and p_k directions, so that we either have

$$\mathbb{I}^{M_i}_{M_i/p_j}[a_i] = p_i\phi(p_j) \text{ for all } a_i \in \mathcal{I}, \tag{9.37}$$

or $\mathbb{I}^{M_i}_{M_i/p_k}[a_i] = p_i\phi(p_k)$ for all $a_i \in \mathcal{I}$.

Assume, without loss of generality, that (9.37) holds, and fix $a_i \in \mathcal{I}$. It follows that $|\mathcal{I} \cap \Pi(a_i, p_k^2)| \geq p_i p_j$. By Lemma 4.3, the last inequality holds as equality, i.e.,

$$|\mathcal{I} \cap \Pi(a_i, p_k^2)| = p_i p_j \text{ and so } A \cap \Pi(a_i, p_k^2) \subset \mathcal{I}. \tag{9.38}$$

We now consider two cases.

- If $\Phi_{p_k^2} | A$, then $A \subset \Pi(a_i, p_k)$. Let $a_k \in \mathcal{K}$. Since $\mathcal{I} \cap \mathcal{K} = \emptyset$, we must have $a_k \notin \Lambda(a_i, D(N_i))$, thus $|A \cap \Pi(a_i, p_k^2)| > p_i p_j$, contradicting (9.38).
- If $\Phi_{p_k} | A$, then
$$|A \cap \Pi(a, p_k)| = p_i p_j \text{ for all } a \in A.$$
By (9.38), we must have $\mathcal{I} \cap \Pi(a_k, p_k) = \emptyset$. Hence $A \cap \Pi(a_k, p_k) \subset \mathcal{J} \cup \mathcal{K}$. Lemma 9.10 (ii) implies now that $p_i > p_k$, contradicting (9.12).

We conclude that $\Phi_{M_i} \nmid A$, hence $\Phi_{N_i} \Phi_{M_i} | B$. This implies that B is $\mathcal{T}$-null with respect to the cuboid type $\mathcal{T} = (M, \vec{\delta}, T)$, where $\vec{\delta} = (0, 1, 1)$ and $T(X) = 1 + X^{M/p_i} + \ldots + X^{(p_i - 1)M/p_i}$. Since $M/p_i \notin \mathrm{Div}(B)$, we have $\mathbb{B}^{\mathcal{T}}[y] \in \{0, 1\}$ for all $y \in \mathbb{Z}_M$. Note that $\mathcal{T}$ is a 2 dimensional cuboid type, so that for every $b \in B$ we either have

$$\mathbb{B}^{\mathcal{T}}[y_j] = \mathbb{B}_M[y_j] + \mathbb{B}_{M/p_i}[y_j] = 1 \text{ for all } y_j \in \mathbb{Z}_M \text{ with } (b - y_j, M) = M/p_j,$$

or

$$\mathbb{B}^{\mathcal{T}}[y_k] = \mathbb{B}_M[y_k] + \mathbb{B}_{M/p_i}[y_k] = 1 \text{ for all } y_k \in \mathbb{Z}_M \text{ with } (b - y_k, M) = M/p_k.$$

As $p_i = \min_\nu p_\nu$, the former implies $M/p_j \in \mathrm{Div}(B)$, and the latter implies $M/p_k \in \mathrm{Div}(B)$, both contradicting (9.11). □

9.5. **Proof of Proposition 9.3.** In this section, we will prove that $\mathcal{I} = \emptyset$ under the following conditions.

Assumption (F2). We have $A \oplus B = \mathbb{Z}_M$, where $M = p_i^2 p_j^2 p_k^2$ is odd. Furthermore, $|A| = |B| = p_i p_j p_k$, $\Phi_M | A$, A is fibered on $D(M)$-grids, $\mathcal{J} \cap \mathcal{K} \neq \emptyset$, and

$$\mathcal{I} \cap \mathcal{J} \cap \mathcal{K} = \emptyset. \tag{9.39}$$

Let $a \in \mathcal{J} \cap \mathcal{K}$. By Lemma 9.5 (i), we have

$$a * F_j * F_k = A \cap \Pi(a, p_i^2), \tag{9.40}$$

and in particular

$$\{M/p_j, M/p_k, M/p_j p_k\} \subset \mathrm{Div}(A). \tag{9.41}$$

We first prove Proposition 9.3 under the assumption that $\Phi_{p_i^2} | A$.

Lemma 9.16. *Assume (F2), and let $a \in \mathcal{J} \cap \mathcal{K}$. If $\Phi_{p_i^2} | A$, then*

$$A \subset \Pi(a, p_i). \tag{9.42}$$

Proof. The assumption $\Phi_{p_i^2} | A$, together with (9.40), implies that $|A \cap \Pi(a, p_i)| = p_i p_j p_k = |A|$. □

Corollary 9.17. *Assume that (F2) holds, and that $\Phi_{p_i^2} | A$. Then $\mathcal{I} = \emptyset$.*

Proof. Let $a \in \mathcal{J} \cap \mathcal{K}$. Suppose, by contradiction, that $\mathcal{I}$ is nonempty. By (9.42), there must exist an element $a_i \in \mathcal{I} \cap \Pi(a, p_i^{n_i})$. It follows from (9.40) that $a_i \in a * F_j * F_k$. But then $a_i \in \mathcal{I} \cap \mathcal{J} \cap \mathcal{K}$, contradicting (9.39). □

In the rest of this section, it remains to consider the following case.

Assumption (F2'). Assume that (F2) holds, and that

$$\Phi_{p_i^2} \nmid A. \tag{9.43}$$

Lemma 9.18. *Assume (F2'). Let $a \in \mathcal{J} \cap \mathcal{K}$, and let $x \in \mathbb{Z}_M \setminus A$ with $(a-x, M) = M/p_i$. Then*

$$A_x \subset \Pi(a, p_i^2). \tag{9.44}$$

Proof. Let $b \in B$. We have $A_{x,b} \subset \Pi(a, p_i^2) \cup \Pi(x, p_i^2)$. Assume, by contradiction, that

$$A_{x,b} \cap \Pi(x, p_i^2) \neq \emptyset. \tag{9.45}$$

Then $|A \cap \Pi(x, p_i^2)| > 0$. Together with (9.40), this implies that $|A \cap \Pi(x, p_i^2)| > p_j p_k$. Since $|A| = p_i p_j p_k$, by Corollary 4.4 we have $\Phi_{p_i^2} | A$, contradicting (9.43). □

Lemma 9.19. *Assume (F2'). Then:*

(i) *For every $b \in B$, and for every $y \in \mathbb{Z}_M$ with $(b-y, M) = M/p_i$, we have*

$$\mathbb{B}_M[y] + \mathbb{B}_{M/p_j}[y] + \mathbb{B}_{M/p_k}[y] + \mathbb{B}_{M/p_j p_k}[y] = 1. \tag{9.46}$$

(ii) *For all d with $p_i^2 | d | (M/p_j p_k)$, we have $\Phi_d | B$. Additionally,*

$$\{M/p_i, M/p_i p_j, M/p_i p_k, M/p_i p_j p_k\} \cap \mathrm{Div}(B) \neq \emptyset. \tag{9.47}$$

(iii) *We have $\Phi_{p_i} | A$. Moreover,*

$$A \cap \Pi(a, p_i) = A \cap \Pi(a, p_i^2) \quad \forall a \in \mathcal{J} \cap \mathcal{K}. \tag{9.48}$$

$$|A \cap \Pi(x', p_i)| \in \{0, p_j p_k\} \quad \forall x' \in \mathbb{Z}_M. \tag{9.49}$$

Proof. Fix $b \in B$, and write $N_{jk} = M/p_j p_k$ for short. Let $a \in \mathcal{J} \cap \mathcal{K}$, and let $x \in \mathbb{Z}_M \setminus A$ with $(a-x, M) = M/p_i$.

By (9.40) and (9.44), we have

$$\begin{aligned}
1 &= \frac{1}{\phi(p_i)} \mathbb{A}_{M/p_i}[x | \Pi(a, p_i^2)]\, \mathbb{B}_{M/p_i}[b] + \frac{1}{\phi(p_i p_j)} \mathbb{A}_{M/p_i p_j}[x | \Pi(a, p_i^2)]\, \mathbb{B}_{M/p_i p_j}[b] \\
&\quad + \frac{1}{\phi(p_i p_k)} \mathbb{A}_{M/p_i p_k}[x | \Pi(a, p_i^2)]\, \mathbb{B}_{M/p_i p_k}[b] + \frac{1}{\phi(p_i p_j p_k)} \mathbb{A}_{M/p_i p_j p_k}[x | \Pi(a, p_i^2)]\, \mathbb{B}_{M/p_i p_j p_k}[b] \\
&= \frac{1}{\phi(p_i)} (\mathbb{B}_{M/p_i}[b] + \mathbb{B}_{M/p_i p_j}[b] + \mathbb{B}_{M/p_i p_k}[b] + \mathbb{B}_{M/p_i p_j p_k}[b]) \\
&= \frac{1}{\phi(p_i)} \sum_{y : (y-b, M) = M/p_i} \mathbb{B}^{N_{jk}}_{N_{jk}}[y].
\end{aligned}$$

On the other hand, by (9.41) and divisor exclusion, any N_{jk}-grid may contain at most one element of B, so that $\mathbb{B}^{N_{jk}}_{N_{jk}}[y] \leq 1$ for all $y \in \mathbb{Z}_M$. Therefore

$$\mathbb{B}^{N_{jk}}_{N_{jk}}[y] = 1 \text{ for all } y \text{ such that } (y-b, M) = M/p_i,$$

which is (9.46). We also note that the first equation in the above calculation implies (9.47).

Next, since $|A| = p_i p_j p_k$, exactly one of Φ_{p_i} and $\Phi_{p_i^2}$ must divide A. By (9.43), we must in fact have $\Phi_{p_i} | A$. Then

$$p_j p_k \geq |A \cap \Pi(a, p_i)| \geq |A \cap \Pi(a, p_i^2)| = p_j p_k,$$

by (9.40). Hence the above must hold with equality, which proves (9.48). It also follows that for any $x' \in \mathbb{Z}_M$, we must have $|A \cap \Pi(x', p_i)| = p_j p_k$ if $p_i | x' - a$ and 0 otherwise, so that (9.49) holds.

Finally, let $d \neq p_i^2$ and $p_i^2 | d | (M/p_j p_k)$, and consider any d-cuboid with one vertex at $a \in \mathcal{J} \cap \mathcal{K}$. By (9.40), we have $\mathbb{A}_d^d[a] \geq p_j p_k$. However, we also have $|A \cap \Pi(a, p_i)| = p_j p_k$ by the third claim in the lemma, so that $\mathbb{A}_{d'}^d[a] = 0$ for all $d' < d$ with $D(d) | d'$. It follows that the cuboid cannot be balanced. Therefore $\Phi_d \nmid A$, which proves (ii). □

Lemma 9.20. *Assume (F2'). Then $\mathcal{I} \cap (\mathcal{J} \cup \mathcal{K}) = \emptyset$.*

Proof. Suppose that the conclusion fails. Without loss of generality, we may assume that $\mathcal{I} \cap \mathcal{K} \neq \emptyset$. Taking into account that $\mathcal{J} \cap \mathcal{K} \neq \emptyset$, we have from (9.41)

$$M/p_i, M/p_j, M/p_k, M/p_i p_k, M/p_j p_k \in \mathrm{Div}(A)$$

so that (9.47) reduces to $\{M/p_i p_j, M/p_i p_j p_k\} \cap \mathrm{Div}(B) \neq \emptyset$, and (9.46) to

$$\mathbb{B}_{M/p_j}[y] + \mathbb{B}_{M/p_j p_k}[y] = 1 \text{ for all } y \text{ with } (y - b, M) = M/p_i.$$

This means we must have

$$p_i < p_j, \tag{9.50}$$

otherwise one must introduce M/p_i or $M/p_i p_k$ as differences in B.

We now repeat the same procedure with i and j interchanged. Let $a' \in \mathcal{I} \cap \mathcal{K}$. We note that $A \cap \Lambda(a', D(M))$ cannot be M-fibered in the p_j direction, since that would contradict (9.39). It follows from Lemma 9.5 (i) that $a' * F_i * F_k \subset A$, and, together with Lemma 4.3, this implies that

$$a' * F_i * F_k = A \cap \Pi(a', p_j^2). \tag{9.51}$$

Let $x' \in \mathbb{Z}_M \setminus A$ satisfy $(a' - x', M) = M/p_j$. As in the proof of Lemma 9.18, we have

$$A_{x'} \subset \Pi(a', p_j^2) \cup \Pi(x', p_j^2).$$

Assume, by contradiction, that there exists a $b \in B$ such that $A_{x',b} \subset \Pi(a', p_j^2)$. Repeating the proof of Lemma 9.19 (i)-(ii) with that b, we get the analogues of (9.46) and (9.47) with i and j interchanged. The same argument as in the proof of (9.50) shows then that $p_j < p_i$, a contradiction.

It follows that $A_{x'} \cap \Pi(x', p_j^2) \neq \emptyset$, and, in particular, $|A \cap \Pi(a', p_j)| > p_i p_k$. By Corollary 4.4, $A \subset \Pi(a', p_j)$. In particular, $\mathcal{J} \subset \Pi(a', p_j)$, with each fiber in $\mathcal{J}$ containing a point in $A \cap \Pi(a', p_j^2)$. But by (9.51), any such point would belong to $\mathcal{I} \cap \mathcal{J} \cap \mathcal{K}$, contradicting (9.39). □

Lemma 9.21. *Assume (F2'). If $\mathcal{I} \neq \emptyset$, then $\Phi_{N_i} \nmid A$.*

Proof. By Lemma 9.8, it suffices to prove that $\Phi_{N_i} \nmid \mathcal{I}$. Let $a_i \in \mathcal{I}$. By Corollary 9.20, we have $a_i \notin \mathcal{J}$ and $a_i \notin \mathcal{K}$, so that there must exist $x_j, x_k \in \mathbb{Z}_M \setminus A$ with $(a_i - x_j, N_i) = N_i/p_j, (a_i - x_k, N_i) = N_i/p_k$ and such that $\mathbb{I}_{N_i}^{N_i}[x_j] = \mathbb{I}_{N_i}^{N_i}[x_k] = 0$.

Consider the N_i cuboid with one face containing vertices at a_i, x_j and x_k, and the other face in $\Pi(a, p_i)$, where $a \in \mathcal{J} \cap \mathcal{K}$. In order for this cuboid to be balanced, $\mathcal{I} \cap \Pi(a, p_i)$ must be nonempty, and in particular $\mathcal{I} \cap \Pi(a, p_i^2) \neq \emptyset$. But this together with (9.40) contradicts (9.39). □

Lemma 9.22. *Assume (F2'). If $p_i = \min_\nu p_\nu$, then $\mathcal{I} = \emptyset$.*

Proof. By Lemma 9.21, $\Phi_{N_i} \nmid A$, hence $\Phi_{N_i}|B$. Since $\mathcal{I} \neq \emptyset$, we have $M/p_i \in \mathrm{Div}(A)$, so that $\mathbb{B}^{N_i}_{N_i}[y] \in \{0,1\}$ for all $y \in \mathbb{Z}_M$. By (9.43), $\Phi_{p_i^2}|B$. Hence $\Phi_{p_i} \nmid B$, and in particular B cannot be N_i-fibered in the p_i direction. It follows that there must exist $b_0 \in B$ and $y \in \mathbb{Z}_M$ with $(b_0 - y, N_i) = N_i/p_i$ and $\mathbb{B}^{N_i}_{N_i}[y] = 0$.

In order to simplify notation, we shall denote $\beta_\nu = \mathbb{B}^{N_i}_{N_i/p_\nu}[b_0]$ for $\nu \in \{j,k\}$. By (9.46) we must have

$$\beta_j + \beta_k + 1 \leq |B \cap \Lambda(b_0, D(M))| \leq p_i \tag{9.52}$$

thus

$$\beta_\nu \leq (p_i - 1)/2 \text{ for some } \nu \in \{j,k\}. \tag{9.53}$$

In addition, considering all N_i cuboids with vertices at b_0 and y such that the vertices at distance N_i/p_j and N_i/p_k from b_0 do not belong to B, we see that

$$\mathbb{B}^{N_i}_{N_i/p_jp_k}[y] \geq (p_j - \beta_j - 1)(p_k - \beta_k - 1).$$

Now, if $\beta_j = 1$ then $\beta_k \leq p_i - 2$ and so

$$\begin{aligned}(p_j - \beta_j - 1)(p_k - \beta_k - 1) &\geq (p_j - 2)(p_k - p_i + 1)\\ &\geq (p_k - p_i + 1)p_i\\ &> p_i\end{aligned}$$

which contradicts (9.46). We may therefore assume $\beta_\nu \geq 2$ for $\nu = j,k$. In this case, however, assuming (9.53) for $\nu = k$, applying (9.52) and the fact that $p_j - p_i \geq 2$, we have

$$\begin{aligned}(p_j - \beta_j - 1)(p_k - \beta_k - 1) &= (p_j - p_i + p_i - \beta_j - 1)(p_k - \beta_k - 1)\\ &\geq (p_j - p_i + \beta_k)(p_k - \beta_k - 1)\\ &\geq 4(p_i - (p_i - 1)/2 - 1)\\ &= 2p_i - 2\end{aligned}$$

The latter exceeds p_i whenever $p_i > 2$. Since M is odd, again we get a contradiction to (9.46) and the lemma follows. □

Lemma 9.22 proves Proposition 9.3, assuming that (F2') holds and that p_i is the smallest prime. From now on, we will therefore assume that

$$p_i > \min_\nu p_\nu. \tag{9.54}$$

The rest of the proof will be split into the following cases:

- Assume (F2'), (9.54), and $\Phi_{N_j}\Phi_{N_k}|A$. This case is addressed in Lemma 9.23 and Corollary 9.24.
- Assume (F2'), (9.54), and (interchanging j and k if necessary) $\Phi_{N_j} \nmid A$, $\Phi_{N_k} \nmid B$. This case is addressed in Lemmas 9.27, 9.28, and 9.29.
- Assume (F2'), (9.54), and $\Phi_{N_j} \nmid A$, $\Phi_{N_k} \nmid A$. This case is addressed in Lemma 9.30, Corollary 9.31, and Lemma 9.32.

Lemma 9.23. *Assume (F2') and (9.54). If $p_k < p_i < p_j$ and $\Phi_{N_k}|A$, then $\mathcal{K} \subset \mathcal{J}$.*

Proof. We first claim that $\mathcal{K}$ is N_k-fibered on each $D(N_k)$-grid in one of the p_j and p_k directions. Indeed, if $\mathcal{K}$ were not N_k-fibered on some $D(N_k)$-grid, then it would follow from Lemma 9.12 that $\{D(M)|m|M\} \subset \mathrm{Div}(A)$; however, that is not compatible with (9.47).

Furthermore, by Lemma 9.20 and Lemma 9.11 (i), $\mathcal{K}$ cannot be N_k-fibered in the p_i direction on any $D(N_k)$ grid. This proves the claim.

Recall from Lemma 9.19 (iii) that for any $a_0 \in A$,

$$|A \cap \Pi(a_0, p_i)| = p_j p_k. \tag{9.55}$$

If $\mathcal{K}$ is N_k-fibered in the p_j direction on some $D(N_k)$ grid Λ_j, then for every $a_k \in \mathcal{K} \cap \Lambda_j$ we have

$$\mathbb{K}^{N_k}_{N_k/p_j}[a_k] = p_k \phi(p_j), \tag{9.56}$$

so that $a_k \in \mathcal{J} \cap \mathcal{K}$. By (9.55), $A \cap \Pi(a_k, p_i) = a_k * F_j * F_k$ is fibered in both directions, and in particular $\Pi(a_k, p_i)$ contains no elements of A outside of Λ_j.

Assume now that there exists $a'_k \in \mathcal{K}$ such that $\mathcal{K}$ is N_k-fibered in the p_k direction on $\Lambda_k := \Lambda(a'_k, D(N_k))$. Then

$$\mathbb{K}^{N_k}_{N_k/p_k}[a'] = p_k \phi(p_k) \text{ for all } a' \in \mathcal{K} \cap \Pi(a'_k, p_i), \tag{9.57}$$

so that

$$\mathbb{A}_M[a'] + \mathbb{A}_{M/p_k}[a'] + \mathbb{A}_{M/p_k^2}[a'] = p_k^2 \text{ for all } a' \in \mathcal{K} \cap \Pi(a'_k, p_i). \tag{9.58}$$

Fix $a'_k \in \mathcal{K}$ satisfying (9.57) and (9.58). We first claim that

$$\mathcal{I} \cap \Pi(a'_k, p_i) = \emptyset. \tag{9.59}$$

Indeed, suppose that (9.59) fails, and let $a_i \in \mathcal{I} \cap \Pi(a'_k, p_i^2)$. Since $\mathcal{K} \cap \Lambda_k$ is N_k fibered in the p_k direction, by Lemma 9.20 we must have $a_i \notin \Lambda_k$, so that a_i must be at distance M/p_j^2 from the fiber chain in the p_k direction rooted at a'_k. We can now extend (9.41) to

$$M/p_j, M/p_k, M/p_j p_k, M/p_j^2, M/p_k^2, M/p_j^2 p_k^2, M/p_j^2 p_k \in \mathrm{Div}(A), \tag{9.60}$$

where all the differences that do not appear in (9.41), come from the interaction between a_i and $\ell_k(a'_k)$.

By (9.60), we see that for every $b \in B$

$$\begin{aligned} |B \cap \Pi(b, p_i^2)| &\leq 1 + \mathbb{B}_{M/p_j p_k^2}[b] \\ &\leq p_k. \end{aligned}$$

But, since $\mathbb{Z}_M$ has only p_i^2 residue classes modulo p_i^2, it follows that

$$p_i p_j p_k = |B| \leq p_i^2 p_k,$$

so that $p_j \leq p_i$, contradicting the assumption that (9.59) fails.

It, therefore, follows that $A \cap \Pi(a'_k, p_i) \subset \mathcal{J} \cup \mathcal{K}$. By (9.55) and Lemma 9.10 (i), we have $A \cap \Pi(a'_k, p_i) \subset \mathcal{K}$. But since all $a' \in \mathcal{K} \cap \Pi(a'_k, p_i)$ satisfy (9.57) and (9.58), we get

$$p_j p_k = c p_k^2$$

for some positive integer c. The latter implies p_k divides p_j, which is not allowed. This completes the proof of the lemma. □

Corollary 9.24. *Assume (F2') and (9.54). Assume further that $\Phi_{N_j}\Phi_{N_k}|A$ and that $p_k < p_j$. Then $\mathcal{I} = \emptyset$, and A is M-fibered in the p_k direction.*

Proof. By (9.54), we have $p_k = \min_\nu p_\nu$. We first apply Corollary 9.13, with j and k interchanged. Since $\Phi_{N_j}|A$ and $p_j \neq \min_\nu p_\nu$, we get that $\mathcal{J} \subset (\mathcal{I} \cup \mathcal{K})$. However, by Lemma 9.20 we have $\mathcal{I} \cap \mathcal{J} = \emptyset$, so that we must in fact have $\mathcal{J} \subset \mathcal{K}$.

Assume, by contradiction, that $\mathcal{I}$ is nonempty. We first prove that this implies

$$p_i < p_j. \tag{9.61}$$

Let $a_i \in \mathcal{I}$. Observe that $\Pi(a_i, p_i)$ cannot contain any elements $a' \in \mathcal{J}$, since any such element would be associated with a grid $a' * F_j * F_k \subset A$, and by (9.55) and Lemma 9.20, we would have $\mathcal{I} \cap \Pi(a_i, p_j) = \emptyset$. Thus $A \cap \Pi(a_i, p_i) \subset \mathcal{I} \cup \mathcal{K}$, and (9.61) follows from Lemma 9.10 (ii).

Applying Lemma 9.23, we see that $\mathcal{K} \subset \mathcal{J}$. This, together with the first part of the proof, implies $\mathcal{K} = \mathcal{J}$. Hence any element $a_k \in \mathcal{K}$ is associated with a grid $a_k * F_j * F_k \subset A$, and as shown above, such grids cannot intersect $\Pi(a_i, p_i)$. Therefore $A \cap \Pi(a_i, p_i)$ must be contained in $\mathcal{I}$. That, however, is clearly false since $p_j p_k = |A \cap \Pi(a_i, p_i)|$ cannot be a multiple of p_i. This contradiction concludes the proof. □

Before we move on to the next two cases, we need two lemmas on the fibering properties of B.

Lemma 9.25. *Assume (F2). If $\Phi_{N_k}|B$, then B is N_k-fibered on each $D(N_k)$ grid, either in the p_k direction or in the p_i direction. The same is true with j and k interchanged.*

Proof. Suppose that $\Phi_{N_k}|B$. By (9.41), we have $\mathbb{B}^{N_k}_{N_k/p_j}[b] = 0$ and $\mathbb{B}^{N_k}_{N_k}[b] = 1$ for all $b \in B$. Since M is odd, the lemma follows from Lemma 4.9. □

Lemma 9.26. *Assume (F2'). If $\mathcal{I} \neq \emptyset$ and B is N_i-fibered on a $D(N_i)$ grid, then it must be fibered in the p_i direction on that grid.*

Proof. We argue by contradiction. Let $\Lambda_0 := \Lambda(b, D(N_i))$ for some $b \in B$, and assume that $B \cap \Lambda_0$ is N_i-fibered in one of the other directions, say p_j. Let also $\Lambda := \Lambda(b, D(M))$. By (9.46), we have

$$|B \cap \Lambda| = p_i. \tag{9.62}$$

On the other hand, the N_i-fibering assumption means that $B \cap \Lambda_0$ can be divided into mutually disjoint N_i-fibers in the p_j direction, each one of cardinality p_j, and each one either entirely contained in Λ or disjoint from it. This implies that p_j divides $|B \cap \Lambda|$. That, however, contradicts (9.62). □

Next, we consider the case $\Phi_{N_j} \nmid A, \Phi_{N_k} \nmid B$. This case will be split further, according to whether p_j or p_k is the smallest prime.

Lemma 9.27. *Assume (F2') and (9.54). Assume further that $\Phi_{N_j} \nmid A$, $\Phi_{N_k} \nmid B$, and $p_j = \min_\nu p_\nu$. If $\mathcal{I} \neq 0$, then:*

- *B is N_j-fibered in the p_j direction,*
- *$\mathcal{K} \subseteq \mathcal{J}$,*
- *$|A \cap \Pi(a', p_i)| = p_j p_k$ for any $a' \in A$,*
- *$p_i < p_k$.*

Proof. We first note that the second claim follows from Corollary 9.13 and Lemma 9.20, and the third one from Lemma 9.19 (iii). Next, let $a_i \in \mathcal{I}$. Then $\mathcal{I} \cap \Pi(a_i, p_i)$ is nonempty and,

by Lemma 9.20, disjoint from $\mathcal{J} \cup \mathcal{K} = \mathcal{J}$. By Lemma 9.10 (ii) with j and k interchanged, it follows that $p_i < p_k$.

It remains to prove the first claim. We have $\Phi_{N_j}|B$, hence by Lemma 9.25, B is N_j-fibered on $D(N_j)$ grids in one of the p_i and p_j directions. Since $p_j < p_i$, it follows that $\mathbb{B}^{N_j}_{N_j/p_i}[b] < \phi(p_i)$ for all $b \in B$, hence B must be N_j-fibered in the p_j direction. □

Lemma 9.28. *Assume (F2') and (9.54). Assume further that $\Phi_{N_j} \nmid A$, $\Phi_{N_k} \nmid B$, and $p_j = \min_\nu p_\nu$, Then $\mathcal{I} = \emptyset$, and A is M-fibered in the p_j direction.*

Proof. Assume, by contradiction, that $\mathcal{I} \neq \emptyset$. By Lemma 9.21, $\Phi_{N_i}|B$.

Assume first that $N_i/p_i \notin \mathrm{Div}(B)$. By Lemma 4.9, B is fibered on $D(N_i)$ grids in one of the p_j and p_k direction. That, however, contradicts Lemma 9.26.

Suppose now that $b, b' \in B$ with $(b - b', N_i) = N_i/p_i$. By Lemma 9.27, B is N_j-fibered in the p_j direction. This together with (9.46) implies that $|B \cap \Pi(b, p_k)| > p_i p_j$. By Corollary 4.4, we have $\Phi_{p_k^2}|B$, hence $\Phi_{p_k}|A$. The latter, in turn, implies

$$|A \cap \Pi(a', p_k)| \leq p_i p_j \quad \forall a' \in A. \tag{9.63}$$

On the other hand, let $a \in \mathcal{J} \cap \mathcal{K}$ with $a * F_j * F_k \subset A$ as provided by (9.40). Then

$$|A \cap \Pi(a, p_k)| \geq p_j p_k > p_i p_j,$$

where at the last step we used Lemma 9.27 again. This contradicts (9.63).

This proves that $\mathcal{I} = \emptyset$. By the second claim in Lemma 9.27, we have $\mathcal{K} \subseteq \mathcal{J}$, hence A is M fibered in the p_j direction. □

Lemma 9.29. *Assume (F2') and (9.54). Assume further that $\Phi_{N_j} \nmid A$, $\Phi_{N_k} \nmid B$, and $p_k = \min_\nu p_\nu$. Then $\mathcal{I} = \emptyset$ and $\mathcal{K} \subseteq \mathcal{J}$. Consequently, A is M-fibered in the p_j direction.*

Proof. The proof splits between two cases.

Case 1: $p_k < p_i < p_j$. In this case, by Lemma 9.23 we have $\mathcal{K} \subseteq \mathcal{J}$. Assume, by contradiction, that $\mathcal{I}$ is nonempty, and let $a_i \in \mathcal{I}$. By Lemma 9.19 (iii), we have $|A \cap \Pi(a_i, p_i)| = p_j p_k$. By Lemma 9.10 (ii) with j and k interchanged, it follows that $p_k > p_i$, contradicting our assumption. Therefore $\mathcal{I} = \emptyset$.

Case 2: $p_k < p_j < p_i$. We first note that if $\mathcal{I}$ is nonempty, then $\mathbb{B}^{N_j}_{N_j/p_i}[b] < \phi(p_i)$ for all $b \in B$. Since $\Phi_{N_j}|B$, by Lemma 9.25

$$B \text{ must be } N_j \text{ fibered in the } p_j \text{ direction.} \tag{9.64}$$

Next, we follow the first part of the proof of Lemma 9.23. (This part does not use the Lemma 9.23 assumption that $p_i < p_j$.) By the same argument as there, $\mathcal{K}$ must be N_k-fibered on every $D(N_k)$ grid in either the p_j or the p_k direction, and for every $a_k \in \mathcal{K}$ we have one of the following:

- (9.56) holds, hence $a_k \in \mathcal{J} \cap \mathcal{K}$ and $a_k * F_j * F_k = A \cap \Pi(a_k, p_i)$,
- (9.57) and (9.58) hold for all $a' \in \mathcal{K} \cap \Lambda(a_k, D(N_k))$.

If (9.56) holds for all $a' \in \mathcal{K}$, then $\mathcal{K} \subseteq \mathcal{J}$, and it follows by the same argument as in Case 1 that $\mathcal{I} = \emptyset$.

We now prove that the second case is impossible. Indeed, assume by contradiction that there exists $a_k \in \mathcal{K}$ such that (9.57) and (9.58) hold for all $a' \in \mathcal{K} \cap \Lambda(a_k, D(N_k))$. We first claim that

$$\mathcal{I} \cap \Pi(a_k, p_i) = \emptyset. \tag{9.65}$$

Indeed, if (9.65) fails, we continue as in the proof of Lemma 9.23 and find an element $a_i \in \mathcal{I}$ at distance M/p_j^2 from the fiber chain through a_k. But (9.64) implies $M/p_j^2 \in \mathrm{Div}(B)$, which is a contradiction. Hence (9.65) holds.

By (9.49), we have $|A \cap \Pi(a_k, p_i)| = p_j p_k$. It follows from (9.65) and Lemma 9.10 (i) that $A \cap \Pi(a_k, p_i) \subset \mathcal{K}$. As in the proof of Lemma 9.23, all elements of A in $\Pi(a_k, p_i)$ must satisfy (9.57). Hence

$$p_j p_k = |A \cap \Pi(a_k, p_i)| = c p_k^2$$

for some positive integer c, so that p_k divides p_j, a contradiction. This completes the proof of the lemma. □

We now address the case in which $\Phi_{N_\nu} \nmid A$ for $\nu \in \{j, k\}$.

Lemma 9.30. *Assume (F2') and (9.54). Assume further that B is N_k-fibered in the p_k direction, $M/p_j^2 \in \mathrm{Div}(B)$, and that $\mathcal{I} \neq \emptyset$. Then for all $a_i \in \mathcal{I}$ we have $\mathcal{K} \cap \Pi(a_i, p_i) = \emptyset$.*

Proof. Assume, by contradiction, that $a_i \in \mathcal{I}$ and $\mathcal{K} \cap \Pi(a_i, p_i) \neq \emptyset$. Replacing a_i by a different element of $a_i * F_i$ if necessary, we may further assume that $p_i^2 | a_i - a_k$ for some $a_k \in \mathcal{K} \cap \Pi(a_i, p_i)$ Moreover, it follows from the fibering assumption on B that the pair (A, B) has a (1,2)-cofibered structure in the p_k direction, with the cofiber in A rooted at a_k.

Suppose that $(a_i - a_k, M) = M/p_j p_k^2$. Applying Lemma 2.9, we could then shift the cofiber $a_k * F_k$ in the p_k direction, obtaining a new T2-equivalent tiling $A' \oplus B = \mathbb{Z}_M$ in which the shifted cofiber $a_k' * F_k$ satisfies $(a_i - a_k, M) = M/p_j$. We claim that A' contains a p_j extended corner structure. Indeed, by Lemma 9.20 we have $\Lambda(a_i, D(M)) \cap (\mathcal{J} \cup \mathcal{K}) = \emptyset$. Hence $A \cap (a_i * F_i * F_k)$ is M-fibered in the p_i direction but not in the p_k direction, and $A \cap (a_k' * F_i * F_k)$ must be empty, so that $A' \cap (a_k' * F_i * F_k) = a_k' * F_k$. This proves the claim. However, Theorem 8.1 now implies that $\Phi_{p_i^2} | A$, contradicting (F2').

Since $M/p_k^2 \in \mathrm{Div}(B)$ by the fibering assumption, we are now left with $(a_i - a_k, M) = M/p_j^2 p_k^2$. But then, by the same fiber-shifting argument as above, we get a T2-equivalent tiling $A'' \oplus B = \mathbb{Z}_M$, where $M/p_j^2 \in \mathrm{Div}(A'')$. This contradicts the assumption that $M/p_j^2 \in \mathrm{Div}(B)$ □

Corollary 9.31. *Assume (F2') and (9.54). If B is N_ν-fibered in the p_ν direction for both $\nu = j$ and $\nu = k$, then $\mathcal{I} = \emptyset$.*

Proof. By the fibering assumption,

$$M/p_j^2, M/p_k^2 \in \mathrm{Div}(B). \tag{9.66}$$

Suppose that $\mathcal{I} \neq \emptyset$, and let $a_i \in \mathcal{I}$. It follows from Lemma 9.30 that $\mathcal{J}$ and $\mathcal{K}$ are both disjoint from $\Pi(a_i, p_i)$. Therefore $A \cap \Pi(a_i, p_i) \subset \mathcal{I}$, and in particular, p_i divides $|A \cap \Pi(a_i, p_i)|$. But this contradicts (9.49). □

Lemma 9.32. *Assume (F2') and (9.54). Assume further that $\Phi_{N_j} \Phi_{N_k} | B$. Then $\mathcal{I} = \emptyset$.*

Proof. Suppose that $\mathcal{I} \neq \emptyset$. Without loss of generality, we may assume that $p_k = \min_\nu p_\nu$. By Lemma 9.25, B is N_k-fibered on $D(N_k)$ grids in one of the p_i and p_k directions. However, B cannot be N_k fibered in the p_i direction on any $D(N_k)$ grid, since the assumptions that $\mathcal{I} \neq \emptyset$ and $p_k < p_i$ imply that $\mathbb{B}^{N_k}_{N_k/p_i}[b] < \phi(p_i)$ for all $b \in B$. Hence

$$B \text{ must be } N_k\text{-fibered in the } p_k \text{ direction.} \tag{9.67}$$

By the same argument as above, if $p_j < p_i$, then B must also be N_j-fibered in the p_j direction, and an application of Corollary 9.31 concludes the proof.

It remains to consider the case when

$$p_k < p_i < p_j. \tag{9.68}$$

We claim that in this case,

$$M/p_ip_k \in \mathrm{Div}(B). \tag{9.69}$$

Indeed, since $\mathcal{I} \neq \emptyset$, the failure of (9.69) would imply that $N_i/p_k \notin \mathrm{Div}_{N_i}(B)$. It would then follow from Lemma 4.9 that B is N_i-fibered on $D(N_i)$ grids. By Lemma 9.26, B can only be N_i-fibered in the p_i direction. In particular, $\Phi_{p_i}|B$, contradicting Lemma 9.19 (ii).

Let $b_1, b_2 \in B$ with

$$(b_1 - b_2, M) = M/p_ip_k. \tag{9.70}$$

By (9.46), one may find $b_3, \ldots, b_{p_i} \in B$ satisfying $(b_\nu - b_{\nu'}, M) \in \{M/p_ip_k, M/p_ip_j, M/p_ip_jp_k\}$ for all $\nu \neq \nu'$.

By Lemma 9.25, B is N_j-fibered on each $D(N_j)$ grid in one of the p_i and p_j directions. However, it is clear from (9.46) and (9.70) that $\mathbb{B}^{N_j}_{N_j/p_i}[b_\nu] < \phi(p_i)$ for all $\nu \in \{1, 2, \ldots, p_i\}$, so that B cannot be N_j fibered in the p_i direction on the grid $\Lambda := \Lambda(b_\nu, D(N_j))$. It follows that B is N_j-fibered in the p_j direction on Λ. Taking also (9.67) into account, we see that

$$|B \cap \Pi(b_\nu, p_i^2)| \geq p_jp_k \text{ for all } \nu \in \{1, 2, \ldots, p_i\}. \tag{9.71}$$

Since $|B| = p_ip_jp_k$, (9.71) must in fact hold with equality for each ν, and

$$B \setminus \bigcup_{\nu=1}^{p_i} \Pi(b_\nu, p_i^2) = \emptyset. \tag{9.72}$$

Let $b \in B$ be arbitrary. We claim that

$$M/p_ip_k \in \mathrm{Div}(B \cap \Lambda(b, D(N_i))). \tag{9.73}$$

To prove this, we start by arguing as in the proof of (9.69) that if (9.73) fails, then B is N_i-fibered in the p_i direction on $\Lambda(b, D(N_i))$. However, if that were the case, then we would have $M/p_i^2 \in \mathrm{Div}(B)$, contradicting (9.72).

We further note that by (9.72), $B \cap \Lambda(b, D(N_i)) = B \cap \Lambda(b, D(M))$, so that in fact we have

$$M/p_ip_k \in \mathrm{Div}(B \cap \Lambda(b, D(M)) \ \ \forall b \in B.$$

With this in place, we repeat the argument starting with (9.70) to prove that B is N_j-fibered in the p_j direction on all $D(N_j)$ grids.

It follows that B is N_ν-fibered in the p_ν direction for both $\nu = j$ and $\nu = k$. By Corollary 9.31, we have $\mathcal{I} = \emptyset$ as claimed. □

9.6. **Proof of Proposition 9.4.** In this section, we are working under the following assumption.

Assumption (F3): We have $A \oplus B = \mathbb{Z}_M$, where $M = p_i^2p_j^2p_k^2$ is odd. Furthermore, $|A| = |B| = p_ip_jp_k$, $\Phi_M|A$, A is fibered on $D(M)$-grids, $\mathcal{I} = \emptyset$, and

$$\text{the sets } \mathcal{J} \setminus \mathcal{K} \text{ and } \mathcal{K} \setminus \mathcal{J} \text{ are nonempty.} \tag{9.74}$$

The proof below works regardless of whether $\mathcal{J}$ and $\mathcal{K}$ are disjoint or not. If $\mathcal{J} \cap \mathcal{K} \neq \emptyset$, then (since $\mathcal{I} = \emptyset$) any element $a \in \mathcal{J} \cap \mathcal{K}$ must satisfy the conditions of Lemma 9.5 (i), so that

$$A \cap \Pi(a, p_i^2) = a * F_j * F_k. \tag{9.75}$$

It follows that the set $\mathcal{J} \setminus \mathcal{K}$ is M-fibered in the p_j direction, and $\mathcal{K} \setminus \mathcal{J}$ is M-fibered in the p_k direction.

We begin with the case when at least one of Φ_{M_j} and Φ_{M_k} divides A.

Lemma 9.33. *Assume (F3), and that $\Phi_{M_k}|A$. Then*

$$\Phi_{p_j^2}|A. \tag{9.76}$$

Furthermore, $\mathcal{K}$ is M_k-fibered in the p_j direction, so that for every $a_k \in \mathcal{K}$ we have

$$\mathbb{K}^{M_k}_{M_k/p_j}[a_k] = p_k \cdot \phi(p_j). \tag{9.77}$$

and

$$A \cap \Pi(a_k, p_i^2) \subset \Lambda(a_k, p_i^2 p_j). \tag{9.78}$$

The same holds with p_k and p_j interchanged.

Proof. Notice first that the fibering statement holds trivially for all $a_k \in \mathcal{K} \cap \mathcal{J}$.

Let now $\mathcal{L} := \mathcal{K} \setminus \mathcal{J}$, with the corresponding N-boxes $\mathbb{L}^N$ for $N|M$. The assumption $\Phi_{M_k}|A$ implies, by (9.5), that $\Phi_{M_k}|\mathcal{L}$. By Lemma 4.7, $\mathcal{L}$ mod M_k is a disjoint union of M_k-fibers in the p_i and p_j direction. Hence, any element $a_0 \in \mathcal{L}$ which does not belong to an M_k-fiber in the p_j direction must satisfy

$$\mathbb{L}^{M_k}_{M_k/p_i}[a_0] = p_k \cdot \phi(p_i) \tag{9.79}$$

and $|A \cap \Pi(a_0, p_j^2)| \geq p_i p_k$. By Lemma 4.3,

$$|A \cap \Pi(a_0, p_j^2)| = p_i p_k, \tag{9.80}$$

in particular

$$A \cap \Pi(a_0, p_j^2) \subset \mathcal{L} \text{ and } \mathcal{J} \cap \Pi(a_0, p_j^2) = \emptyset. \tag{9.81}$$

We first prove (9.76). Indeed, assume for contradiction that $\Phi_{p_j}|A$. Then

$$|A \cap \Pi(a, p_j)| = p_i p_k \text{ for all } a \in A. \tag{9.82}$$

Let $a_j \in \mathcal{J}$, and consider the plane system $\Pi(a_j, p_j)$. These planes cannot contain any elements $a \in \mathcal{L}$ satisfying (9.79), since any such element would belong to an M_k-fiber in the p_i direction in $\mathcal{L}$, of cardinality $p_i p_k$ and contained in $\Pi(a_j, p_j)$, and this would leave no room for the additional element $a_j \notin \mathcal{L}$.

Thus every element of $A \cap \Pi(a_j, p_j)$ must either belong to $\mathcal{J}$, or else it must be an element of $\mathcal{L}$ belonging to an M_k-fiber in the p_j direction in $\mathcal{L}$, of cardinality $p_j p_k$. Since the two sets are disjoint, (9.82) implies that

$$p_i p_k = |A \cap \Pi(a, p_j)| = c_1 p_j + c_2 p_j p_k,$$

where c_1, c_2 are nonnegative integers. But then p_j divides either p_i or p_k, a contradiction.

Therefore $\Phi_{p_j^2}|A$. Suppose now that there actually exists an element $a_0 \in \mathcal{L}$ such that (9.79) holds. Then $A \subset \Pi(a_0, p_j)$. Since $\mathcal{J}$ is nonempty, it follows that $\mathcal{J}$ must intersect $\Pi(a_0, p_j^2)$, contradicting (9.81). This proves the fibering conclusion of the lemma. $\square$

Lemma 9.34. *Assume (F3). The following holds true:*

(i) *If $\Phi_{p_i^2}\Phi_{M_k}|A$, then A is contained in a subgroup.*

(ii) *If $\Phi_{p_i}|A$, then $|A\cap\Pi(a,p_i)|=p_jp_k$ for all $a\in A$. Moreover, for every $a\in A$ we have either $A\cap\Pi(a,p_i)\subset\mathcal{J}$ or $A\cap\Pi(a,p_i)\subset\mathcal{K}$.*

Proof. For part (i), by Lemma 9.33 every $a_k\in\mathcal{K}$ satisfies (9.77). By Lemma 4.3,

$$|A\cap\Pi(a_k,p_i^2)|=p_jp_k,\text{ hence }A\cap\Pi(a_k,p_i^2)\subset\mathcal{K}. \tag{9.83}$$

If $\Phi_{p_i^2}|A$, then $A\subset\Pi(a_k,p_i)$. This proves the first part of the lemma.

For part (ii), assume that $\Phi_{p_i}|A$. Then $|A\cap\Pi(a,p_i)|=p_jp_k$ for all $a\in A$. The second part follows from Lemma 9.10 (i) with $\alpha_i=1$. □

Lemma 9.35. *Assume (F3) and*

$$\Phi_{p_i}\Phi_{M_k}|A. \tag{9.84}$$

Then $\Phi_{N_j}\nmid A$.

Proof. Assume, by contradiction, that $\Phi_{N_j}|A$. By (9.74) and Corollary 9.13, we must have $p_j=\min_\nu p_\nu$. We first claim that

$$\text{if }\Phi_{N_j}|A,\text{ then }\mathcal{J}\text{ must be }N_j\text{-fibered on }D(N_j)\text{ grids.} \tag{9.85}$$

Indeed, suppose that (9.85) fails. By Lemma 9.12, we have

$$\{D(M)|m|M\}\subset\mathrm{Div}(A). \tag{9.86}$$

Applying Corollary 9.13 and (9.74) again in the p_k direction, we get $\Phi_{N_k}|B$. By (9.86) and Lemma 9.11 (ii), B is N_k-fibered in the p_k direction, implying a (1,2)-cofibered structure for (A,B) with all fibers in $\mathcal{K}$ as cofibers.

Fix $a_k\in\mathcal{K}$, and recall that it must satisfy (9.77). This produces a family of M-fibers in the p_k direction, all contained in $A\cap\Pi(a_k,p_i^2)$. Using Lemma 2.9 to shift and align these fibers if necessary, we get a T2-equivalent set A' such that $A'\oplus B=\mathbb{Z}_M$ and $a_k*F_j*F_k\subset A'$. By T2-equivalence, $\Phi_{p_i}|A'$, and by Lemma 4.3,

$$a_k*F_j*F_k=A\cap\Pi(a_k,p_i^2). \tag{9.87}$$

Since $\mathcal{I}=\emptyset$, a_k cannot belong to an M-fiber in the p_i direction, and in particular there exists an $x\in\mathbb{Z}_M\setminus A$ with $(a_k-x,M)=M/p_i$. We have $A'_x\subset\Pi(a_k,p_i^2)\cup\Pi(x,p_i^2)$. However, if $A'_x\cap\Pi(x,p_i^2)\neq\emptyset$, then $|A'\cap\Pi(x,p_i^2)|>0$. By Corollary 4.4 and (9.83), we have $\Phi_{p_i^2}|A'$, which contradicts the fact that $\Phi_{p_i}|A'$.

Thus $A'_x\subset\Pi(a_k,p_i^2)$, and by (9.87), for any $b\in B$ we have

$$1=\sum_{m\in\{1,p_j,p_k,p_jp_k\}}\frac{1}{\phi(mp_i)}\mathbb{A}_{M/mp_i}[x|\Pi(a_k,p_i^2)]\mathbb{B}_{M/mp_i}[b].$$

Hence $\{D(M)|m|M\}\cap\mathrm{Div}(B)$ is nonempty, contradicting (9.86). This proves (9.85).

By (9.74), we may find $a_j\in\mathcal{J}\setminus\mathcal{K}$. It follows from Lemma 9.11 (i) that $A\cap\Lambda(a_j,D(N_j))$ can only be N_j-fibered in the p_j direction. Recall from Lemma 9.34 that $|A\cap\Pi(a_j,p_i)|=p_jp_k$ and $A\cap\Pi(a_j,p_i)\subset\mathcal{J}\setminus\mathcal{K}$. But the fibering in $\mathcal{J}\setminus\mathcal{K}$ implies that

$$p_jp_k=c\cdot p_j^2,$$

thus p_j must divide p_k, which is clearly false. The lemma follows. □

Lemma 9.36. *Assume (F3) and (9.84). Then $\Phi_{M_k/p_i}|A$.*

Proof. Denote $N = M_k/p_i$, and write

$$A(X) = \mathcal{J}(X) - (\mathcal{J} \cap \mathcal{K})(X) + \mathcal{K}(X).$$

By Lemma 9.5 (i) and (9.4), Φ_N divides both $\mathcal{J}$ and $\mathcal{J} \cap \mathcal{K}$. Hence it suffices to prove that $\Phi_N | \mathcal{K}$.

Let $a_k \in \mathcal{K}$. By Lemma 9.33, we have $\mathbb{K}^{M_k}_{M_k}[a_k] = \mathbb{K}^{M_k}_{M_k}[x_j] = p_k$ for all $x_j \in \mathbb{Z}_M$ with $(a_k - x_j, M) = M/p_j$. In particular, $|\mathcal{K} \cap \Pi(a_k, p_i^2)| = p_j p_k$, and, since $\Phi_{p_i} | A$, there are no other elements of A in $\Pi(a_k, p_i)$. It follows that, for any x_j as above,

$$\mathbb{K}^N_N[a_k] = \mathbb{K}^N_N[x_j] = p_k \tag{9.88}$$

We need to prove that $\mathbb{K}^N_N[\Delta] = 0$ for every N-cuboid Δ. It suffices to check this under the assumption that at least one vertex of Δ belongs to $\mathcal{K}$, so that two of its vertices are at points a_k and x_j as above. The other two vertices are at $x, x' \in \mathbb{Z}_M$ with

$$(a_k - x, N) = (x_j - x', N) = N/p_i \text{ and } (x - x', N) = N/p_j.$$

By (9.88), the cuboid face containing a_k and x_j is balanced. Consider now the face containing x and x'. By Lemma 9.34 (ii) we need to consider two cases. If $A \cap \Pi(x, p_i) \subset \mathcal{J}$, then this face must be balanced on the scale N, since $\Phi_N | \mathcal{J}$. Otherwise, we must have $A \cap \Pi(x, p_i) \subset \mathcal{K}$, and then by the same argument as above, either $\mathbb{K}^N_N[x] = \mathbb{K}^N_N[x'] = p_k$ or $\mathbb{K}^N_N[x] = \mathbb{K}^N_N[x'] = 0$. In both cases, the cuboid is balanced, which proves the lemma. □

Lemma 9.37. *Assume (F3) and (9.84). Then*

$$\Phi_d | B \quad \forall d \in \{N_j, N_k, M/p_j p_k, M/p_j^2 p_k, M/p_j p_k^2\}.$$

Proof. For $d = N_j$, this follows from Lemma 9.35.

Assume, by contradiction, that $\Phi_{N_k} | A$. By Lemma 9.8, we must have $\Phi_{N_k} | \mathcal{K}$. As in the proof of Lemma 9.36, we have $|\mathcal{K} \cap \Pi(a_k, p_i^2)| = p_j p_k$ for all $a \in \mathcal{K}$, with each line $\ell_k(x)$ for $x \in a_k * F_j$ containing an M-fiber in the p_k direction. In particular,

$$|\mathcal{K} \cap \ell_k(a_k)| = p_k \quad \forall a_k \in \mathcal{K}. \tag{9.89}$$

On the other hand, let $a'_k \in \mathcal{K} \setminus \mathcal{J}$. Then $\Phi_{N_k} | \mathcal{K}$ implies one of the following:

- $\mathcal{K}$ is not fibered on $\Lambda(a'_k, D(N_k))$. By Lemma 9.12, there exists an $x \in \mathbb{Z}_M$ such that $\mathbb{K}^{N_k}_{N_k/p_k}[x] = \phi(p_k^2)$.
- $\mathcal{K}$ is N_k-fibered on $\Lambda(a'_k, D(N_k))$. By Lemma 9.11 (i), it can only be fibered in the p_k direction.

Both of these are clearly incompatible with (9.89). This proves that $\Phi_{N_k} | B$.

To prove that $\Phi_{M/p_j p_k} \nmid A$, let $a_k \in \mathcal{K} \setminus \mathcal{J}$, and consider $M/p_j p_k$-cuboids with vertices at a_k and $x \in \ell_k(a_k)$ with $(x - a_k, M) = M/p_k^2$. By (9.78) and Lemma 9.34 (ii), we have $\mathbb{A}^{M/p_j p_k}_{M/p_j p_k}[v] = 0$ for any cuboid vertex v other than a_k and x. Varying x as above, we see that in order for all such cuboids to be balanced we must have

$$|A \cap \Pi(a_k, p_i^2)| = p_k \mathbb{A}^{M/p_j p_k}_{M/p_j p_k}[a_k].$$

But this is not possible, since the left side is equal to $p_j p_k$ and the right side is divisible by p_k^2. The same argument, with the cuboids collapsed further to scale $M/p_j^2 p_k$, proves that $\Phi_{M/p_j^2 p_k} \nmid A$.

Finally, we prove that $\Phi_{M_k/p_j} \nmid A$. Consider any M_k/p_j-cuboid with one vertex at $a_k \in \mathcal{K} \setminus \mathcal{J}$. By the same argument as above, we have $\mathbb{A}^{M_k/p_j}_{M_k/p_j}[v] = 0$ for all vertices $v \neq a_k$, hence the cuboid cannot be balanced. □

Lemma 9.38. *Assume (F3) and (9.84). Then the conditions of Theorem 2.6 are satisfied in the p_i direction, after interchanging A and B.*

Proof. By assumption, $\Phi_{p_i^2}|B$. We need to verify that for every d such that $p_i^{n_i}|d|M$ and $\Phi_d \nmid B$, we have

$$\Phi_{d/p_i^\alpha}|A, \quad \alpha \in \{1,2\}. \tag{9.90}$$

By Lemma 9.37, it remains to check (9.90) for $d \in \{M, M_j, M_k\}$.

- For $d = M$, (9.90) follows from Lemma 9.8 since $\mathcal{I} = \emptyset$.
- For $d = M_k$, we have $\Phi_{M_k/p_i}\Phi_{p_j^2}|A$ by Lemmas 9.33 and 9.36.
- For $d = M_j$, if $\Phi_{M_j}|B$, there is nothing to prove. If on the other hand $\Phi_{M_j}|A$, then Lemma 9.33 and Lemma 9.36 hold with j and k interchanged, and so (9.90) also holds in this case.

□

This resolves the case $\Phi_{M_k}|A$. The case $\Phi_{M_j}|A$ is similar, with j and k interchanged. It remains to prove Proposition 9.4 under the assumption that

$$\Phi_{M_\nu} \nmid A \text{ for } \nu \in \{j,k\}. \tag{9.91}$$

Without loss of generality, we may also assume that

$$p_k > p_j. \tag{9.92}$$

By Corollary 9.13, this implies that $\Phi_{N_k}\Phi_{M_k}|B$. It follows that B is $\mathcal{T}$-null with respect to the cuboid type $\mathcal{T} = (N_k, \vec{\delta}, 1)$, where $\vec{\delta} = (1,1,0)$. Since cuboids of this type are 2-dimensional, it follows by Lemma 4.7 that for every $b \in B$ at least one of the following holds:

$$\mathbb{B}_M[y] + \mathbb{B}_{M/p_k}[y] = 1 \text{ for every } y \in \mathbb{Z}_M \text{ with } (b-y, M) = M/p_j, \tag{9.93}$$

$$\mathbb{B}_M[y] + \mathbb{B}_{M/p_k}[y] = 1 \text{ for every } y \in \mathbb{Z}_M \text{ with } (b-y, M) = M/p_i. \tag{9.94}$$

In particular, this implies that

$$\{D(M)|m|M\} \cap \mathrm{Div}(B) \neq \emptyset. \tag{9.95}$$

Lemma 9.39. *Assume (F3), (9.91), and (9.92). Then $\Phi_{p_i}|A$ and $\Phi_{N_j} \nmid A$.*

Proof. We start with the second part. Assume for contradiction that $\Phi_{N_j}|A$. By Lemma 9.12 applied to p_j, (9.95), and Lemma 9.11 (i), the set $\mathcal{J} \setminus \mathcal{K}$ must be N_j fibered in the p_j direction. Let $a_j \in \mathcal{J} \setminus \mathcal{K}$. We now consider two cases.

- Suppose that $\Phi_{p_i}|A$. By Lemma 9.34 (ii), we have $A \cap \Pi(a_j, p_i) \subset \mathcal{J} \setminus \mathcal{K}$ and $|A \cap \Pi(a_j, p_i)| = p_j p_k$. But then the fibering of $\mathcal{J} \setminus \mathcal{K}$ implies that $p_j p_k$ is divisible by p_j^2, a contradiction.
- Assume now that $\Phi_{p_i^2}|A$. Let A' be a translate of A such that $a_j \in A'_{p_i}$. By the cyclotomic divisibility assumption, we have $|A'_{p_i}| = p_j p_k$. On the other hand, by the fibering properties of A,

$$p_j p_k = |A'_{p_i}| = c_j p_j^2 + c_k p_k, \quad c_j > 0.$$

Thus $c_j = p_k c_j'$ and $p_j = c_j' p_j^2 + c_k$ with $c_j' > 0$, a contradiction.

Therefore $\Phi_{N_j} \nmid A$. By (9.91), we have $\Phi_{N_j}\Phi_{M_j}|B$. Applying the same argument as in (9.93), (9.94) to p_j instead of p_k, we get that every $b \in B$ must satisfy at least one of

$$\mathbb{B}_M[y] + \mathbb{B}_{M/p_j}[y] = 1 \text{ for every } y \in \mathbb{Z}_M \text{ with } (b-y, M) = M/p_k,$$

$$\mathbb{B}_M[y] + \mathbb{B}_{M/p_j}[y] = 1 \text{ for every } y \in \mathbb{Z}_M \text{ with } (b-y, M) = M/p_i. \tag{9.96}$$

But since $p_k > p_j$, if the former holds for some $b \in B$, we must have $M/p_k \in \mathrm{Div}(B)$, which is not allowed. Thus (9.96) holds for all $b \in B$. Hence the assumptions of Lemma 4.2 hold for B, with $m = M/p_i p_j$ and $s = p_i^2$. It follows that $\Phi_{p_i^2}|B$, and therefore $\Phi_{p_i}|A$. □

Lemma 9.40. *Assume (F3), (9.91), and (9.92). Then the conditions of Theorem 2.6 are satisfied in the p_i direction, after interchanging A and B.*

Proof. We verify the conditions of Theorem 2.6. By (9.91) and Lemma 9.39, we have $\Phi_d|B$ for $d \in \{p_i^2, M_j, M_k, N_j\}$. Next, we claim that

$$\Phi_d|B \text{ for } d \in \{M/p_j p_k, M/p_j^2 p_k, M/p_j p_k^2\}. \tag{9.97}$$

Bince (9.96) holds for all $b \in B$, we may write B as

$$B(X) = B^i(X) + Q(X)(X^{M/p_j} - 1) \tag{9.98}$$

for some polynomial $Q(X)$, where B^i is M-fibered in the p_i direction. By (9.4), we have $\Phi_d|B^i$ for all $p_i^2|d$. Using also that $\Phi_d|(X^{M/p_j} - 1)$ for all $d|M/p_j$, we get (9.97).

Finally, since $\Phi_M|A$, we need to prove that $\Phi_{M/p_i^\alpha}|A$ for $\alpha \in \{1, 2\}$. Indeed, since $\mathcal{I}$ is empty, this follows from Lemma 9.8. □

This concludes the proof of Proposition 9.4.

10. Acknowledgement

Both authors were supported by NSERC Discovery Grants.

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