

# Incidence theorems for pseudoflats

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## Abstract

We prove Pach-Sharir type incidence theorems for a class of curves in  $\mathbb{R}^n$  and surfaces in  $\mathbb{R}^3$ , which we call *pseudoflats*. In particular, our results apply to a wide class of generic irreducible real algebraic sets of bounded degree.

## 1 Introduction

One of the most intriguing, and most actively studied, problems in combinatorial geometry is finding upper bounds on the number of point-curve and point-surface incidences. The best known such result, which has since become an indispensable tool with a wide variety of applications in discrete and combinatorial geometry, is the Szemerédi-Trotter theorem [18] on point-line incidences in the plane.

There are now many extensions and generalizations of the Szemerédi-Trotter theorem. In one direction, Pach and Sharir [14] have obtained an analogous incidence bound for *pseudolines*, i.e. planar curves which are uniquely determined by a certain fixed number  $r$  of points (for lines, we have  $r = 2$ ). Another line of work concerns point-surface, especially point-hyperplane, incidences in higher dimensions; see [15] for an excellent survey. Recently, Elekes and Tóth [10] obtained a sharp Szemerédi-Trotter type bound for point-hyperplane incidences in  $\mathbb{R}^n$ , assuming a certain non-degeneracy condition; this bound was refined further by Solymosi and Tóth

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[16] under the additional assumption that the point set in question is *homogeneous* (see below).

The purpose of this paper is to propose a common generalization of the results of [14] and [10], [16], for homogeneous point sets in  $\mathbb{R}^3$ . Specifically, we first obtain Pach-Sharir type bounds for a class of curves (which we will also call pseudolines) in  $\mathbb{R}^n$ , recovering a special case of the bound of [14] for  $n = 2$ . We then use it to prove the main result of our paper, namely an analogous incidence theorem for a class of 2-dimensional surfaces in  $\mathbb{R}^3$  (*pseudoplanes*). For 2-dimensional planes in  $\mathbb{R}^3$ , our bound differs from that of [10], [16] only by the additional logarithmic factors; on the other hand, our non-degeneracy assumption is weaker than that of [10], [16].

**Definition 1.1.** (*cf. [17], [16], [11]*) *A finite point set  $P \subset \mathbb{R}^n$  is called homogeneous if  $P$  lies in the interior of an  $n$ -dimensional cube  $Q = [0, a]^n$  of volume  $\Theta(|P|)^1$ , and if any unit cube in  $\mathbb{R}^n$  contains at most  $O(1)$  points of  $P$ .*

Fix  $P$  as above. We will say that a set  $S \subset \mathbb{R}^n$  (usually a curve or surface) is  *$m$ -rich* if it contains at least  $m$  points of  $P$ . As in [10], [16], our results are stated in terms of a bound on the number of  $m$ -rich pseudolines and pseudoplanes for a fixed point set  $P$  of cardinality  $N$ .

Our first result concerns incidences for one-dimensional *pseudolines*, which we now define.

**Definition 1.2.** (*cf. [14]*) *Let  $\mathcal{V}$  be a family of subsets in  $\mathbb{R}^n$ . We say that  $\mathcal{V}$  is a type  $r$  family of pseudolines if the following two conditions are satisfied:*

(i) (rectifiability) *Let  $t \in \mathbb{N}$ . If the enclosing cube  $Q$  is subdivided into  $t^n$  congruent and disjoint (except for boundary) subcubes, then each  $V \in \mathcal{V}$  has nonempty intersection with at most  $O(t)$  subcubes.*

(ii) (type  $r$ ) *For any distinct  $r$  points in  $\mathbb{R}^n$ , there is at most one  $V \in \mathcal{V}$  which contains them all.*

In Section 2, we verify that the conditions of Definition 1.2 hold if  $\mathcal{V}$  is a family of irreducible one-dimensional algebraic varieties defined by polynomial equations of degree at most  $d$  (with an explicit value of  $r$ , depending on  $n$  and  $d$ ). It is useful to think of the elements of  $\mathcal{V}$  as curves, since this description applies to the main cases of interest (such as the algebraic varieties just mentioned). Note that the rectifiability condition (i) implies that

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<sup>1</sup>Here and below, all constants depend only on  $n$  and  $r$ .

each  $V \in \mathcal{V}$  has Hausdorff dimension at most 1. However, no continuity or smoothness assumptions are actually required.

**Theorem 1.3.** *Let  $\mathcal{V}$  be a type  $r$  family of pseudolines in  $\mathbb{R}^n$ , and let  $P$  be a homogeneous set of  $N$  points. Then:*

(i) *if  $k \geq CN^{1/n}$  for large enough  $C$ , then there are no  $k$ -rich pseudolines in  $\mathcal{V}$ ;*

(ii) *if  $k \leq CN^{1/n}$ , then the number of  $k$ -rich pseudolines in  $\mathcal{V}$  is bounded by  $O(N^r/k^{n(r-1)+1})$ .*

For  $n = 2$ , we recover a special case (for homogeneous point sets) of the Pach-Sharir theorem on incidences for pseudolines [14]. For  $n \geq 3$ , Theorem 1.3 extends a result of Solymosi and Vu [17] on incidences for lines in  $\mathbb{R}^n$ . Our proof is in fact very similar to that of [17].

Our main result concerns 2-dimensional surfaces in  $\mathbb{R}^3$ . (Again, the rectifiability assumption (i) implies Hausdorff dimension at most 2, but otherwise our “surfaces” could be quite arbitrary.)

**Definition 1.4.** *Let  $\mathcal{S}$  be a family of subsets of  $\mathbb{R}^3$ . We say that  $\mathcal{S}$  is a type  $r$  family of pseudoplanes if the following holds:*

(i) (rectifiability) *Let  $t \in \mathbb{N}$ . If the enclosing cube  $Q$  is subdivided into  $t^3$  congruent and disjoint (except for boundary) subcubes, then each  $S \in \mathcal{S}$  has nonempty intersection with at most  $O(t^2)$  subcubes.*

(ii) (pairwise intersections are pseudolines) *Let*

$$\mathcal{V} = \{S \cap S' : S, S' \in \mathcal{S}, S \neq S'\}.$$

*Then  $\mathcal{V}$  is a type  $r$  family of pseudolines.*

**Theorem 1.5.** *Let  $P$  be a homogeneous point set in a cube  $Q$ ,  $|P| = N$ . Let  $\mathcal{S}$  be a type  $r$  family of pseudoplanes in  $\mathbb{R}^3$ . Assume that each  $S \in \mathcal{S}$  contains a defining  $r + 1$ -tuple, i.e.  $r + 1$  distinct points  $x_1, \dots, x_{r+1}$  such that no pseudoplane in  $\mathcal{S}$ , other than  $S$  itself, contains them all. Then:*

(i) *if  $k \geq CN^{2/3}$  for large enough  $C$ , then there are no  $k$ -rich pseudoplanes in  $\mathcal{S}$ ;*

(ii) *if  $k \leq CN^{2/3}$ , then the number of  $k$ -rich pseudoplanes in  $\mathcal{S}$  is bounded by  $O(N^{r+1}(\log N \log k)^{3r/2+2}/k^{3r/2+1})$ .*

**Remark.** It is well known (cf. [10], Section 2) that bounds on the number of  $k$ -rich curves or surfaces are equivalent to the more standard formulation in terms of a bound on the total number of incidences between the point set and the objects in question. Essentially (i.e. modulo the endpoints), we obtain bounds  $O(M^{\frac{n(r-1)}{n(r-1)+1}} N^{\frac{r}{n(r-1)+1}})$  on the number of incidences between  $M$  type  $r$  pseudolines and a homogeneous set of  $N$  points in  $\mathbb{R}^n$ , and  $O(M^{\frac{3r}{3r+2}} N^{\frac{2r+2}{3r+2}})$  on the number of incidences between  $M$  type  $r$  pseudoplanes and a homogeneous set of  $N$  points in  $\mathbb{R}^3$ .

We briefly explain how our results fit in with the existing literature on point-plane incidences. It is clear that if  $m$  points of  $P$  lie on one line, then any plane containing this line will be  $m$ -rich, and there is no bound on the number of such planes. Therefore any non-trivial Szemerédi-Trotter type results for higher-dimensional surfaces must make some non-degeneracy assumption on the plane-point configuration. The special cases considered in the literature include configurations where there are no three collinear points [7], the incidence graph does not contain a  $K_{r,r}$  [5], and when all hyperplanes are spanned by the point set [1] (see also [9]).

Our result is closest to those of [10] and [16]. Elekes and Tóth [10] give a sharp bound  $O(\frac{N^n}{k^{n+1}} + \frac{N^{n-1}}{k^{n-1}})$  on the number of  $k$ -rich  $n - 1$ -dimensional hyperplanes in  $\mathbb{R}^n$ , with respect to a point set  $P$  of cardinality  $N$ , provided that the hyperplanes are *not-too-degenerate* in the following sense: there is an  $\alpha < 1$  such that for each hyperplane  $S$ , no more than  $\alpha|P \cap S|$  points of  $P \cap S$  lie in a lower-dimensional flat. This estimate was strengthened by Solymosi and Tóth [16] to  $O(\frac{N^n}{k^{n+1}})$ , under the same non-degeneracy assumption and with the additional condition that the point set  $P$  is homogeneous; the homogeneity assumption is necessary here, as Elekes and Tóth observe that for certain ranges of  $N, k$  the second term in the bound of [10] is dominant and sharp.

The bound in Theorem 1.5 with  $r = 2$  matches that of [16] for  $n = 3$ , modulo the extra logarithmic factors. On the other hand, the nondegeneracy condition of [10], [16] is stronger than that of Theorem 1.5, where it suffices for each plane to contain just one defining triple (i.e. non-trivial triangle) of points in  $P$ . We do not know whether the logarithmic factors can be dropped without additional assumptions (such as those in [10], [16]) on the distribution of *all* points in each plane.

If we only assume that each  $n - 1$ -dimensional hyperplane contains  $n$  affinely independent points of  $P$ , but no further conditions are imposed on

the hyperplane-point configuration, the bounds of [10], [16] are known to be false. In this case, an optimal incidence bound is due to Agarwal and Aronov [1], namely there are at most  $O(\frac{N^n}{k^3} + \frac{N^{n-1}}{k})$   $k$ -rich hyperplanes spanned by  $P$ . Examples of [6], [8] show that this estimate is sharp. This also shows that our theorem would already fail for  $r = 2$  without the homogeneity assumption on  $P$ .

Incidence bounds for curves and surfaces, other than lines and planes, in dimensions 3 and higher, are in general not well understood. Non-trivial bounds have been obtained only in certain special cases, e.g. circles, spheres, convex plane curves lying in distinct planes. See [15] for an overview.

A very rough outline of the proof of Theorem 1.5 is as follows. Consider a sequence of nested cell decompositions of the enclosing cube  $Q$ . In a coarse decomposition (e.g. when the entire  $Q$  is a single cell), there are many same-cell defining  $r + 1$ -tuples of points of  $P$ . In a very fine decomposition, e.g. if each cell contains only one point of  $P$ , there are no such  $r + 1$ -tuples. We find an intermediate scale at which the transition takes place for most points and surfaces. At that scale, if each cell contains few points of  $P \cap S$  for most  $S$ , we simply double-count the number of defining same-cell  $r + 1$ -tuples. If on the other hand each cell contains many points of  $P \cap S$ , then these points must in fact live on the intersection pseudolines in  $\mathcal{V}$ . Our bound is now obtained by applying Theorem 1.3 to  $\mathcal{V}$  and to the points of  $P$  in each cell.

Our argument does not seem to extend easily to hypersurfaces in higher dimensions, and in fact it is not clear how one should define higher-dimensional pseudoflats. For example, our proof relies heavily on the assumption that pairwise intersections of surfaces are pseudolines defined uniquely by  $r$  points; this condition fails for generic higher-dimensional hypersurfaces where pairwise intersections may have dimension 2 or more. There seems to be no easy way to circumvent this by considering multiple intersections.

The plan of the paper is as follows. In Section 2, we discuss the applicability of Theorems 1.3 and 1.5 to algebraic sets. We then prove Theorems 1.3 and 1.5 in Sections 3 and 4, respectively.

We use Roman letters to denote Cartesian coordinates of points in  $\mathbb{R}^n$ , eg.  $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ .

## 2 Algebraic varieties as pseudoflats

In this section we state conditions on families of algebraic varieties under which the assumptions of Definitions 1.2 and 1.4 are satisfied.

We briefly recall a few basic definitions from algebraic geometry, restricting our attention to *real* algebraic sets. The reader is cautioned that the terminology and features of real algebraic geometry are sometimes quite different from the complex case; see e.g. [4] for more details.

For the purposes of this paper, a *real algebraic set* or *real algebraic variety* (we will usually omit the qualifier “real” in what follows) is the zero set in  $\mathbb{R}^n$  of a finite family of polynomials  $F_1(x), \dots, F_s(x)$  with real coefficients<sup>2</sup>. We will say that an algebraic set  $S$  is *reducible* if there are two algebraic sets  $S', S''$ , neither equal to  $S$ , such that  $S = S' \cup S''$ ; otherwise, we will say that  $S$  is *irreducible*.

There are several equivalent ways of defining the *dimension* of an algebraic set. The easiest one for us to use is the following: the dimension  $k$  of an algebraic set  $S$  is the length of the longest chain of irreducible varieties  $S_j$  such that

$$\emptyset \neq S_1 \subsetneq S_2 \subsetneq \dots \subsetneq S_k \subset S.$$

In particular, an irreducible variety does not contain any proper subvariety of the same dimension.

If an algebraic set  $S$  is a  $C^\infty$   $k$ -dimensional submanifold of  $\mathbb{R}^n$ , then its algebraic dimension is  $k$ . Note, however, that there are irreducible real algebraic sets which consist of several components of different topological dimensions (see e.g. the examples in [4], pp. 60–61). In such cases, the algebraic dimension of the set will be the largest of the dimensions of its components.

**Proposition 2.1.** *Let  $\mathcal{V}$  be a family of irreducible one-dimensional varieties in  $\mathbb{R}^n$ , defined by a polynomial equations of degree at most  $d$ . Then  $\mathcal{V}$  is a type  $r$  family of pseudolines, with  $r = d^2 + 1$  if  $n = 2$ , and with  $r = d(2d - 1)^{n-1} + 1$  if  $n \geq 3$ .*

**Proposition 2.2.** *Let  $\mathcal{S}$  be a family of 2-dimensional algebraic varieties in  $\mathbb{R}^3$ , each given by a polynomial equation of degree no more than  $d$ . Assume*

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<sup>2</sup>In real algebraic geometry, one also considers algebraic sets over *real closed fields*. Furthermore, it is useful to distinguish between an algebraic set and an algebraic variety, the latter being an algebraic set equipped with a sheaf of regular functions. However, we do not need to make this distinction here.

that the intersection  $V = S \cap S'$  of any two varieties  $S, S' \in \mathcal{S}$ ,  $S \neq S'$ , is an irreducible one-dimensional variety. Then  $\mathcal{S}$  is a type  $r$  family of pseudo-planes, with  $r = d(2d - 1)^2 + 1$ .

The proofs of both propositions will rely on the following result from real algebraic geometry [2], [3]. Let  $V \subset \mathbb{R}^n$  be a  $k$ -dimensional variety defined by polynomials of degree at most  $d$ . Let also  $P_1, \dots, P_s$  be polynomials in  $n$  variables of degree at most  $d$ . A *sign condition* for the set  $\mathcal{P} = \{P_1, \dots, P_s\}$  is a vector  $\sigma \in \{-1, 0, 1\}^s$ . We write

$$\sigma_{\mathcal{P}, V} = \{x \in V, (\text{sign}(P_1(x)), \dots, \text{sign}(P_n(x))) = \sigma\},$$

and call its non-empty semi-algebraically connected components *cells* of the sign condition  $\sigma$  for  $\mathcal{P}$  over  $V$ . Let  $|\sigma_{\mathcal{P}, V}|$  be the number of such cells, then

$$C(\mathcal{P}, V) = \sum_{\sigma} |\sigma_{\mathcal{P}, V}|$$

is the number of all cells defined by all possible sign conditions. Let  $f(d, n, k, s)$  be the maximum of  $C(\mathcal{P}, V)$  over all varieties  $V \subset \mathbb{R}^d$  and sets of polynomials  $\mathcal{P}$  as described above. Then the main result of [2] (see also [3]) is that

$$f(d, n, k, s) = \binom{s}{k} (O(d))^n. \quad (1)$$

*Proof of Proposition 2.1.* If  $\mathcal{V}$  is a family of one-dimensional irreducible varieties in  $\mathbb{R}^2$ , each defined by a polynomial equation of degree at most  $d$ , it follows from Bezout's theorem that any two distinct varieties in  $\mathcal{V}$  intersect in no more than  $d^2$  points, hence  $\mathcal{V}$  is type  $r$  for  $r = d^2 + 1$ .

Suppose now that  $\mathcal{V}$  is a family of one-dimensional irreducible varieties in  $\mathbb{R}^n$ ,  $n \geq 3$ , each one defined by a system of polynomial equations in  $n$  variables of degree at most  $d$ . The classic results on the sum of Betti numbers of algebraic sets [13], [12], [19] imply in particular that the intersection of two such distinct varieties has no more than  $d(2d - 1)^{n-1}$  connected components; since the varieties are irreducible, the intersection is a variety of dimension 0, hence each connected component is a single point. Thus we may take  $r = d(2d - 1)^{n-1} + 1$ .

It remains to verify rectifiability. Let  $t \in \mathbb{N}$ . We subdivide the enclosing cube  $Q$  into  $t^n$  congruent open subcubes  $Q_j = \{(j_i - 1)a/t < x_i < j_i a/t, i =$

$1, \dots, n\}$ , indexed by  $j = (j_1, \dots, j_n) \in \{1, \dots, t\}^n$ . Fix a  $V \in \mathcal{V}$ , and consider the sets  $V_j = V \cap Q_j$ . We have  $P \cap V \subset \bigcup_j (P \cap V_j)$ . Each nonempty  $V_j$  contains a cell of  $V$  associated with a suitable sign condition for the system of polynomials  $P_{i,s}(x) = x_i - sa/t$ ,  $i = 1, \dots, n$ ,  $s = 1, \dots, t$ . By (1), the number of such cells is bounded by  $ntO(d)^n$ , as required.  $\square$

*Proof of Proposition 2.2.* Let  $\mathcal{V} = \{S \cap S' : S, S' \in \mathcal{S}, S \neq S'\}$ , then  $\mathcal{V}$  is a type  $r$  family of pseudolines by Proposition 2.1. The proof of rectifiability is the same as in the proof of Proposition 2.1, except that this time (1) yields the bound  $\binom{nt}{2}O(d)^n$  on the number of non-empty cells.  $\square$

### 3 Proof of Theorem 1.3

We assume that the enclosing cube for  $P$  is  $Q = [0, a]^n$  for some positive integer  $a$ , and that all points in  $P$  have irrational coordinates. We always let  $N$  be sufficiently large. Without loss of generality, we assume that all  $V \in \mathcal{V}$  are  $k$ -rich.

We first prove (i). Let  $t$  be an integer to be fixed later. We subdivide the enclosing cube  $Q$  into  $t^n$  congruent open subcubes  $Q_j = \{(j_i - 1)a/t < x_i < j_i a/t, i = 1, \dots, n\}$ , indexed by  $j = (j_1, \dots, j_n) \in \{1, \dots, t\}^n$ . By the assumption from the last paragraph, no points in  $P$  lie on the boundary of any  $Q_j$ . For each  $V \in \mathcal{V}$ , we let  $V_j = V \cap Q_j$ . We have  $P \cap V \subset \bigcup_j (P \cap V_j)$ . By the rectifiability assumption, the number of non-empty  $V_j$ 's is at most  $O(t)$ .

We now choose  $t$  so that  $t^n = \Theta(N)$ . Since  $P$  is homogeneous, each  $Q_j$  contains no more than  $O(1)$  points of  $P$ . Hence the cardinality of  $P \cap V$  is bounded by  $O(t) = O(N^{1/n})$ , as claimed.

It remains to prove (ii). We divide  $Q$  into  $t^n$  subcubes  $Q_j$  as in the proof of (i), except that this time we will choose

$$t = \Theta(k), \tag{2}$$

with the implicit constants small enough (depending on  $n, r$ ). In particular, by (i) we may assume that  $t^n \leq N$ , since otherwise there is nothing left to prove.

A  $r$ -tuple of distinct points  $x_1, \dots, x_r \in P$  is *good* if all  $x_i$  belong to the same subcube  $Q_j$ . We count the number  $M$  of good  $r$ -tuples in two ways. On



one hand, since  $P$  is homogeneous, each  $Q_j$  contains no more than  $O(N/t^n)$  points of  $P$ . Thus

$$M = O((N/t^n)^r \cdot t^n) = O(N^r/t^{n(r-1)}).$$

On the other hand, let  $V_j = V \cap Q_j$  as in the proof of (i). Each  $r$ -tuple of points in  $P \cap V$  contained in one  $P \cap V_j$  is good. By rectifiability, the number  $K$  of distinct and non-empty  $V_j$ 's is bounded by  $k/r$  (provided that the constants in (2) were chosen appropriately small). Thus the number of good  $r$ -tuples in  $P \cap V$  is  $\Omega((k/K)^r \cdot K) = \Omega(k^r/K^{r-1}) = \Omega(k)$ .

Summing over all  $V \in \mathcal{V}$  and remembering that any  $r$ -tuple can belong to only one  $V$  (since  $\mathcal{V}$  is type  $r$ ), we see that

$$M = \Omega(|\mathcal{V}|k).$$

Comparing the upper and lower bounds for  $M$ , and using (2), we see that

$$|\mathcal{V}| = O\left(\frac{N^r}{kt^{n(r-1)}}\right) = O\left(\frac{N^r}{k^{n(r-1)+1}}\right)$$

as claimed.

## 4 Proof of Theorem 1.5

The proof of (i) is identical to that of Theorem 1.3(i), except that the 2-dimensional rectifiability condition with  $n = 3$  yields the exponent  $2/3$  as indicated in the theorem. We omit the details.

We now prove (ii). Let  $|\mathcal{S}| = X$ . We assume that  $Q = [0, a]^3$  for some positive integer  $a$ , and that all points in  $P$  have irrational coordinates.

For  $i = 0, 1, 2, \dots, I$ , we define the  $i$ -th cutting of  $Q$  to be the subdivision of  $Q$  into  $2^{3i}$  congruent open subcubes  $Q_{i,j}$  of sidelength  $a/2^i$ . We let  $I = \Theta(\log N)$  so that each subcube in the  $I$ -th cutting contains at most 1 point of  $P$ . Note that no points in  $P$  lie on the boundary of any  $Q_{i,j}$ .

For each  $i$  and each  $S \in \mathcal{S}$ , the  $i$ -th cutting divides  $S$  into subsets  $S_{i,j} = S \cap Q_{i,j}$ . By the rectifiability assumption, we have

$$|\{j : S_{i,j} \neq \emptyset\}| = O(2^{2i}), \tag{3}$$

with constants uniform in  $i$ .

Let  $S \in \mathcal{S}$ . We will say that a  $r+1$ -tuple of points  $x_1, \dots, x_{r+1}$  is *defining for  $S$  at level  $i$*  if  $x_1, \dots, x_{r+1}$  are distinct points in  $P \cap S$  which all belong to the same subcube of the  $i$ -th cutting, and if moreover there is no other surface  $S' \in \mathcal{S}$ ,  $S' \neq S$ , such that  $x_1, \dots, x_{r+1} \in S'$ . Thus a defining  $r+1$ -tuple at level 0 is simply a defining  $r+1$ -tuple for  $S$  as in the statement of the theorem.

We define the *index*  $i(x, S)$  of a pair  $(x, S)$ , where  $S \in \mathcal{S}$ ,  $x \in P \cap S$ , to be the least value of  $i$  such that  $x$  does not belong to a defining  $r+1$ -tuple for  $S$  at level  $i$ .

**Lemma 4.1.** *For all  $S \in \mathcal{S}$ ,  $x \in P \cap S$ , we have  $1 \leq i(x, S) \leq I$ .*

*Proof.* Clearly there are no defining  $r+1$ -tuples at level  $I$ , hence  $i(x, S) \leq I$ . It remains to prove that  $i(x, S) \geq 1$  for all  $x \in P \cap S$ ,  $S \in \mathcal{S}$ . Indeed, fix  $S \in \mathcal{S}$  and  $x_0 \in P \cap S$ . We need to prove that  $x_0$  belongs to a defining  $r+1$ -tuple for  $S$ . By the non-degeneracy assumption,  $S$  contains a defining  $r+1$ -tuple  $T = \{x_1, \dots, x_{r+1}\}$ . If  $x_0 \in T$ , we are done. Otherwise, let  $T_j = (T \setminus \{x_j\}) \cup \{x_0\}$ , and suppose that  $T_1, \dots, T_{r+1}$  are all non-defining. This means that for each  $j = 1, \dots, r+1$  there is a  $S_j \in \mathcal{S}$ ,  $S_j \neq S$ , such that  $T_j \subset V_j := S \cap S_j$ . In particular, if  $j \geq 2$ , then  $V_1$  and  $V_j$  share the  $r$  points  $x_m$ ,  $m \in \{0, 1, \dots, r+1\} \setminus \{1, j\}$ . By the  $r$ -type assumption,  $V_1 = V_j$ ,  $j = 2, \dots, r+1$ . But then all the points  $x_1, \dots, x_{r+1}$  belong to  $V_1 = S \cap S_1$ , hence  $T$  is not defining, contradicting our assumption. It follows that at least one of  $T_1, \dots, T_{r+1}$  is a defining  $r+1$ -tuple for  $S$  containing  $x_0$ , as required.  $\square$

For each  $S$ , we choose  $i(S)$  to be the least value of  $i$  such that

$$|\{x : x \in P \cap S, i(x, S) = i(S)\}| \geq k/I.$$

We then choose an  $\mathbf{i} \in \{1, \dots, I\}$  and a subset  $\mathcal{S}_1 \subset \mathcal{S}$  such that

$$|\mathcal{S}_1| \geq |\mathcal{S}|/I, \quad i(S) = \mathbf{i} \text{ for all } S \in \mathcal{S}_1.$$

**Case 1:**  $k \leq C_0 2^{21} \log N \log k$ . We count the number  $M$  of all defining  $r+1$ -tuples for all  $S \in \mathcal{S}_1$  at level  $\mathbf{i}-1$ . Each  $S \in \mathcal{S}_1$  contains at least  $k/I$  points  $x \in P$  with index  $i(x, S) = \mathbf{i}$ . Each such point must belong to a defining  $r+1$ -tuple for  $S$  at level  $\mathbf{i}-1$ , and each  $r+1$ -tuple can be defining for only one  $S$ . Thus

$$M \geq \frac{X}{I} \cdot \frac{k}{I} \cdot \frac{1}{r+1}.$$

On the other hand,  $M$  is trivially bounded from above by the total number of the  $r + 1$ -tuples that belong to the same cube of the  $\mathbf{i} - 1$ -th cutting,

$$M = O\left(\left(\frac{N}{2^{3\mathbf{i}-3}}\right)^{r+1} \cdot 2^{3\mathbf{i}-3}\right) = O(N^{r+1}/2^{3\mathbf{i}r}).$$

Comparing the upper and lower bounds, and using the assumption on  $k$  for Case 1, we get

$$X = O\left(\frac{N^{r+1}I^2}{k \cdot 2^{3\mathbf{i}r}}\right) = O\left(\frac{N^{r+1}(\log N \log k)^{3r/2}(\log N)^2}{k^{3r/2+1}}\right)$$

as required.

**Case 2:**  $k \geq C_0 2^{2\mathbf{i}} \log N \log k$ . In this case, points of  $P$  tend to be aligned along the one-dimensional intersection curves; we will therefore use our one-dimensional incidence bound. We first do some pigeonholing to fix the values of certain parameters. For each  $S \in \mathcal{S}_1$ , we let

$$P(S) = \{x \in P \cap S : i(x, S) = \mathbf{i}\},$$

then  $|P(S)| \geq k/I$ . We then choose a subset  $P_0(S) \subset P(S)$  such that  $|P_0(S)| \in [\frac{k}{I}, \frac{k}{I} + 1)$ . Let  $L \geq 1$  be an integer such that  $2^{L-1} \leq k < 2^L$  (hence  $L = \Theta(\log k)$ ). Note that for each  $j$ ,

$$|P_0(S) \cap S \cap \mathcal{Q}_{\mathbf{i},j}| \leq |P_0(S)| \leq \frac{k}{I} + 1 \leq k + 1 \leq 2^{L+1}.$$

Thus if we let

$$m(l, S) = |\{j : |P_0(S) \cap S \cap \mathcal{Q}_{\mathbf{i},j}| \in [2^l, 2^{l+1}]\}|, \quad l = 0, 1, \dots, L-1,$$

then for each  $S \in \mathcal{S}_1$ ,

$$\sum_{l=0}^{L-1} m(l, S) \cdot 2^{l+1} \geq |P_0(S)| \geq k/I,$$

hence we may choose  $l(S)$  such that

$$m(l(S), S) \cdot 2^{l(S)+1} \geq k(IL)^{-1}.$$

Pigeonholing again, we find a value of  $\mathbf{l} \in \{0, \dots, L-1\}$  and a set  $\mathcal{S}_2 \subset \mathcal{S}_1$  such that

$$|\mathcal{S}_2| \geq |\mathcal{S}_1|/L, \quad l(S) = \mathbf{l} \text{ for all } S \in \mathcal{S}_2.$$

Let  $S \in \mathcal{S}_2$ , and let  $S_j = S \cap Q_{\mathbf{i},j}$ . Relabelling the subcubes if necessary, we may assume that

$$|P_0(S) \cap S_j| \geq 2^{\mathbf{l}}, \quad j = 1, \dots, m,$$

where

$$m \cdot 2^{\mathbf{l}+1} \geq k(IL)^{-1}. \quad (4)$$

By rectifiability, we have  $m \leq C \cdot 2^{2\mathbf{i}}$ . Thus it follows that

$$2^{\mathbf{l}} \geq k(2ILm)^{-1} \geq r + 1, \quad (5)$$

provided that the constant  $C_0$  in the assumption of Case 2 was chosen large enough.

We now claim that for each  $j = 1, \dots, m$ , there is a unique  $V_j \in \mathcal{V}$  such that  $P_0(S) \cap S_j \subset V_j$ . Indeed, let  $x_1, \dots, x_r, x_{r+1} \in P_0(S) \cap S_j$ . By the definition of  $P_0(S)$  and  $\mathbf{i}$ ,  $x_1, \dots, x_{r+1}$  is not a defining  $r+1$ -tuple for  $S$ , hence there is a  $S' \in \mathcal{S}$ ,  $S' \neq S$  such that  $x_1, \dots, x_{r+1} \in S'$ . Thus  $x_1, \dots, x_{r+1} \in V_j := S \cap S'$ . Let now  $x \in P_0(S) \cap S_j$ ,  $x \neq x_1, \dots, x_{r+1}$ . Then  $x_1, \dots, x_r, x$  is another non-defining  $r$ -tuple, hence  $x_1, \dots, x_r, x \in V'_j := S \cap S''$  for some  $S'' \in \mathcal{S}$ . But then  $V'_j$  intersects  $V_j$  in  $r$  distinct points  $x_1, \dots, x_r$ , hence  $V'_j = V_j$  since  $\mathcal{V}$  is type  $r$ . It follows that  $x \in V_j$  for all  $x \in P_0(S) \cap S_j$ , as claimed.

Thus for each  $S \in \mathcal{S}_2$ , there are at least  $m$  subcubes  $Q_{\mathbf{i},j}$  with the following property: there is a  $V_j = V_j(S) \in \mathcal{V}$  which contains at least  $2^{2\mathbf{l}}$  points of  $S \cap Q_{\mathbf{i},j}$  with index  $i(x, S) = \mathbf{i}$ . Moreover, we have (4) and (5).

Now for the main argument. We count the number  $M'$  of ‘‘admissible’’ triples  $(S, V, j)$  such that:

- $S \in \mathcal{S}$ ,  $V \in \mathcal{V}$ ,  $V = S \cap S'$  for some  $S' \in \mathcal{S}$ ;
- $|V \cap P \cap Q_{\mathbf{i},j}| \geq 2^{\mathbf{l}}$ .

**Lower bound:** From the above construction, for each  $S \in \mathcal{S}_2$  there are at least  $m$  values of  $j$  such that

$$|V \cap P \cap Q_{\mathbf{i},j}| \geq |P_0(S) \cap S_j| \geq 2^{\mathbf{l}}$$

for some  $V \in \mathcal{V}$  (depending on  $j$ ). Hence

$$M' \geq |\mathcal{S}_2| \cdot m = \Omega(X \cdot k/IL).$$

**Upper bound:** There are three ingredients.

- First, there are at most  $2^{3i}$  values of  $j$ .
- For each  $j$ , we estimate the number of eligible  $V$ 's by applying Theorem 1.3 to  $\mathcal{V}$  and to point sets  $P_j := P \cap Q_{\mathbf{i},j}$ , homogeneous in  $Q_{\mathbf{i},j}$  and of cardinality  $\Theta(N/2^{3i})$ . Thus the number of  $V \in \mathcal{V}$  containing at least  $2^{\mathbf{1}}$  points of  $P_j$  is bounded by

$$O\left(\frac{(N/2^{3i})^r}{(2^{\mathbf{1}})^{3r-2}}\right).$$

- Finally, we claim that for each such fixed  $j$  and  $V$ , there are at most  $O(N/2^{3i})$  surfaces  $S \in \mathcal{S}$  such that  $(S, V, j)$  is admissible. Indeed, define the *parent cube* of  $Q_{\mathbf{i},j}$  to be the unique cube in the  $\mathbf{i} - 1$ -th cutting which contains  $Q_{\mathbf{i},j}$ . Suppose that  $(S, V, j)$  is admissible. Then  $V \cap Q_{\mathbf{i},j}$  contains at least  $2^{\mathbf{1}}$  points  $x \in P$  with  $i(x, S) = \mathbf{i}$ . Fix such an  $x$ , then by the definition of index,  $x$  belongs to a defining  $r + 1$ -tuple for  $S$  at level  $\mathbf{i} - 1$ , i.e. contained in the parent cube. This  $r + 1$ -tuple must contain at least one point, say  $x_0$ , which is not in  $V$ . It remains to prove that  $V$  and  $x_0$  define  $S$  uniquely; this implies the claim, since the parent cube contains at most  $O(N/2^{3i})$  points of  $P$ .

By (5), there are at least  $r + 1$  distinct points  $x_1, \dots, x_{r+1}$  in  $V \cap P_j$ . It suffices to prove that  $x_0, x_1, \dots, x_r$  is a defining  $r + 1$ -tuple for  $S$  at level  $\mathbf{i} - 1$ . Indeed, suppose to the contrary that there is a  $S'' \in \mathcal{S}$ ,  $S'' \neq S$ , such that  $x_0, x_1, \dots, x_r \in V' := S \cap S''$ . But then  $x_1, \dots, x_r \in V \cap V'$ . Since  $\mathcal{V}$  is type  $r$ , it follows that  $V' = V$ , and in particular that  $x_0 \in V$ , contrary to our choice of  $x_0$ .

Combining the three estimates, we obtain the upper bound

$$M' = O\left(\frac{(N/2^{3i})^r}{(2^{\mathbf{1}})^{3r-2}} \cdot \frac{N}{2^{3i}} \cdot 2^{3i}\right) = O\left(\frac{N^{r+1}2^{-3ir}}{(2^{\mathbf{1}})^{3r-2}}\right).$$

**Conclusion:** Comparing the upper and lower bounds on  $M'$ , we get that

$$X = O\left(\frac{N^{r+1}2^{-3ir} \log N \log k}{(2^{\mathbf{1}})^{3r-2m}}\right).$$

By (4), we have  $2^{\mathbf{1}} \geq k(m \log N \log k)^{-1}$ . Hence

$$X = O\left(\frac{N^{r+1}2^{-3ir} \log N \log k}{(k/m \log N \log k)^{3r-2m}}\right)$$

$$= O\left(\frac{N^{r+1}2^{-3ir}m^{3r-3}}{k^{3r-2}}(\log N \log k)^{3r-1}\right).$$

By rectifiability, we have  $m = O(2^{2i})$ , so that

$$X = O\left(\frac{N^{r+1}2^{(3r-6)i}}{k^{3r-2}}(\log N \log k)^{3r-1}\right).$$

Finally, the assumption of Case 2 is that  $2^{2i} = O(k/\log N \log k)$ . Thus

$$\begin{aligned} X &= O\left(C \frac{N^{r+1}(k/\log N \log k)^{3r/2-3}}{k^{3r-2}}(\log N \log k)^{3r-1}\right) \\ &= O\left(\frac{N^{r+1}}{k^{3r/2+1}}(\log N \log k)^{3r/2+2}\right). \end{aligned}$$

This completes the proof of the theorem.

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