From harmonic analysis to arithmetic combinatorics: a brief survey

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The purpose of this note is to showcase a certain line of research that connects harmonic analysis, specifically restriction theory, to other areas of mathematics such as PDE, geometric measure theory, combinatorics, and number theory. There are many excellent in-depth presentations of the various areas of research that we will discuss, see e.g. the references below. The emphasis here will be on highlighting the connections between these areas.

Our starting point will be restriction theory in harmonic analysis on Euclidean spaces. The main theme of restriction theory, in this context, is the connection between the decay at infinity of the Fourier transforms of singular measures and the geometric properties of their support, including (but not necessarily limited to) curvature and dimensionality. For example, the Fourier transform of a measure supported on a hypersurface in $\mathbb{R}^d$ need not, in general, belong to any L^p with $p < \infty$, but there are positive results if the hypersurface in question is curved. A classic example is the restriction theory for the sphere, where a conjecture due to E.M. Stein asserts that the Fourier transform maps $L^\infty(S^{d-1})$ to $L^q(\mathbb{R}^d)$ for all $q > 2d/(d-1)$. This has been proved in dimension 2 (Fefferman-Stein, 1970), but remains open otherwise, despite the impressive and often groundbreaking work of Bourgain, Wolff, Tao, Christ, and others. We recommend [8] for a thorough survey of restriction theory for the sphere and other curved hypersurfaces.

Restriction-type estimates have been immensely useful in PDE theory – in fact much of the interest in the subject stems from PDE applications. Harmonic analysis techniques are used to prove spacetime $L^p_{x,t}$ or mixed norm $L^q_t L^p_x$ estimates for solutions to linear PDE, and these in turn have become

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basic tools in proving existence and regularity results for wide classes of nonlinear equations. In the simplest cases, solutions to linear equations can be rewritten as oscillatory integrals where the main contribution comes from a hypersurface in the Fourier space. A restriction estimate applied to this hypersurface yields an L^p PDE estimate. Many more such estimates rely on direct applications of restriction-type arguments (as opposed to using a restriction estimate as a black box), usually in combination with additional analytic tools. This is an extremely rich and active area of research, going back to the early 1970s (Strichartz estimates) and including for example work by Bourgain, Colliander, Keel, Kenig, Klainerman, Machedon, Smith, Sogge, Stafillani, Takaoka, Tao, Wolff, and many others. Gigliola Stafillani's survey will say more about this.

What is particularly striking about restriction and related PDE estimates is the sophisticated interplay between a wide variety of techniques from analysis, combinatorics, and geometry. The analytic methods range from the traditional, such as L^2 orthogonality or stationary phase, to the more recent additions: L^p orthogonality, local restriction estimates, bilinear estimates, induction on scales. However, the best restriction estimates available today involve also advanced geometrical and combinatorial input. The search for the latter inspired many harmonic analysts to explore such areas as graph theory, combinatorial geometry, and additive number theory. It has to be said that, for the most part, we are still searching. Nonetheless, the ensuing exchange of knowledge and ideas between experts in different fields has led to breathtaking progress in other directions, to be discussed shortly.

The restriction conjecture for the sphere, which we mentioned earlier, turns out to depend on the geometry of Kakeya sets. The latter, also known as Besicovitch sets, are subsets of $\mathbb{R}^d$ which contain a unit line segment in each direction in S^{d-1}. A classic construction from the 1920s, due to Besicovitch, shows that such sets may have d-dimensional Lebesgue measure 0; The *Kakeya conjecture* asserts that Kakeya sets must nonetheless be "large" in that they cannot have Hausdorff dimension lower than d. This has been proved in dimension 2 in the 1970s (Davies, Cordoba), but remains open in higher dimensions, although hard-won partial results are available (Bourgain, Wolff, Katz, Łaba, Tao) See [1] for a glimpse of the fascinating history of the Kakeya problem, and [12] for a modern exposition and a review of the work on the subject up to 2000.

It is known, mostly from the work of Fefferman and Cordoba in the 1970s, that the restriction conjecture implies the Kakeya conjecture. Very roughly speaking, the line segments in a Besicovitch set E are identified with "wave packets" in a suitable decomposition of the Fourier transform of a sphere-supported measure; the small size of E forces the relevant L^p norms to be large. Conversely, Bourgain's discovery in the early 1990s that this implications can be partially reversed (i.e. partial results on Kakeya can be used to make progress on restriction) led to renewed interest in the area, with significant results due to Tao, Wolff, Vargas, S. Lee, and others.

It has since become a recurring theme in this area that harmonic-analytic and PDE estimates can be partially reduced, via a wave packet decomposition, to geometrical problems involving packing a large number of thin objects (circles, thin annuli, plates) into sets of small size. For example, a circle-packing problem played a major role in Wolff's deep work (2000) on the *local smoothing conjecture* of Sogge, an outstanding open problem concerning the spacetime regularity of solutions to the wave equation. However, it seems unlikely that restriction or similar problems will be resolved by geometrical arguments alone. The best current methods involve a fusion of analytic and geometric techniques, for instance combining oscillatory estimates on fine scales with Kakeya-type information on coarse scales.

Turning to the question of obtaining the requisite Kakeya-type geometric information, we enter rather different areas of mathematics, notably combinatorial geometry and additive number theory.

Incidence geometry is a part of combinatorial geometry which deals with counting incidences between objects, typically points and curves or surfaces. (A curve is *incident* to a point if the point lies on the curve.) A classic result of this type is the Szemerédi-Trotter theorem giving a bound $O(n + m + n^{2/3}m^{2/3})$ on the number of incidences between n lines and m points in $\mathbb{R}^2$; there are also similar results involving other curves (such as circles) and higher-dimensional configurations. The use of incidence geometry in harmonic analysis, specifically treating the "wave packets"' as thin geometric objects and applying combinatorial methods to deduce information about their possible arrangements, was pioneered by Wolff in the 1990s. While the Kakeya problem resisted this approach, Wolff was much more successful with other questions, for example the local smoothing problem just mentioned. Just as importantly, ongoing communication was gradually established between discrete geometers and harmonic analysts. Many more intriguing connections between the two areas have since been uncovered and continue to be pursued. See e.g. [7] for a review of some of the recent work in incidence geometry, and [5] for a discussion of some of the connections to analysis.

Additive number theory is a mixture of combinatorics, number theory, and often incredible ingenuity and insight. The questions of interest are often stated in the language of first-grade arithmetic – addition, multiplication, and counting of integers – yet, starting with those most basic ingredients, one weaves a surprisingly rich tapestry of techniques and results. Two results that are central to, and representative of, the field are Freiman's theorem and Szemerédi's theorem. The former describes the structure of sets with small sumset, i.e. sets A such that the set of pairwise sums $\{a + b : a, b \in A\}$ has size comparable to that of A. The latter asserts (in one formulation) that a set of integers which has positive relative density must contain arbitrarily long arithmetic progressions. The statements of both theorems are elementary; the proofs are hard and deep, drawing on several different areas of mathematics. Szemerédi's theorem, by far the more difficult of the two, has in fact four remarkably distinct proofs, each of which was a milestone in combinatorics in its own right: the original combinatorial proof by Szemerédi (1974), Furstenberg's ergodic-theoretic proof (1975), Gowers's proof based on harmonic analysis (1998), and the most recent hypergraph proof, due independently to Gowers and Nagle-R"odl-Schacht-Skokan (2004). We cannot do justice to any of this work here, referring the reader to e.g. [10] instead for an overview, but we emphasize the wide diversity of techniques and ideas involved. For a broader perspective on this area, see for example [2] or [11].

A new generation of harmonic analysts was introduced to all this when Bourgain, in a brilliant leap of thought, applied additive number-theoretic techniques to the Kakeya problem. Roughly speaking, a hypothetical Kakeya set of low dimension was made to generate a set with a small (partial) sumset. A variant of Freiman's theorem developed by Gowers in the course of his work on Szemerédi's theorem - just off the press at the time - could then be applied to obtain a contradiction if the Kakeya set was too small. This led to further work on the Kakeya problem (Katz, Łaba, Tao), but also, more importantly in hindsight, attracted many of us to additive number theory and inspired us to work on its problems.

It may have been no surprise that increased interest in the subject led to progress on a variety of questions. But Green and Tao exceeded all expectations by proving, in a stunning breakthrough, that the primes contain arbitrarily long arithmetic progressions (2004). This work later became a starting point for an even more ambitious research program, addressing a very general conjecture of Hardy and Littlewood with far-reaching implications in analytic number theory. There are now many expositions and reviews of various aspect of this work, see e.g. [3], [4], [6], [9], [10]. The focus of this note will remain on connections to harmonic analysis, and so we return to restriction theory.

Restriction estimates for finite exponential sums, as opposed to continuous Fourier transforms, were first derived by Bourgain (1993) in the context of proving Strichartz estimates for solutions of evolution PDEs (Schrödinger, KdV) on the torus $\mathbb{T}^d$. They were then revisited by Green in his proof of existence of 3-term arithmetic progressions in dense subsets of primes (2003). Roughly, all proofs of Szemerédi's theorem rely on a certain dichotomy between "randomness"' and "structure". A random set contains many arithmetic progressions; if on the other hand the set is not random, one identifies a "structured" part of it and then proceeds by induction. Restriction estimates are used to show that the set of primes is sufficiently random.

Although this type of Fourier analysis is not directly applicable to Szemerédi-type problems for progressions of length 4 and more, it was reportedly a major source of ideas for Green and Tao, as well as Gowers before them. Green and Tao are currently working to develop a "quadratic Fourier analysis" that could be applied to finding 4-term progressions, or more generally solutions to systems of 2 linear equations, in suitable sets such as the primes or their dense subsets. This is a rapidly developing area and many more exciting developments are sure to follow.

It would be impossible to even attempt a complete survey of this extensive and complex body of work in a few pages. Each of the many areas of research that we mentioned can easily be the subject of a much longer article. Many fascinating problems and issues were not mentioned at all. For example, the Bochner-Riesz problem in harmonic analysis which is closely related to restriction theory, or the intriguing questions in geometric measure theory concerning projections and distance sets, with connections to both restriction theory and discrete geometry, or the applications of incidence geometry to additive number theory – the list could go on. We hope that we have at least aroused the reader's interest and inspired her to learn more about this fascinating area.

References

[1] A.S. Besicovitch, *The Kakeya problem*, Amer. Math. Monthly **70** (1963), 697-706.

[2] W.T. Gowers, *Some unsolved problems in additive and combinatorial number theory*, preprint, 2001 (available at `http://www.dpmms.cam.ac.uk/ wtg10/papers.html`)

[3] B. Green, *Generalising the Hardy-Littlewood method for primes*, Proc. Intern. Congress. Math., Madrid 2006.

[4] B. Green, *Long arithmetic progressions of primes*, submitted to Proceedings of the Gauss-Dirichlet conference, Gottingen 2005

[5] A. Iosevich, *Curvature, combinatorics and the Fourier transform*, Notices Amer. Math. Soc. **46** (2001), 577–583.

[6] B. Kra, *The Green-Tao theorem on arithmetic progressions in the primes: an ergodic point of view*, Bull. Amer. Math. Soc.

[7] J. Pach, M. Sharir, *Geometric incidences*, in: Towards a Theory of Geometric Graphs (J. Pach, ed.), Contemporary Mathematics, vol. 342, Amer. Math. Soc. 2004

[8] T. Tao, *Recent progress on the restriction conjecture*, to appear in Park City conference proceedings.

[9] T. Tao, *Arithmetic progressions and the primes - El Escorial lectures*, Collectanea Mathematica (2006), Vol. Extra., 37-88 (Proceedings of the 7th International Conference on Harmonic Analysis and Partial Differential Equations, El Escorial)

[10] T. Tao, *The dichotomy between structure and randomness, arithmetic progressions, and the primes*, Proc. Intern. Congress. Math., Madrid 2006.

[11] T. Tao, V. Vu, *Additive Combinatorics*, Cambridge University Press, 2006, to appear.

[12] T. Wolff, *Recent progress connected with the Kakeya problem*, in *Prospects in Mathematics*, H. Rossi, ed., Amer. Math. Soc., Providence, R.I. (1999), 129–162.

[13] T. Wolff, *Lectures on Harmonic Analysis*, edited by I. Łaba and C. Shubin, Amer. Math. Soc., Providence, R.I. (2003).

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