# Tiling and spectral properties of near-cubic domains

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**Abstract**

We prove that if a measurable domain tiles $\mathbb{R}$ or $\mathbb{R}^2$ by translations, and if it is "close enough" to a line segment or a square respectively, then it admits a lattice tiling. We also prove a similar result for spectral sets in dimension 1, and give an example showing that there is no analogue of the tiling result in dimensions 3 and higher.

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## 1 Introduction

Let $E$ be a measurable set in $\mathbb{R}^n$ such that $0 < |E| < \infty$. We will say that $E$ *tiles* $\mathbb{R}^n$ *by translations* if there is a set $T \subset \mathbb{R}^n$ such that, up to sets of measure 0, the sets $E+t : t \in T$ are mutually disjoint and $\bigcup_{t\in T}(E+t) = \mathbb{R}^n$. We call any such $T$ a *translation set* for $E$, and write $E + T = \mathbb{R}^n$. A tiling $E + T = \mathbb{R}^n$ is called *periodic* if it admits a period lattice of rank $n$; it is a *lattice tiling* if $T$ itself is a lattice. Here and below, a *lattice* in $\mathbb{R}^n$ will always be a set of the form $T\mathbb{Z}^n$, where $T$ is a linear transformation of rank $n$.

It is known [19], [18] that if a convex set $E$ tiles $\mathbb{R}^n$ by translations, it also admits a lattice tiling. A natural question is whether a similar result holds if $E$ is "sufficiently close" to being convex, e.g. if it is close enough (in an appropriate sense) to a $n$-dimensional cube. In this paper we prove that this is indeed so in dimensions 1 and 2; we also construct a counterexample in dimensions $n \geq 3$.

A major unresolved problem in the mathematical theory of tilings is the *periodic tiling conjecture*, which asserts that any $E$ which tiles $\mathbb{R}^n$ by translations must also admit a periodic tiling. (See [3] for an overview of this and other related questions.) The conjecture has been proved for all bounded measurable subsets of $\mathbb{R}$ [16], [12] and for topological discs in $\mathbb{R}^2$ [2], [8]. Our Theorem 2 and Corollary 1 prove the conjecture for near-square domains in $\mathbb{R}^2$. We emphasize that no assumptions on the topology of $E$ are needed; in particular, $E$ is not required to be connected and may have infinitely many connected components.

Our work was also motivated in part by a conjecture of Fuglede [1]. We call a set $E$ *spectral* if there is a discrete set $\Lambda \subset \mathbb{R}^n$, which we call a *spectrum* for $E$, such that $\{e^{2\pi i\lambda\cdot x} : \lambda \in \Lambda\}$ is an orthogonal basis for $L^2(E)$. Fuglede conjectured that $E$ is spectral if and only if it tiles $\mathbb{R}^n$ by translations, and proved it under the assumption that either the translation set $T$ or the spectrum $\Lambda$ is a lattice. This problem was addressed in many recent papers (see e.g. [4], [7], [10], [13], [14], [15], [16], [17]), and in particular the conjecture has been proved for convex regions in $\mathbb{R}^2$ [9], [5], [6].

It follows from our Theorem 1 and from Fuglede's theorem that the conjecture is true for $E \subset \mathbb{R}$ such that $E$ is contained in an interval of length strictly less than $3|E|/2$. (This was proved in [15] in the special case when $E$ is a union of finitely many intervals of equal length.) In dimension 2, we obtain the "tiling $\Rightarrow$ spectrum" part of the conjecture for near-square domains. Namely, if $E \subset \mathbb{R}^2$ tiles $\mathbb{R}^2$ and satisfies the assumptions of Theorem 2 or Corollary 1, it also admits a lattice tiling, hence it is a spectral set by Fuglede's theorem on the lattice case of his conjecture. We do not know how to prove the converse implication.

Our main results are the following.

**Theorem 1** *Suppose* $E \subseteq [0, L]$ *is measurable with measure* 1 *and* $L = 3/2 - \epsilon$ *for some* $\epsilon > 0$. *Let* $\Lambda \subset \mathbb{R}$ *be a discrete set containing* 0. *Then*
*(a) if* $E + \Lambda = \mathbb{R}$ *is a tiling, it follows that* $\Lambda = \mathbb{Z}$.
*(b) if* $\Lambda$ *is a spectrum of* $E$*, it follows that* $\Lambda = \mathbb{Z}$.

The upper bound $L < 3/2$ in Theorem 1 is optimal: the set $[0, 1/2] \cup [1, 3/2]$ is contained in an interval of length $3/2$, tiles $\mathbb{Z}$ with the translation set $\{0, 1/2\} + 2\mathbb{Z}$, and has the spectrum $\{0, 1/2\} + 2\mathbb{Z}$, but does not have either a lattice translation set or a lattice spectrum. This example has been known to many authors; an explicit calculation of the spectrum is given e.g. in [14].

**Theorem 2** *Let $E \subset \mathbb{R}^2$ be a measurable set such that $[0,1]^2 \subset E \subset [-\epsilon, 1+\epsilon]^2$ for $\epsilon > 0$ small enough. Assume that $E$ tiles $\mathbb{R}^2$ by translations. Then $E$ also admits a tiling with a lattice $\Lambda \subset \mathbb{R}^2$ as the translation set.*

Our proof works for $\epsilon < \epsilon_0 \approx 0.05496$; we do not know what is the optimal upper bound for $\epsilon$.

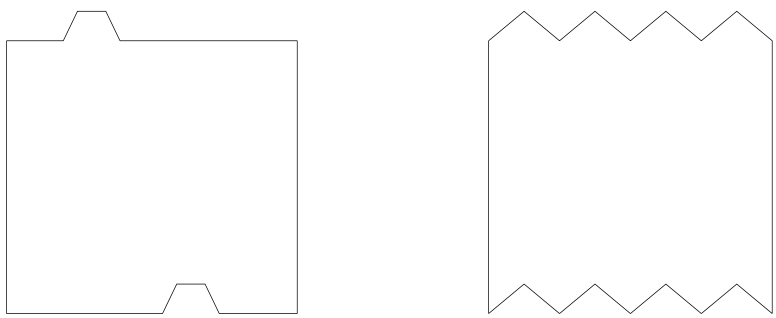

Figure 1: Examples of near-square regions which tile $\mathbb{R}^2$. Note that the second region also admits aperiodic (hence non-lattice) tilings.

**Corollary 1** *Let $E \subset \mathbb{R}^2$ be a measurable set such that $|E| = 1$ and $E$ is contained in a square of sidelength $1+\epsilon$ for $\epsilon > 0$ small enough. If $E$ tiles $\mathbb{R}^2$ by translations, then it also admits a lattice tiling.*

**Theorem 3** *Let $n \geq 3$. Then for any $\epsilon > 0$ there is a set $E \subset \mathbb{R}^n$ with $[0,1]^n \subset E \subset [-\epsilon, 1+\epsilon]^n$ such that $E$ tiles $\mathbb{R}^n$ by translations, but does not admit a lattice tiling.*

## 2 The one-dimensional case

In this section we prove Theorem 1. We shall need the following crucial lemma.

**Lemma 1** *Suppose that $E \subseteq [0, L]$ is measurable with measure 1 and that $L = 3/2 - \epsilon$ for some $\epsilon > 0$. Then*

$$|E \cap (E + x)| > 0 \quad \textit{whenever } 0 \le x < 1. \tag{1}$$

**Proof of Lemma 1.** We distinguish the cases (i) $0 \le x \le 1/2$, (ii) $1/2 < x \le 3/4$ and (iii) $3/4 < x < 1$.

<u>(i) $0 \le x \le 1/2$</u>

This is the easy case as $E \cup (E + x) \subseteq [0, L + 1/2] = [0, 2 - \epsilon]$. Since this interval has length less than 2, the sets $E$ and $E + x$ must intersect in positive measure.

<u>(ii) $1/2 < x \le 3/4$</u>

Let $x = 1/2 + \alpha$, $0 < \alpha \le 1/4$. Suppose that $|E \cap (E + x)| = 0$. Then $1 + 2\alpha \le 3/2$ and

$$|(E \cap [0, x]) \cup (E \cap [x, 2x])| \le x,$$

as the second set does not intersect the first when shifted back by $x$. This implies that

$$|E| \le x + (3/2 - \epsilon - 2x) = 3/2 - \epsilon - x = 1 - \epsilon - \alpha < 1,$$

a contradiction as $|E| = 1$.

<u>(iii) $3/4 < x < 1$</u>

Let $x = 3/4 + \alpha$, $0 < \alpha < 1/4$. Suppose that $|E \cap (E + x)| = 0$. Then

$$|(E \cap [0, 3/4 - \alpha - \epsilon]) \cup (E \cap [3/4 + \alpha, 3/2 - \epsilon])| \le 3/4 - \alpha - \epsilon,$$

for the second set translated to the left by $x$ does not intersect the first. This implies that

$$|E| \le (3/4 - \alpha - \epsilon) + 2\alpha + \epsilon = 3/4 + \alpha < 1,$$

a contradiction.
$\square$

We need to introduce some terminology. If $f$ is a nonnegative integrable function on $\mathbb{R}^d$ and $\Lambda$ is a subset of $\mathbb{R}^d$, we say that $f + \Lambda$ is a packing if, almost everywhere,

$$\sum_{\lambda \in \Lambda} f(x - \lambda) \le 1. \tag{2}$$

We say that $f+\Lambda$ is a tiling if equality holds almost everywhere. When $f=\chi_E$ is the indicator function of a measurable set, this definition coincides with the classical geometric notions of packing and tiling.

We shall need the following theorem from [10].

**Theorem 4** *If $f, g \geq 0$, $\int f(x)dx = \int g(x)dx = 1$ and both $f+\Lambda$ and $g+\Lambda$ are packings of $\mathbb{R}^d$, then $f+\Lambda$ is a tiling if and only if $g+\Lambda$ is a tiling.*

**Proof of Theorem 1.** (a) Suppose $E+\Lambda$ is a tiling. From Lemma 1 it follows that any two elements of $\Lambda$ differ by at least 1. This implies that $\chi_{[0,1]}+\Lambda$ is a packing, hence it is also a tiling by Theorem 4. Since $0 \in \Lambda$, we have $\Lambda = \mathbb{Z}$.

(b) Suppose that $\Lambda$ is a spectrum of $E$. Write

$$\delta_\Lambda = \sum_{\lambda \in \Lambda} \delta_\lambda$$

for the measure of one unit mass at each point of $\Lambda$. Our assumption that $\Lambda$ is a spectrum for $E$ implies that

$$\left|\widehat{\chi_E}\right|^2 + \Lambda = \mathbb{R}$$

is a tiling (see, for example, [10]). This, in turn, implies that $\Lambda$ had density 1. Here and below, we say that a set $A \subset \mathbb{R}$ has density $\rho$ if

$$\lim_{N\to\infty} \frac{\#(A \cap [-N,N])}{2N} = \rho.$$

**Notation.** The definition of the Fourier Transform we use is

$$\widehat{f}(\xi) = \int_{\mathbb{R}} e^{-2\pi i \xi \cdot x} f(x)\ dx,$$

for an $L^1$ function $f$. If $T$ is a tempered distribution (a bounded linear functional on the Schwarz space $\mathcal{S}$) then its Fourier Transform is defined by duality as the tempered distribution $\widehat{T}$ given by

$$\widehat{T}(\phi) = T(\widehat{\phi}), \quad \phi \in \mathcal{S}.$$

We now use the following result from [10]:

**Theorem 5** *Suppose that $f \ge 0$ is not identically $0$, that $f \in L^1(\mathbb{R}^d)$, $\widehat{f} \ge 0$ has compact support and $\Lambda \subset \mathbb{R}^d$. If $f + \Lambda$ is a tiling then*

$$\operatorname{supp} \widehat{\delta_\Lambda} \subseteq \left\{ \widehat{f} = 0 \right\} \cup \{0\}. \tag{3}$$

Let us emphasize here that the object $\widehat{\delta_\Lambda}$, the Fourier Transform of the tempered measure $\delta_\Lambda$, is in general a tempered distribution and need not be a measure.

For $f = |\widehat{\chi_E}|^2$ Theorem 5 implies

$$\operatorname{supp} \widehat{\delta_\Lambda} \subseteq \{0\} \cup \{\chi_E * \widetilde{\chi_E} = 0\}, \tag{4}$$

since $\chi_E * \widetilde{\chi_E}$ is the Fourier transform of $|\widehat{\chi_E}|^2$ (where $\widetilde{g}(x) = \overline{g(-x)}$). But

$$\{\chi_E * \widetilde{\chi_E} = 0\} = \{x : \ |E \cap (E + x)| = 0\}.$$

This and Lemma 1 imply that

$$\operatorname{supp} \widehat{\delta_\Lambda} \cap (-1, 1) = \{0\}.$$

Let

$$K_\delta(x) = \max\{0, 1 - (1+\delta)|x|\} = (1+\delta)\chi_{I_\delta} * \widetilde{\chi_{I_\delta}}(x),$$

where $I_\delta = [0, \frac{1}{1+\delta}]$, be a Fejér kernel (we will later take $\delta \to 0$). Then

$$\widehat{K_\delta} = (1+\delta)|\widehat{\chi_{I_\delta}}|^2 = \frac{1+\delta}{\pi^2 x^2} \sin^2 \frac{\pi x}{1+\delta}$$

is a non-negative continuous function and it follows that

$$\widehat{K_\delta}(0) = \frac{1}{1+\delta}$$

and

$$\left\{ x : \widehat{K_\delta}(x) = 0 \right\} = (1+\delta)(\mathbb{Z} \setminus \{0\}). \tag{5}$$

Next, we use the following result from [11] (proved there in a more general setting):

**Theorem 6** *Suppose that $\Lambda \in \mathbb{R}$ is a set with density $\rho$, $\delta_\Lambda = \sum_{\lambda\in\Lambda} \delta_\lambda$, and that $\widehat{\delta_\Lambda}$ is a measure in a neighborhood of* $0$. *Then $\widehat{\delta_\Lambda}(\{0\}) = \rho$.*

**Remark.** The proof of Theorem 6 shows that the assumption of $\widehat{\delta_\Lambda}$ being a measure in a neighborhood of zero is superfluous, if one knows a priori that $\widehat{\delta_\Lambda}$ is supported only at zero, in a neighborhood of zero. Indeed, what is shown in that proof is that, as $t \to \infty$, the quantity $\widehat{\delta_\Lambda}(\phi(tx))$ remains bounded, for any $C_c^\infty$ test function $\phi$. If $\widehat{\delta_\Lambda}$ were not a measure near 0 but had support only at 0, locally, this quantity would grow like a polynomial in $t$ of degree equal to the degree of the distribution at 0.

Applying Theorem 6 and the remark following it we obtain that $\widehat{\delta_\Lambda}$ is equal to $\delta_0$ in a neighborhood of 0, since $\Lambda$ has density 1.

Next, we claim that

$$\sum_{\lambda\in\Lambda} \widehat{K_\delta}(x-\lambda) = 1, \quad \text{for all } x \in \mathbb{R}.$$

Indeed, take $\psi_\epsilon$ to be an even, smooth, positive-definite approximate identity, supported in $(-\epsilon, \epsilon)$, and take $\epsilon = \epsilon(\delta)$ to be small enough so that $\operatorname{supp} \psi_\epsilon * K_\delta \subset (-1, 1)$. We have then, for fixed $x$,

$$\begin{aligned}
\sum_{\lambda\in\Lambda} \widehat{K_\delta}(x-\lambda) &= \lim_{\epsilon\to 0} \sum_{\lambda\in\Lambda} \widehat{\psi_\epsilon}(x-\lambda)\widehat{K_\delta}(x-\lambda) \\
&= \lim_{\epsilon\to 0} \delta_\Lambda \left( (\widehat{\psi_\epsilon}\widehat{K_\delta})(x-\cdot) \right) \quad \text{(by definition of } \delta_\Lambda) \\
&= \lim_{\epsilon\to 0} \widehat{\delta_\Lambda} \left( e^{2\pi i x t} (\psi_\epsilon * K_\delta)(t) \right) \quad \text{(by the definition of the FT of } \delta_\Lambda) \\
&= \lim_{\epsilon\to 0} \delta_0 \left( e^{2\pi i x t} (\psi_\epsilon * K_\delta)(t) \right) \quad \text{(for } \epsilon \text{ small enough)} \\
&= \lim_{\epsilon\to 0} (\psi_\epsilon * K_\delta)(0) \\
&= K_\delta(0) \\
&= 1,
\end{aligned}$$

which establishes the claim. Applying this for $x = 0$ and isolating the term $\lambda = 0$ we get

$$1 = \frac{1}{1+\delta} + \sum_{0\neq\lambda\in\Lambda} \widehat{K_\delta}(-\lambda).$$

Letting $\delta \to 0$ we obtain that $\widehat{K_\delta}(-\lambda) \to 0$ for each $\lambda \in \Lambda \setminus \{0\}$, which implies that each such $\lambda$ is an integer, as $\mathbb{Z} \setminus \{0\}$ is the limiting set of the zeros of $\widehat{K_\delta}$.

To get that $\Lambda = \mathbb{Z}$ notice that $\chi_{[0,1]} + \Lambda$ is a packing. By Theorem 4 again we get that $\chi_{[0,1]} + \Lambda$ is in fact a tiling, hence $\Lambda = \mathbb{Z}$.
$\square$

# 3 Planar regions

**Proof of Theorem 2.** We denote the coordinates in $\mathbb{R}^2$ by $(x_1, x_2)$. For $0 \le a \le b \le 1$ we will denote

$$E_1(a,b) = (E \cap \{a \le x_1 \le b,\ x_2 \le 0\}) \cup \{a \le x_1 \le b,\ x_2 \ge 0\},$$

$$E_2(a,b) = (E \cap \{a \le x_1 \le b,\ x_2 \ge 0\}) \cup \{a \le x_1 \le b,\ x_2 \le 0\},$$
$$F_1(a,b) = (E \cap \{a \le x_2 \le b,\ x_1 \le 0\}) \cup \{a \le x_2 \le b,\ x_1 \ge 0\},$$
$$F_2(a,b) = (E \cap \{a \le x_2 \le b,\ x_1 \ge 0\}) \cup \{a \le x_2 \le b,\ x_1 \le 0\}.$$

We will also use $S_{a,b}$ to denote the vertical strip $[a,b] \times \mathbb{R}$. Let $v = (v_1, v_2) \in \mathbb{R}^2$. We will say that $E_2(a,b)$ *complements* $E_1(a',b') + v$ if $E_1(a',b') + v$ is positioned above $E_2(a,b)$ so that (up to sets of measure 0) the two sets are disjoint and their union is $S_{a,b}$. In particular, we must have $a' + v_1 = a$ and $b' + v_1 = b$. We will also say that $F_2(a,b)$ *complements* $F_1(a',b') + v$ if the obvious analogue of the above statement holds. We will write $\widetilde{E}_1(a,b) = S_{a,b} \setminus E_1(a,b)$, and similarly for $E_2$. Finally, we write $A \sim B$ if the sets $A$ and $B$ are equal up to sets of measure 0.

**Lemma 2** *Let $0 < s'' < s' < s < 2s''$. Suppose that $E_1(a, a+s) + v$, $E_1(a, a+s') + v'$, $E_1(a, a+s'') + v''$ complement $E_2(b-s, b)$, $E_2(b-s', b)$, $E_2(b-s'', b)$ respectively. Then the points $v, v', v''$ are collinear. Moreover, the absolute value of the slope of the line through $v, v''$ is bounded by $\epsilon(2s''-s)^{-1}$.*

Applying the lemma to the symmetric reflection of $E$ about the line $x_2 = 1/2$, we find that the conclusions of the lemma also hold if we assume that

$E_2(a,a+s)+v$, $E_2(a,a+s')+v'$, $E_2(a,a+s'')+v''$ complement $E_1(b-s,b)$, $E_1(b-s',b)$, $E_1(b-s'',b)$ respectively. Furthermore, we may interchange the $x_1$ and $x_2$ coordinates and obtain the analogue of the lemma with $E_1, E_2$ replaced by $F_1, F_2$.

**Proof of Lemma 2.** Let $v=(v_1,v_2)$, $v'=(v_1',v_2')$, $v''=(v_1'',v_2'')$. We first observe that if $v_1=v_1''$, it follows from the assumptions that $v=v''$ and there is nothing to prove. We may therefore assume that $v_1\neq v_1''$. We do, however, allow $v'=v$ or $v'=v''$.

It follows from the assumptions that $E_2(b-s'',b)$ complements each of $E_1(a,a+s'')+v''$, $E_1(a+s'-s'',a+s')+v'$, $E_1(a+s-s'',a+s)+v$. Hence

$$E_1(a+s'-s'',a+s')\sim E_1(a,a+s'')+(v''-v'),$$

$$E_1(a+s-s'',a+s)\sim E_1(a,a+s'')+(v''-v).$$

Let $n$ be the unit vector perpendicular to $v-v''$ and such that $n_2>0$. For $t\in\mathbb{R}$, let $P_t=\{x:\ x\cdot n\le t\}$. We define for $0\le c\le c'\le 1$:

$$\alpha_{c,c'}=\inf\{t\in\mathbb{R}:\ |E_1(c,c')\cap P_t|>0\},$$

$$\beta_{c,c'}=\sup\{t\in\mathbb{R}:\ |\widetilde{E}_1(c,c')\setminus P_t|>0\}.$$

We will say that $x$ is a *low point* of $E_1(c,c')$ if $x\in S_{c,c'}$, $x\cdot n=\alpha_{c,c'}$, and for any open disc $D$ centered at $x$ we have

$$|D\cap E_1(c,c')|>0. \tag{6}$$

Similarly, we call $y$ a *high point* of $\widetilde{E}_1(c,c')$ if $y\in S_{c,c'}$, $y\cdot n=\beta_{c,c'}$, and for any open disc $D$ centered at $y$ we have

$$|D\cap \widetilde{E}_1(c,c')|>0. \tag{7}$$

It is easy to see that such points $x,y$ actually exist. Indeed, by the definition of $\alpha_{c,c'}$ and an obvious covering argument, for any $\alpha>\alpha_{c,c'}$ there are points $x'$ such that $x'\cdot n\le\alpha$ and that (6) holds for any disc $D$ centered at $x'$. Thus the set of such points $x'$ has at least one accumulation point $x$ on the line $x\cdot n=\alpha_{c,c'}$. It follows that any such $x$ is a low point of $E_1(c,c')$. The same argument works for $y$.

The low and high points need not be unique; however, all low points $x$ of $E_1(c,c')$ lie on the same line $x \cdot n = \alpha_{c,c'}$ parallel to the vector $v - v''$, and similarly for high points. Furthermore, the low and high points of $E_1(c,c')$ do not change if $E_1(c,c')$ is modified by a set of measure 0.

Let now $A = E_1(a, a+s'')$, and let $x$ be a low point of $A$. Since $s < 2s''$, we have

$$B := E_1(a, a+s) = E_1(a, a+s'') \cup E_1(a+s-s'', a+s) \sim A \cup (A + v'' - v),$$

hence $x$ is also a low point of $B$ with respect to $v - v''$. Now note that

$$E_1(a+s'-s'', a+s') \sim A + (v'' - v')$$

intersects any open neighbourhood of $x + (v'' - v')$ in positive measure. But on the other hand, $E_1(a+s'-s'', a+s') \subset B$. By the extremality of $x$ in $B$, $x + (v'' - v')$ lies on or above the line segment joining $x$ and $x + (v'' - v)$, hence $v'' - v'$ lies on or above the line segment joining 0 and $v'' - v$.

Repeating the argument in the last paragraph with $x$ replaced by a high point $y$ of $\widetilde{E}_1(a, a+s'')$, we obtain that $v'' - v'$ lies on or below the line segment joining 0 and $v'' - v$. Hence $v, v', v''$ are collinear.

Finally, we estimate the slope of the line through $v, v''$. We have to prove that

$$\frac{2s'' - s}{s - s''}\,|v_2'' - v_2| \le \epsilon \tag{8}$$

(recall that $v_1'' - v_1 = s - s''$). Define $x$ as above, and let $k \in \mathbb{Z}$. Iterating translations by $v - v''$ (in both directions), we find that $x + k(v - v'')$ is a low point of $B$ as long as it belongs to $B$, i.e. as long as

$$a \le x_1 + k(s - s'') \le a + s.$$

The number of such $k$'s is at least $\frac{s}{s-s''} - 1$. On the other hand, all low points of $B$ lie in the rectangle $a \le x_1 \le a+s, -\epsilon \le x_2 \le 0$. Hence

$$(\frac{s}{s - s''} - 2)|v_2'' - v_2| \le \epsilon,$$

which is (8).
$\square$

We return to the proof of Theorem 2. Since $E$ is almost a square, we know roughly how the translates of $E$ can fit together. Locally, any tiling by $E$ is essentially a tiling by a "solid" $1 \times 1$ square with "margins" of width between 0 and $2\epsilon$ (see Fig. 2).

We first locate a "corner". Namely, we may assume that the tiling contains $E$ and its translates $E + u$, $E + v$, where

$$1 \leq u_1 \leq 1 + 2\epsilon, \ -2\epsilon \leq u_2 \leq 2\epsilon, \tag{9}$$

$$0 \leq v_1 \leq \frac{1}{2} + \epsilon, \ 1 \leq v_2 \leq 1 + 2\epsilon. \tag{10}$$

This can always be achieved by translating the tiled plane and taking symmetric reflections of it if necessary.

Let $E + w$ be the translate of $E$ which fits into this corner:

$$v_1 + 1 \leq w_1 \leq v_1 + 1 + 2\epsilon, \ u_2 + 1 \leq w_2 \leq u_2 + 1 + 2\epsilon. \tag{11}$$

We will prove that $w = u + v$ (without the $\epsilon$-errors).

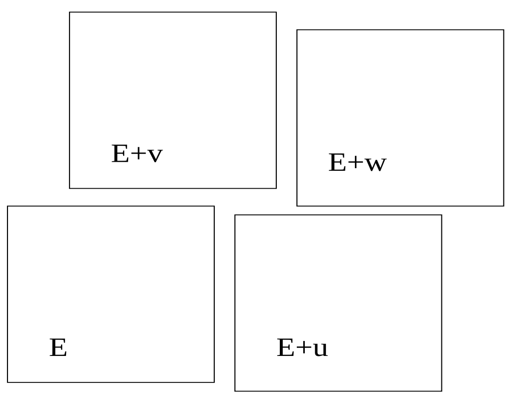


Figure 2: A "corner" and a fourth near-square.

From (11), (9), (10) we have

$$1 \leq w_1 \leq \frac{3}{2} + 3\epsilon, \ -4\epsilon \leq w_2 - v_2 \leq 4\epsilon.$$

Observe also that any points $(x_1, x_2)$ between $E + u$ and $E + w$ that belong to tiles other than $E + u$ or $E + w$ must have $x_1 \leq w + \epsilon$ or $x_1 \geq u + 1 - \epsilon$, since otherwise the solid square belonging to the same tile would overlap with at least one of the solid squares belonging to $E+u$ or $E+w$. A similar statement holds for $E+v$ and $E+w$. Hence $w$ satisfies both of the following.

(A) $E_1(\epsilon, 1-(w_1-u_1)-\epsilon)$ complements $E_2(w_1-u_1+\epsilon, 1-\epsilon)+(u-w)$, and

$$1-(w_1-u_1)-2\epsilon \geq 1+1-(\frac{3}{2}+3\epsilon)-2\epsilon = \frac{1}{2}-5\epsilon,$$

$$|w_1-v_1-1| \leq 2\epsilon.$$

(B) $-4\epsilon \leq w_2-v_2 \leq 4\epsilon$, $u_2+1 \leq w_2 \leq u_2+1+2\epsilon$, and $F_2(r,t)$ complements $F_1(\tilde{r},\tilde{t})+(w-v)$, where

$$r = \max(0, w_2-v_2)+\epsilon,\ \tilde{r} = \max(0, v_2-w_2)+\epsilon,$$

$$t = 1-\max(0, v_2-w_2)-\epsilon,\ \tilde{t} = 1-\max(0, w_2-v_2)-\epsilon.$$

If $w = u+v$, we have $w-u = v$, $w-v = u$, hence by considering the "corner" $E, E+u, E+v$ we see that both (A) and (B) hold. Assuming that $\epsilon$ is small enough, we shall prove that:

1. All points $w$ satisfying (A) lie on a fixed straight line $l_1$ with slope $m_1$, where $|m_1| \leq \epsilon(\frac{1}{2}-9\epsilon)^{-1}$.

2. All points $w$ satisfying (B) lie on a fixed straight line $l_2$ with slope $m_2$, where $|m_2| \geq \epsilon^{-1}(1-8\epsilon)$.

If $\epsilon < (13-3\sqrt{3})/142 \approx 0.05496$ (the smaller root of the equation $71\epsilon^2 - 13\epsilon + \frac{1}{2} = 0$), the upper bound for $|m_1|$ is less than the lower bound for $|m_2|$. It follows that there can be at most one $w$ which satisfies both (A) and (B), since $l_1$ and $l_2$ intersect only at one point. Consequently, if $E+w$ is the translate of $E$ chosen as above, we must have $w = u+v$. Now it is easy to see that $E+\Lambda$ is a tiling, where $\Lambda$ is the lattice $\{ku+mv:\ k,m \in \mathbb{Z}\}$.

We first prove 1. Suppose that $w, w', w'', \ldots$ (not necessarily all distinct) satisfy (A). By the assumptions in (A), we may apply Lemma 2 with $E_1$ and $E_2$ interchanged and with $a=0$, $b=1$, $s = 1-(w_1-u_1), s' = 1-(w'_1-u_1), \ldots \geq \frac{1}{2}-5\epsilon$. From the second inequality in (A) and the triangle inequality we also have $|s - s''| \leq 4\epsilon$. We find that all $w$ satisfying (A) lie on a line $l_1$ with slope bounded by

$$\frac{\epsilon}{|2s'' - s|} \leq \frac{\epsilon}{s'' - |s'' - s|} \leq \frac{\epsilon}{1/2 - 9\epsilon}.$$

To prove 2., we let $w, w', w''$ be three (not necessarily distinct) points satisfying (B) and such that $w_2 \leq w_2' \leq w_2''$. Observe that $r \leq r' \leq r''$ and $t \geq t' \geq t''$ (the notation is self-explanatory). We then apply the obvious analogue of Lemma 2 with $E_1, E_2$ replaced by $F_1, F_2$ and with $a = r''$, $s = t - r''$, $s' = t' - r''$, $s'' = t'' - r''$, $b = \tilde{t}''$. From the estimates in (B) we have

$$|s - s''| = |t - t''| \leq |w_2 - w_2''| \leq 2\epsilon,$$

$$s'' = t'' - r'' = 1 - \max(0, v_2 - w_2'') - \max(0, w_2'' - v_2) - 2\epsilon \geq 1 - 6\epsilon,$$

hence $|2s'' - s| \geq s'' - |s - s''| \geq 1 - 8\epsilon$. We conclude that all $w$ satisfying (B) lie on a line $l_2$ such that the inverse of the absolute value of its slope is bounded by $\frac{\epsilon}{1-8\epsilon}$.
□

**Proof of Corollary 1.** Let $Q = [0,1] \times [0,1]$. By rescaling, it suffices to prove that for any $\epsilon > 0$ there is a $\delta > 0$ such that if $E \subset Q$, $E$ tiles $\mathbb{R}^2$ by translations, and $|E| \geq 1 - \delta$, then $E$ contains the square

$$Q_\epsilon = [\epsilon, 1 - \epsilon] \times [\epsilon, 1 - \epsilon]$$

(up to sets of measure 0). The result then follows from Theorem 2.

Let $E$ be as above, and suppose that $Q_\epsilon \setminus E$ has positive measure. Since $E$ tiles $\mathbb{R}^2$, there is a $v \in \mathbb{R}^2$ such that $|E \cap (E + v)| = 0$ and $|Q_\epsilon \cap (E + v)| > 0$. We then have

$$|E \cup (E + v)| = |E| + |E + v| \geq 2 - 2\delta,$$

but also

$$|E \cup (E + v)| \leq |Q \cup (Q + v)| \leq 2 - \epsilon^2,$$

since $E \subset Q$, $E + v \subset Q + v$, and $Q_\epsilon \cap (Q + v) \neq \emptyset$ so that $|Q \cap (Q + v)| \geq \epsilon^2$. This is a contradiction if $\delta$ is small enough.
□

# 4 A counterexample in higher dimensions

In this section we prove Theorem 3. It suffices to construct $E$ for $n = 3$, since then $E \times [0,1]^{n-3}$ is a subset of $\mathbb{R}^n$ with the required properties.

Let $(x_1, x_2, x_3)$ denote the Cartesian coordinates in $\mathbb{R}^3$. It will be convenient to rescale $E$ so that $[\epsilon, 1]^3 \subset E \subset [0, 1+\epsilon]^3$.

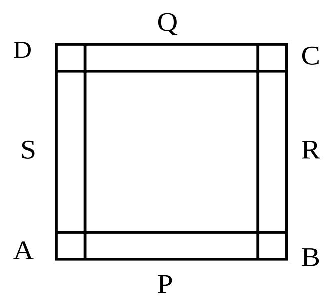


Figure 3: The construction of $E$.

We construct $E$ as follows. We let $E$ be bounded from below and above by the planes $x_3 = 0$ and $x_3 = 1$ respectively. The planes $x_1 = \epsilon, x_1 = 1, x_2 = \epsilon, x_2 = 1$ divide the cube $[0, 1+\epsilon]^3$ into 9 parts (Figure 3). The middle part is entirely contained in $E$. We label by $A, B, C, D, P, Q, R, S$ the remaining 8 segments as shown in Figure 3. We then let

$$E \cap P = P \cap \left\{ 0 \le x_3 \le \frac{1}{8} \text{ or } \frac{1}{2} \le x_3 \le \frac{5}{8} \right\},$$

$$E \cap R = R \cap \left\{ 0 \le x_3 \le \frac{1}{8} \text{ or } \frac{1}{2} \le x_3 \le \frac{5}{8} \right\},$$

$$E \cap Q = Q \cap \left\{ 0 \le x_3 \le \frac{1}{4} \text{ or } \frac{3}{8} \le x_3 \le \frac{3}{4} \text{ or } \frac{7}{8} \le x_3 \le 1 \right\},$$

$$E \cap S = S \cap \left\{ 0 \le x_3 \le \frac{1}{4} \text{ or } \frac{3}{8} \le x_3 \le \frac{3}{4} \text{ or } \frac{7}{8} \le x_3 \le 1 \right\},$$

and

$$E \cap A = A \cap \left\{ 0 \le x_3 \le \frac{1}{16} \right\}, \; E \cap C = A \cap \left\{ \frac{1}{2} \le x_3 \le \frac{9}{16} \right\},$$

$$E \cap B = B \cap \left\{ \frac{5}{16} \le x_3 \le \frac{3}{4} \right\}, \; E \cap D = D \cap \left\{ 0 \le x_3 \le \frac{1}{4} \text{ or } \frac{13}{16} \le x_3 \le 1 \right\}.$$

We also denote $K = \bigcup_{j\in\mathbb{Z}}(E + (0,0,j))$.

Let $E+T$ be a tiling of $\mathbb{R}^3$, and assume that $0 \in T$. Suppose that $E+v$ and $E+w$ are neighbours in this tiling so that the vertical sides of $(E\cap P)+v$ and $(E\cap Q)+w$ meet in a set of non-zero two-dimensional measure. Then we must have $v-w = (0,1,(v-w)_3)$, where $(v-w)_3 \in \{\pm\frac{1}{4}, \pm\frac{3}{4}\}$. A similar statement holds with $P, Q$ replaced by $R, S$ and with the $x_1, x_2$ coordinates interchanged. We deduce that the tiling consists of copies of $E$ stacked into identical vertical "columns" $K_{ij} = K + (i, j, t_{ij})$, arranged in a rectangular grid in the $x_1x_2$ plane and shifted vertically so that $t_{i+1,j} - t_{ij}$ and $t_{i,j+1} - t_{ij}$ are always $\pm\frac{1}{4}$. We will use matrices $(t_{ij})$ to encode such a tiling or portions thereof.

It is easy to see that $(t_{ij})$, where $t_{ij} = 0$ if $i+j$ is even and $\frac{1}{4}$ if $i+j$ is odd, is indeed a tiling. It remains to show that $E$ does not admit a lattice tiling. Indeed, the four possible choices of the generating vectors in any lattice $(t_{ij})$ with $t_{ij} = \pm\frac{1}{4}$ produce the configurations

$$\begin{pmatrix} 0 & t \\ t & 2t \end{pmatrix}, \begin{pmatrix} 2t & t \\ t & 0 \end{pmatrix}, \begin{pmatrix} 0 & t \\ -t & 0 \end{pmatrix}, \begin{pmatrix} 0 & -t \\ t & 0 \end{pmatrix}.$$

But it is easy to see that the corners $A, B, C, D$ do not match if so translated.

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