FALCONER'S DISTANCE SET CONJECTURE FOR POLYGONAL NORMS

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Abstract. A conjecture of Falconer [F86] asserts that if E is a planar set with Hausdorff dimension strictly greater than 1, then its Euclidean distance set $\Delta(E)$ has positive one-dimensional Lebesgue measure. We review recent work on the analogous question with the Euclidean distance replaced by non-Euclidean norms $\|\cdot\|_X$ in which the unit ball is a polygon, and construct explicit examples of sets with large Hausdorff dimension whose distance set has Lebesgue measure 0.

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§0. INTRODUCTION

A conjecture of Falconer [F86] asserts that if a set $E \subset \mathbb{R}^2$ has Hausdorff dimension strictly greater than 1, then its Euclidean distance set

$$\Delta(E) = \Delta_{l_2^2}(E) = \left\{ \|x - x'\|_{l_2^2} : x, x' \in E \right\}$$

has positive one-dimensional Lebesgue measure. The current best result in this direction is due to Wolff [W99], who proved that the conclusion is true if E has Hausdorff dimension greater than 4/3. Erdogan [Er03], [E04] extended this result to higher dimensions, proving that the same conclusion holds for subsets of $\mathbb{R}^d$ with Hausdorff dimension greater than $\frac{d}{2} + \frac{1}{3}$. This improves on the earlier results of Falconer [F86], Mattila [M87], and Bourgain [B94].

An analogous question may be posed for more general n-dimensional normed spaces. Let X be the n-dimensional vector space over $\mathbb{R}$ equipped with a norm $\|\cdot\|_X$. We define the X-distance set of a set $E \subset X$:

$$\Delta_X(E) = \{\|x - x'\|_X : x, x' \in E\},$$

and ask how the size of $\Delta_X(E)$ depends on the dimension of E as well as on the properties of the norm $\|\cdot\|_X$. Simple examples show that Falconer's conjecture as stated above, but with $\Delta(E)$ replaced by $\Delta_X(E)$, cannot hold for all normed spaces X. For instance, let X be the 2-dimensional plane with the norm

$$\|x\|_{l_\infty^2} = \max(|x_1|, |x_2|)$$

and let $E = F \times F$, where F is a subset of $[0,1]$ with Hausdorff dimension 1 such that $F - F := \{x - x' : x, x' \in F\}$ has measure 0. (It is an easy exercise to modify the Cantor set construction to produce such a set.) Then E has Hausdorff dimension 2, but its l_∞^2-distance set $F - F$ has measure 0.

Here and below, we use $\dim(E)$ to denote the Hausdorff dimension of E, $|F|_d$ to denote the d-dimensional Lebesgue measure of F, and $|A|$ to denote the cardinality of a finite set A.

Definition 0.1. *Let X be a 2-dimensional normed space, and let $0 < \alpha < 2$. We will say that the α-Falconer conjecture holds in X if for any set $E \subset X$ with $dim(E) > \alpha$ we have $|\Delta_X(E)|_1 > 0$.*

The above considerations indicate that the range of α for which the α-Falconer conjecture holds in X will depend on the properties of the norm on X, and in particular that the curvature of the distance function should play a role. Indeed, let

$$BX = \{x \in X : \|x\|_X \leq 1\}$$

be the unit ball in X. In the example with the product of Cantor sets, the unit ball was a square (no curvature), and the α-Falconer conjecture fails for all $\alpha < 2$. On the other hand, we do have an α-Falconer conjecture with $\alpha > 4/3$ (and expect it to hold with $\alpha > 1$) in a 2-dimensional plane is equipped with the Euclidean norm, where the unit ball is strictly convex and its boundary ∂BX has nonvanishing curvature. This motivates several natural questions: For what range of α does the α-Falconer conjecture hold in X if ∂BX has everywhere nonvanishing curvature? What if only know that ∂BX is strictly convex, but make no curvature assumptions? Does the α-Falconer conjecture with $\alpha < 2$ fail for all polygons and for all $\alpha < 2$?

With regard to the first two questions, the following partial results are known.

Theorem 1. *(Iosevich-Łaba* [IŁ04]*) The $3/2$-Falconer conjecture holds in any 2-dimensional vector space X over $\mathbb{R}$ such that BX is strictly convex and ∂BX has the property that the diameter of the chord*

$$\{x \in BX : x \cdot v \geq \max_{y \in BX} (y \cdot v) - \epsilon\},$$

where v is a unit vector and $\epsilon > 0$, is bounded by $C\sqrt{\epsilon}$ uniformly for all v and ϵ.

Erdogan [Er03], [Er04] observes that if we make the stronger assumption that BX is strictly convex and that ∂BX is smooth and has nonvanishing Gaussian curvature, then his arguments for the Euclidean case extend to X, with only minor changes. Thus, with these assumptions, the $4/3$-Falconer conjecture holds in X, and moreover this result extends to higher dimensions.

Theorem 2. *(Erdogan* [Er03], [Er04]*) If X is an n-dimensional vector space over $\mathbb{R}$ equipped with a norm $\|\cdot\|_X$ such that the unit ball BX has a smooth boundary with nonvanishing Gaussian curvature, and if $E \subset X$ has $dim X > \frac{n}{2} + \frac{1}{3}$, then $\Delta_X(E)$ has positive measure.*

The methods of [W99], [Er03], [Er04], [IŁ04] are Fourier-analytic. The general strategy, due to Falconer [F86] and Mattila [M87], employs decay estimates on the Fourier transform of measures supported on ∂BX. In [Er03], [Er04], Mattila's approach is combined with a weighted modification of the bilinear restriction estimate of Tao [T03]. [IŁ04] uses recent stationary-phase type estimates available for non-smooth surfaces, see eg. [BRT98].

We do not know what the optimal range of α should be for the strictly convex case. However, there are no known counterexamples to the 1-Falconer conjecture in normed spaces with BX strictly convex.

On the other hand, the polygonal case has been resolved entirely by Falconer [Fa04], along with its higher-dimensional analogue:

Theorem 3. *If X is an n-dimensional vector space over $\mathbb{R}$ equipped with a norm $\|\cdot\|_X$ such that the unit ball BX is a polytope with finitely many faces, then there is a compact set $E \subset X$ with $dimE = n$ and $|\Delta_X(E)|_1 = 0$.*

In particular, if X is a 2-dimensional space and BX is a polygon with finitely many sides, then the α-Falconer conjecture fails for all $\alpha < 2$. The proof is based on consideration of "typical" intersections of homothetic copies of fixed Borel subsets of $\mathbb{R}^n$, and, as such, is not constructive.

The purpose of the remainder of this paper is to give explicit examples of subsets of X with large dimension whose X-distance set has measure zero, for large classes of 2-dimensional spaces X such that BX is a symmetric polygon with finitely many sides. Throughout the sequel, we will always assume that X is 2-dimensional, with BX as above. In general, we do not know how to construct explicit sets E with $\dim E = 2$ and $|\Delta_X(E)|_1 = 0$. However, we have the following construction.

Theorem 3A. *Let BX be a symmetric convex polygon with $2K$ sides. Then there is a set $E \subset [0,1]^2$ with Hausdorff dimension $\geq K/(K-1)$ such that $|\Delta_X(E)|_1 = 0$.*

Using recent results on Diophantine approximations, we can improve this for almost all polygons BX. Fixing a coordinate system, we can define for any non-degenerate segment $I \subset X$ its *slope* $Sl(I)$: if the line containing I is given by an equation $u_1x_1 + u_2x_2 + u_0 = 0$, then we set $Sl(I) = -u_1/u_2$. We write $Sl(I) = \infty$ if $u_2 = 0$.

Theorem 3B. *For any integer $K \geq 3$ and for almost all $\gamma_1, \ldots, \gamma_K$ (satisfying the Diophantine condition stated in Section 2), the following is true. If BX is a symmetric convex polygon with $2K$ sides, and the slopes of non-parallel sides are equal to $\gamma_1, \ldots, \gamma_K$, then there is an explicit compact set $E \subset X$ such that the Hausdorff dimension of E is 2 and the Lebesgue measure of $\Delta_X(E)$ is 0.*

Actually, we will prove the stronger result: if $K \geq 3$ and if the slopes of 3 non-parallel sides of BX are fixed, then for almost all choices of slopes of other $K-3$ non-parallel sides we can construct the set E as claimed. More specifically, the construction can be carried out provided that the slopes $\gamma_1, \ldots, \gamma_K$ satisfy a certain Diophantine condition stated in Section 2. (Note that for $K \leq 3$ Theorem 3B follows from Theorem 3C below.).

The sets we construct are Cantor-type sets E defined as intersections of a sequence of sets E_j, each of which is a union of balls of radii decreasing to 0 as $j \to \infty$. The main step in the construction is finding a suitable set A_j of the centers of the balls used at j-th step. On one hand, for the distance set of E to be small we need estimates on the size of certain projections (depending on BX) of the difference set $A_j - A_j$. On the other hand, for the lower bound on the dimension of E we require that A_j be well separated, i.e. we need a suitable bound from below on $|a - a'|$ for all $a, a' \in A_j$, $a \neq a'$. This is done in Section 1 in the setting of Theorem 3A. The proof of Theorem 3B is given in Section 2: there, we use the Diophantine condition just mentioned to improve the separation constants.

If we assume that there is a coordinate system in which the slopes of all sides of K are algebraic, then a stronger result is known [KŁ04]. Note in particular that Theorem 3C applies to all polygons BX with 4 or 6 sides.

Theorem 3C. [KŁ04] *If BX is a polygon with finitely many sides, and if there is a coordinate system in which all sides of BX have algebraic slopes, then there is* *a compact $E \subset X$ such that the Hausdorff dimension of E is 2 and the Lebesgue measure of $\Delta_X(E)$ is 0.*

In fact, [KŁ04] gives a recipe for an explicit construction of the set E claimed in the theorem. First, a suitable discrete set of points is constructed in [KŁ04]; to obtain the Cantor-type set E, one then follows the procedure described in [IŁ04].

§1. PROOF OF THEOREM 3A

We may assume that $K \geq 4$, since otherwise Theorem 3C applies. We use $B(x, r)$ to denote the closed Euclidean ball with center at x and with radius r. We also denote $A - A = \{a - a' : \ a, a' \in A\}$ and $A \cdot v = \{a \cdot v : \ a \in A\}$.

Let $b_1, \dots, b_K$ be vectors such that

$$BX = \bigcap_{k=1}^{K} \{x : \ |x \cdot b_k| \leq 1\}.$$

Then for any $x \in X$,

$$\|x\|_X = \max_{1 \leq k \leq K} |x \cdot b_k|. \tag{1.3}$$

Let also $a_1, \dots, a_K$ be unit vectors parallel to the K sides of BX, so that

$$a_j \cdot b_j = 0, \ j = 1, \dots, K. \tag{1.4}$$

Lemma 1.1. *Assume that $K \geq 4$. Then there are arbitrarily large integers n for which we may choose sets $A = A(n) \subset B(0, 1/2)$ such that $|A| = n$ and*

$$|(A - A) \cdot b_k| \ll n^{1-1/K}, \ k = 1, 2, \dots, K, \tag{1.1}$$

(in particular, $|\Delta_X(A)| \ll n^{1-1/K}$), and

$$\|x - x'\|_X \gg n^{-1/2}, \ x, x' \in A, \ x \neq x', \tag{1.2}$$

with the implicit constants independent of n.

Proof. Fix a large integer N, and let $u_1, \dots, u_K$ be numbers in $[1, 2]$, to be determined later. Define

$$S = \Big\{ \sum_{k=1}^{K} \frac{j_k}{N} u_k a_k, \ j_k \in \{1, \dots, N\} \Big\}.$$

We claim that the set

$$U = \{(u_1, \dots, u_K) \in \mathbb{R}^K : \ |S| < N^K\}$$

has K-dimensional measure 0. Indeed, if $|S| < N^K$, then we must have

$$\sum_{k=1}^{K} \frac{j_k}{N} u_k a_k = 0$$

for some $j_1, \dots, j_K \in \{1-N, \dots, N-1\}$, not all zero. Fix such $j_1, \dots, j_K$. Then the $2 \times K$ matrix with columns $\frac{j_k}{N} u_k a_k$, $k = 1, \dots, K$, has rank at least 1, hence its nullspace has dimension at most $K-1$. It follows that U is a union of a finite number of hyperplanes of dimension at most $K-1$, therefore has K-dimensional measure 0 as claimed.

We will assume henceforth that $(u_1, \dots, u_K) \notin U$. Then $|S| = N^K$ and $S \subset B(0, 2K)$. Our goal is to obtain (1.1), (1.2) for $n = N^K$ and $A = (4K)^{-1} S$.

We first prove that (1.1) holds, i.e.

$$|(S-S) \cdot b_k| \ll N^{K-1} \ll n^{1-1/K},\ k = 1, 2, \dots, K. \tag{1.5}$$

Indeed, let $x \in S-S$, then $x = \sum_{k=1}^K \frac{j_k}{N} u_k a_k$ for some $j_1, \dots, j_K \in \{1-N, \dots, N-1\}$. Fix $k_0 \in \{1, \dots, k\}$, then

$$x \cdot b_{k_0} = \sum_{k=1}^K \frac{j_k}{N} u_k a_k \cdot b_{k_0} = \sum_{k \neq k_0} \frac{j_k}{N} u_k a_k \cdot b_{k_0},$$

where we also used (1.4). The last sum can take at most $(2N)^{K-1}$ possible values, which proves (1.5).

It remains to verify that there is a choice of $u_1, \dots, u_K$ for which (1.2) also holds. We will do so by proving that if t is a sufficiently small constant, depending only on K and on the angles between the non-parallel sides of BX, then the set

$$\{(u_1, \dots, u_K) \in [1,2]^K :\ \|x\|_X \leq tN^{-K/2} \text{ for some } x \in S-S\} \tag{1.6}$$

has K-dimensional Lebesgue measure strictly less than 1.

Let $x \in S-S$, then $x = \sum_{k=1}^K \frac{j_k}{N} u_k a_k$ for some $j_k \in \{1-N, \dots, N-1\}$. Suppose that $x \neq 0$ and

$$\|x\|_X \leq tN^{-K/2}. \tag{1.7}$$

Assume that $|j_{k_1}| \geq |j_{k_2}| \geq \dots \geq |j_{k_K}|$, and that $|j_{k_1}| \in [2^s, 2^{s+1})$ for some integer s such that $1 \leq 2^s \leq N$. If we had $|j_{k_2}| < 2^{s-2}/K$, then we would also have

$$\|x\|_X \geq \|\frac{j_{k_1}}{N} u_{k_1} a_{k_1}\|_X - \sum_{k \neq k_1} \|\frac{j_k}{N} u_k a_k\|_X \geq \frac{2^s}{N} - K \cdot \frac{2 \cdot 2^{s-2}}{KN} = \frac{2^{s-1}}{N} \geq \frac{1}{2N}.$$

But if $K \geq 4$, then (1.7) implies that $\|x\|_X \leq tN^{-2}$, which contradicts the last inequality if $t \leq 1$ and $N > 2$. It follows that

$$|j_{k_1}| \geq 2^s,\ |j_{k_2}| \geq 2^{s-2}/K. \tag{1.8}$$

Fix j_{k_1}, j_{k_2} as in (1.8). Fix also $y = \sum_{k \neq k_1, k_2} \frac{j_k}{N} u_k a_k$, and consider the set of $(u_{k_1}, u_{k_2}) \in \mathbb{R}^2$ such that (1.7) holds, i.e.

$$\|\frac{j_{k_1}}{N} u_{k_1} a_{k_1} + \frac{j_{k_2}}{N} u_{k_2} a_{k_2} + y\|_X \leq tN^{-K/2}.$$

By (1.8), this set has 2-dimensional measure

$$\leq c_1(tN^{-K/2})^2 \cdot \frac{N}{2^s} \cdot \frac{NK}{2^{s-2}} = 4c_1K \cdot t^2N^{2-K}/2^{2s}.$$

Here and through the rest of the proof of the lemma, c_1, c_2, c_3 denote constants which may depend on K and on the angles between the non-parallel sides of BX, but are independent of t and N.

Integrating over u_k, $k \neq k_1, k_2$, we see that the set

$$\Big\{(u_1, \dots, u_K) \in [1,2]^K : \ \|\sum_{k=1}^{K} \frac{j_k}{N} u_k a_k\|_X \leq tN^{-K/2}\Big\},$$

with fixed $j_1, \dots, j_K$ such that

$$2^s \leq \max_{k=1,\dots,K} |j_k| < 2^{s+1}, \tag{1.9}$$

has K-dimensional measure $\leq 4c_1K \cdot t^2N^{2-K}/2^{2s}$.

The number of K-tuples $j_1, \dots, j_K$ satisfying (1.9) is $\leq (2^{s+2})^K$, hence summing over all such K-tuples we get a set of measure

$$\leq c_2t^2N^{2-K}2^{(K-2)s}.$$

Now sum over all s with $2^s \leq N$. We find that the measure of the set in (1.6) is

$$\leq c_2 \sum_{s:1\leq 2^s \leq N} t^2N^{2-K}2^{(K-2)s} \leq c_3t^2N^{2-K}N^{K-2} = c_3t^2.$$

This is less than 1 if $t < \sqrt{c_3}$, as claimed.

Proof of Theorem 3A. We construct E as follows. Take a small positive number c which will be specified later. Let $A_j = A(n_j)$ be as in Lemma 1.1, where a nondecreasing sequence $\{n_j\}$ and a sequence $\{N_j\}$ are such that

$$N_j = \prod_{\nu=1}^{j} n_\nu, \quad n_j \to \infty \, (j \to \infty), \quad \log n_{j+1}/\log N_j \to 0 \, (j \to \infty). \tag{1.10}$$

(We consider that the empty product for $j = 0$ is equal to 1.) Also, fix $s = (K-1)/K > 1/2$. Let also c be small enough so that for any j the discs $B(x, cn_j^{-s})$, $x \in A_j$, are mutually disjoint and contained in $B(0,1)$; this is possible by (1.2). Denote

$$\delta_j = cn_j^{-s}, \quad \Delta_j = \prod_{\nu=1}^{j} \delta_j = c^jN_j^{-s}.$$

Let $E_1 = \bigcup_{x\in A_1} B(x, \delta_1)$. We then define $E_2, E_3, \dots$ by induction. Namely, suppose that we have constructed E_j which is a union of N_j disjoint closed discs B_i of radius Δ_j each. Then E_{j+1} is obtained from E_j by replacing each B_i by the image of $\bigcup_{x\in A_{j+1}} B(x, \delta_{j+1})$ under the unique affine mapping which takes $B(0,1)$ to B_i and preserves direction of vectors. We then let $E = \bigcap_{j=1}^{\infty} E_j$.

We will first prove that E has Hausdorff dimension at least $1/s$. The calculation follows closely that in [F85], pp. 16–18.

Let $\mathcal{B}_j$ be the family of all discs of radius Δ_j used in the construction of E_j, and let $\mathcal{B} = \bigcup_{j=0}^{\infty} \mathcal{B}_j$, where we set $\mathcal{B}_0 = \{B(0,1)\}$. We then define

$$\mu(F) = \inf \Big\{ \sum_{i=1}^{\infty} N_{j(i)}^{-1} : \ F \subset \bigcup_{i=1}^{\infty} B(x_i, r_i), B(x_i, r_i) \in \mathcal{B}_{j(i)} \Big\}, \tag{1.11}$$

for all $F \subset E$. Clearly, μ is an outer measure on subsets of E. Observe that if $B = B(x, \Delta_j) \in \mathcal{B}_j$, then

$$N_j^{-1} = n_{j+1} \cdot N_{j+1}^{-1} = \sum_{B' \in \mathcal{B}_{j+1} : B' \subset B} (N_{j+1})^{-1}, \tag{1.12}$$

hence the sum in (1.11) does not change if we replace a disc $B \in \mathcal{B}_j$ by all its subdiscs from the next iteration $\mathcal{B}_{j+1}$. In particular, we may assume that all the discs in the covering of F in (1.11) have radius less than δ for any $\delta > 0$.

We first claim that if $B_0 = B_0(x_0, r_0) \in \mathcal{B}_j$ then

$$\mu(E \cap B_0) = N_j^{-1}. \tag{1.13}$$

The inequality $\mu(E \cap B_0) \leq N_j^{-1}$ is obvious, by taking a covering of $E \cap B_0$ by the single ball B_0. Let now $E \cap B_0 \subset \bigcup_i B_i$, where $B_i \in \mathcal{B}$ has radius $r_i = \Delta_{j(i)}$. We need to prove that

$$\sum r_i^{1/s} \geq r_0^{1/s}. \tag{1.14}$$

Since E is compact and B_i are open relative to E, we may assume that the covering is finite. We may also assume that all B_i are disjoint, since otherwise we may simply remove any discs contained in any other disc of the covering. If the covering consists of the single disc B_0, we are done. Otherwise, let B_I be one of the covering discs with smallest r_i, say $B_I \in \mathcal{B}_j$, and let $\tilde{B}_I \in \mathcal{B}_{j-1}$ be such that $B_I \subset \tilde{B}_I$. Then $\tilde{B}_I \subset B_0$, hence all discs in $\mathcal{B}_j$ contained in $\tilde{B}_I$ are also contained in B_0. By the minimality of r_I, these discs belong to the covering $\{B_i\}$. We then replace all these discs by the single disc $\tilde{B}_I$; by (1.12), the sum on the left side of (1.14) does not change. Iterating this procedure, we eventually arrive at a covering consisting only of B_0, which proves (1.14).

Next, we prove that for any $s' > s$

$$\mu(E \cap B) \ll r^{1/s'} \tag{1.15}$$

for any disc $B = B(x, r)$, not necessarily in $\mathcal{B}$, where the constant in $\ll$ may depend on s'. We may assume that $r \leq 1$, since otherwise we have from (1.13) with $B_0 = B(0,1)$

$$\mu(E \cap B) \leq \mu(E) = 1 \leq r^{1/s'},$$

which proves (1.15). Let $j \geq 0$ be such that $r \in (\Delta_{j+1}, \Delta_j]$, and consider all discs in $\mathcal{B}_j$ which intersect $E \cap B$. They are closed, mutually disjoint discs which intersect

B and have radius no less than r; hence there are at most 6 such discs. Applying (1.13) to each of these discs and summing up, we have

$$\mu(E \cap B) \leq 6N_j^{-1}.$$

Moreover,

$$r > \Delta_{j+1} = N_j^{-s} n_{j+1}^{-s} c^{-j-1},$$

and we get (1.15) using (1.10).

Thus, if $s' > s$ and $\{B_i\}_{i=1}^{\infty}$ is a covering of E by discs of radii r_i, then from (1.15) we have

$$\sum_{i=1}^{\infty} r_i^{1/s'} \gg \sum_{i=1}^{\infty} \mu(E \cap B_i) \geq \mu(E).$$

Taking the infimum over all such coverings, we see that

$$H_{1/s'}(E) > 0.$$

Since $s' > s$ is arbitrary, we conclude that the Hausdorff dimension of E is at least $K/(K-1)$.

It remains to prove that $|\Delta_X(E)|_1 = 0$. From (1.1) we have

$$|(A - A) \cdot b_k| \leq Cn^{1-1/K}, \ k = 1, 2, \ldots, K, \tag{1.16}$$

with C independent of n. We choose c small enough so that

$$cC < 1/2. \tag{1.17}$$

Let D_j be the set of the centers of the discs in $\mathcal{B}_j$. We claim that

$$|(D_j - D_j) \cdot b_k| \leq C^j N_j^s, \ k = 1, 2, \ldots, K. \tag{1.18}$$

Indeed, for $j = 1$ this is (1.16). Assuming (1.18) for j, we now prove it for $j+1$. Let $x, x' \in D_{j+1}$. Then $x \in B(y, \Delta_j)$, $x' \in B(y', \Delta_j)$, $y, y' \in D_j$. We write

$$(x - x') \cdot b_k = (y - y') \cdot b_k + ((x - y) - (x' - y')) \cdot b_k. \tag{1.19}$$

The first term on the right is in $(D_j - D_j) \cdot b_k$, hence has at most $C^j N_j^s$ possible values. Also, by construction $x - y, x' - y'$ are in $\Delta_j A_{j+1}$, hence the second term is in $\Delta_j (A_{j+1} - A_{j+1}) \cdot b_k$ and has at most Cn_{j+1}^s possible values, by (1.16). This gives at most $C^{j+1} N_{j+1}^s$ possible values for (1.19), as required.

By (1.18), (1.3) and the triangle inequality, $\Delta_X(E_j)$ can be covered by at most $KC^j N_j^s$ intervals of length $2c_0 \Delta_j = 2c_0 c^j N_j^{-s}$, where c_0 is the X-diameter of $B(0,1)$. It follows that

$$|\Delta_X(E_j)|_1 \leq 2Kc_0 (cC)^j \leq 2Kc_0 (1/2)^j,$$

by (1.17). The last quantity goes to 0 as $j \to \infty$. Since $\Delta_X(E) \subset \Delta_X(E_j)$, this proves our claim that $|\Delta_X(E)|_1 = 0$. The proof of the theorem is complete.

Remark. It is easy to check that the set constructed in the proof of Theorem 3A has the Hausdorff dimension exactly $K/(K-1)$.

§2. PROOF OF THEOREM 3B

The case $K \le 3$ is covered by Theorem 3C. We consider that $K > 3$ and denote $d = K - 3$. Denote

$$\bar{l} = (l_1, \dots, l_d) \in \mathbb{Z}_+^d,$$

$$\mathcal{L}(L) = \{\bar{l} : 0 \le l_k < L \, (k = 1, \dots, d)\}.$$

For a real vector $\overline{\gamma} = (\gamma_1, \dots, \gamma_d)$ we write $\overline{\gamma} \in (KM)$ if for any positive integer L and for any $\varepsilon > 0$

$$\inf \left| \sum_{\bar{l} \in \mathcal{L}(L)} n_{\bar{l}} \gamma_1^{l_1} \dots \gamma_d^{l_d} \right| \left(\max_{\bar{l} \in \mathcal{L}(L)} |n_{\bar{l}}| \right)^{(1+\varepsilon)L^d} > 0,$$

where infimum is taken over all nonzero integral vectors $\{n_{\bar{l}} : \bar{l} \in \mathcal{L}\}$. The following theorem easily follows from the results of Kleinbock and Margulis [KM98].

Theorem A. *For almost all $\overline{\gamma} \in \mathbb{R}^d$ we have $\overline{\gamma} \in (KM)$.*

The results of [KM98] have been refined in [BKM01], [Be02], [BBKM02].

Now we formulate the main result of this section.

Theorem 4. *Let $\overline{\gamma} \in (KM)$, $K = d+3$, and let BX be a symmetric convex polygon with $2K$ sides, and the slopes of non-parallel sides are equal to $\gamma_1, \dots, \gamma_d, 0, 1,$ and ∞, then there is a compact $E \subset X$ such that the Hausdorff dimension of E is 2 and the Lebesgue measure of $\Delta_X(E)$ is 0.*

Formally, Theorem 4 deals with polygons BX of special kind, but it is easy to see that for any polygon we can make slopes of three sides of it equal to $0, 1, \infty$ by a choice of a coordinate system. Indeed, if I_1, I_2, I_3 are 3 non-parallel sides of BX, then, taking the x_1-coordinate axis and the x_2-coordinate axis of a new coordinate system parallel to I_1 and I_3 respectively, we get $Sl(I_1) = 0$, $Sl(I_3) = \infty$; moreover, the slope of I_2 can be made equal to 1 by scaling and, if necessary, reflecting, the x_2-coordinate axis. Thus, combining Theorem A and Theorem 4 we get Theorem 3 (and also its stronger version mentioned in the end of §0).

We use notation introduced in the beginning of §1. To prove Theorem 4, we need a lemma similar to Lemma 1.1.

Lemma 2.1. *Assume that $K, d, \overline{\gamma}, BX$ satisfy the conditions of Theorem 4. Then for any $\varepsilon > 0$ there are arbitrarily large integers n for which we may choose sets $A = A(n) \subset B(0, 1/2)$ such that $|A| = n$ and*

$$|(A - A) \cdot b_k| \ll n^{(1/2)+\varepsilon}, \ k = 1, 2, \dots, K, \tag{2.1}$$

(in particular, $|\Delta_X(A)| \ll n^{(1/2)+\varepsilon}$), and

$$\|x - x'\|_X \gg n^{-1/2-\varepsilon}, \ x, x' \in A, \ x \ne x', \tag{2.2}$$

where the implicit constants may depend on ϵ but are independent of n.

Proof. Fix a positive integer $L > 1/\varepsilon$. Next, fix a large integer N. Define

$$S_0 = \left\{ \sum_{\bar{l} \in \mathcal{L}(L)} \frac{j_{\bar{l}}}{N} \gamma_1^{l_1} \dots \gamma_d^{l_d} : j_{\bar{l}} \in \{1, \dots, N\} \right\}. \tag{2.3}$$

and $S = S_0 \times S_0$, that is

$$S = \{(x_1, x_2) : x_1, x_2 \in S_0\}.$$

For any $x \in S_0$ we have

$$|x| \le \sum_{\bar{l} \in \mathcal{L}(L)} |\gamma_1|^{l_1} \dots |\gamma_d|^{l_d} = \sum_{l=0}^{L-1} |\gamma_1|^l \cdots \sum_{l=0}^{L-1} |\gamma_d|^l \le \gamma^{dL},$$

where

$$\gamma = \max(|\gamma_1|, \dots, |\gamma|_d) + 1.$$

Therefore, $S \subset B(0, 2\gamma^{dL})$. Our goal is to check that $|S| = n$ and to obtain (2.1), (2.2) for $n = N^{2L^d}$ and $A = (4\gamma^{dL})^{-1} S$.

We consider that a_k ($k = 1, \dots, d$) are parallel to the sides with slopes $\gamma_1, \dots, \gamma_d$ respectively and $a_{d+1}, a_{d+2}, a_{d+3}$ are parallel to the sides with slopes $0, 1, \infty$ respectively. Thus, we can take $b_k = (-\gamma_k, 1)$ for $k = 1, \dots, d$, $b_{d+1} = (0, 1)$, $b_{d+2} = (-1, 1)$, $b_{d+3} = (1, 0)$.

We first prove (2.1) for $k = 1, \dots, d$, i.e.

$$|(S - S) \cdot b_k| \ll n^{(1/2)+\varepsilon}. \tag{2.4}$$

Indeed, for $x \in (S - S) \cdot b_{k_0}$, $k_0 = 1, 2, \dots, d$, we have a representation

$$x \cdot b_{k_0} = -\gamma_k \sum_{\bar{l} \in \mathcal{L}(L)} \frac{j'_{\bar{l}}}{N} \gamma_1^{l_1} \dots \gamma_d^{l_d} + \sum_{\bar{l} \in \mathcal{L}(L)} \frac{j''_{\bar{l}}}{N} \gamma_1^{l_1} \dots \gamma_d^{l_d},$$

where

$$j'_{\bar{l}}, j''_{\bar{l}} \in \{1 - N, \dots, N - 1\} \quad (\bar{l} \in \mathcal{L}(L)).$$

Denote

$$\mathcal{L}(L, k_0) = \{\bar{l} : 0 \le l_k < L \, (k = 1, \dots, d; k \ne k_0), \, 0 \le l_{k_0} \le L\}.$$

Then we have

$$x \cdot b_{k_0} = \sum_{\bar{l} \in \mathcal{L}(L, k_0)} \frac{j_{\bar{l}}}{N} \gamma_1^{l_1} \dots \gamma_d^{l_d}$$

with

$$j_{\bar{l}} \in \{2 - 2N, \dots, 2N - 2\} \quad (\bar{l} \in \mathcal{L}(L, k_0)).$$

Hence,

$$|(S - S) \cdot b_{k_0}| \ll (4N)^{L^d + L^{d-1}}.$$

By the choice of L we have $L^d + L^{d-1} < (1+\varepsilon)L^d$, and we get (2.4). for $k = 1, \dots, d$. Next, (2.4) holds for $k = d+1, d+2, d+3$ because for those k and for $x \in (S-S) \cdot b_k$ we have a representation

$$x \cdot b_{k_0} = \sum_{\bar{l} \in \mathcal{L}(L)} \frac{j_{\bar{l}}}{N} \gamma_1^{l_1} \dots \gamma_d^{l_d}$$

with

$$j_{\bar{l}} \in \{2 - 2N, \dots, 2N - 2\} \quad (\bar{l} \in \mathcal{L}(L)).$$

Hence,

$$|(S - S) \cdot b_{k_0}| \le (4N)^{L^d},$$

and we again get (4.2) for sufficiently large N. So, (2.1) is proved.

Now observe that the supposition $\overline{\gamma} \in (KM)$ implies that elements of S_0 with different representations (2.3) are distinct. This gives $|S_0| = N^{L^d}$ and thus $|S| = |S_0|^2 = n$ as required. Moreover, since for any $x, x' \in S_0$ there is a representation

$$x - x' = \sum_{\bar{l} \in \mathcal{L}(L, k_0)} \frac{j_{\bar{l}}}{N} \gamma_1^{l_1} \dots \gamma_d^{l_d}$$

with

$$j_{\bar{l}} \in \{1 - N, \dots, N - 1\} \quad (\bar{l} \in \mathcal{L}(L, k_0)).$$

we conclude from the supposition $\overline{\gamma} \in (KM)$ that for $x \neq x'$

$$|x - x'| \gg (2N)^{-(1+0.1\varepsilon)L^d - 1}. \tag{2.5}$$

By the choice of L, we have $(1 + 0.1\varepsilon)L^d + 1 \le (1 + 1.1\varepsilon)L^d$, and from (2.5) we get for sufficiently large N and distinct $y, y' \in A$

$$\|y - y'\|_X \gg (4\gamma^{dL})^{-1}(2N)^{-(1+1.1\varepsilon)L^d} \gg N^{-(1+2\varepsilon)L^d} = n^{-1/2-\varepsilon}.$$

This completes the proof of Lemma 2.1.

Proof of Theorem 4. We construct E as follows. Let $A_j = A(n_j)$ be as in Lemma 2.1 with $\varepsilon = \varepsilon_j$, where a nondecreasing sequence $\{n_j\}$, a sequence $\{N_j\}$, and a sequence $\{\varepsilon_j\}$ are such that

$$N_j = \prod_{\nu=1}^{j} n_\nu, \quad n_j \to \infty \, (j \to \infty), \quad \log n_{j+1} / \log N_j \to 0, \, \varepsilon_j \to 0 \, (j \to \infty).$$

(We consider that the empty product for $j = 0$ is equal to 1.) Let also all n_j be large enough so that for any j the discs $B(x, n_j^{-1/2-2\varepsilon_j})$, $x \in A_j$, are mutually disjoint and contained in $B(0, 1)$; this is possible by (2.2). Denote

$$\delta_j = n_j^{-1/2-2\varepsilon_j}, \quad \Delta_j = \prod_{\nu=1}^{j} \delta_j.$$

Let $E_1 = \bigcup_{x \in A_1} B(x, \delta_1)$. We then define $E_2, E_3, \dots$ by induction. Namely, suppose that we have constructed E_j which is a union of N_j disjoint closed discs B_i of radius Δ_j each. Then E_{j+1} is obtained from E_j by replacing each B_i by the image of $\bigcup_{x \in A_{j+1}} B(x, \delta_{j+1})$ under the unique affine mapping which takes $B(0,1)$ to B_i and preserves direction of vectors. We then let $E = \bigcap_{j=1}^{\infty} E_j$. The verification of properties $\dim(E) = 2$ and $|\Delta_X(E)| = 0$ is exactly as in the proof of Theorem 3A.

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